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BUBBLES OF LANDAU-LIFSHITZ EQUATIONS WITH APPLIED FIELDS

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Abstract In this paper, we discuss the Landau-Lifshitz equations with applied magnetic fields. The equations describing the bubbles in the ferromagnets and the behaviors of the solutions near the singularities are given. We found that the applied fields do not affect the bubbles and we have the same conclusions as in reference [1].
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1. Introduction

Let M be a two dimensional manifold without boundary. We consider the following Landau-Lifshitz equation describing the evolution of spin fields in continuum ferromagnets with applied magnetic fields:

$$\partial_t u = -u \times (u \times \Delta u) + u \times \Delta u + u \times h(u), (x,t) \in M \times (0,+\infty) \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x) \tag{1.2}$$

where $|u_0(x)| = 1$ for $x \in \Omega$, $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the spin chain vector and $h(u)$ denotes the applied fields.

Using $|u| = 1$ and $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, we know that (1.1) is equivalent to

$$\partial_t u = \Delta u + u \times \Delta u + |\nabla u|^2 u + u \times h(u), \quad (x,t) \in M \times (0,+\infty). \tag{1.3}$$

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This equation was first derived on phenomenological grounds by Landau-Lifshitz [2]. It plays a fundamental role in the understanding of non-equilibrium magnetism.

Let the applied field $h(u)$ satisfy: $h(u) \in L^{\infty}(W^{1,\infty}(M), \mathbb{R}_+)$. Then for any $u_0 \in H^1(M, S^1)$, the Cauchy problem (1.1) and (1.2) admits unique solution [3] which is regular within finite time and develops singular points beginning at some time, $t = T$ for example, but with at most finitely many points on the plane $t = T$.

In this note, we want to know what will happen near these points and what is the local behavior of the solution near its singularities.

For the solutions of harmonic map heat flow, Struwe [4] has shown that for any $u_0 \in H^1(M, N)$, the solution exists and is unique which is smooth away from at most finitely many points in $M \times \mathbb{R}_+$. Moreover, if the solution u develops a singularity at (x_0, T), by choosing a suitable sequence $t_i \uparrow T$ and rescaling $u(\cdot, t_i)$ properly near x_0, one can obtain finitely many nonconstant harmonic maps ϕ_i $(1 \le i \le L)$ from $\mathbb{R}^2 \to N$ and they can be extended to the harmonic maps from S^2 to N referred as bubbles. Qing [5] proved that if the target manifold N is a sphere, these bubbles are responsible for the energy loss at the singular times $T = \infty$. This is the so called energy identity:

$$\lim_{t_i \uparrow T} \int_{B_\delta(x_0)} |\nabla u|^2(x, t_i)dx = \int_{B_\delta(x_0)} |\nabla u|^2(x, T)dx + \sum_{i=1}^{L} \int_{\mathbb{R}^2} |\nabla \phi_i|^2(x)dx \tag{1.4}$$

where $B_\delta(x_0)$ is a small neighborhood of x_0, which does not contain any other singular points of u. Recently Qing's results have been generated to the flow of harmonic maps to arbitrary compact target manifold in [6–8] In [7] the energy identity (1.4) has been proved for any general target manifolds and for any finite singular time $T < \infty$.

For the ferromagnetic equation without applied fields, [1] proves the similar results as above. However, in our case, as stated in [3], since we do not know whether the energy is decreasing with time, we proved in [3] that the singular points the solution develops at time T may keep singular with time increasing, the exception set of singular points in $M \times \mathbb{R}_+$ may be not a finite set but some lines. So our discussions only apply to the first time $t = T$ at which the solution first develops singular points. Our studies show that the bubbles at layer $t = T$ can be described by the same method as in [1].

In this paper, the following notations are used. For a point $z_0 = (x_0, t_0)$, $P_r(z_0)$ denotes the cylinder

$$P_r(z_0) = \{(x, t) \in \mathbb{R}^2 \times \mathbb{R}_+ : \ |x - x_0| < r, \ t_0 - r^2 < t < t_0\}$$

and $B_r(x_0)$ denotes the ball centered at $x = x_0$ with radius r. If $z_0 = (0, 0)$ or $x_0 = 0$, we simply denote $P_r = P_r(0)$ and $B_r = B_r(0)$. For $Q \subset \mathbb{R}_+ \times \mathbb{R}^2$, $C^{\alpha, 2\alpha}(Q)$ denotes the Hölder space on Q and

$$W_p^{1,2} = \{u \in L^p(Q) : \ u_t, \nabla u, \nabla^2 u \in L^p(Q)\}.$$

The energy of $u(x, t)$ on $\Omega \subset \mathbb{R}^2$ at the time t is denoted by $E(u, \Omega)(t)$ i.e.

$$E(u, \Omega)(t) = \int_{\Omega} e(u(x, t))dx \tag{1.5}$$

where $e(u) = |\nabla u|^2$.

2. Some Lemmas

In the following, we always assume $M = \mathbb{R}^2$ and denote $E(u, M)(t)$ by $E(u)(t)$. Similarly to [1], we may prove the following lemmas.

Lemma 2.1 (Energy Inequality) *Let $u \in C^\infty(M \times (0,T), S^2)$ be a solution of* (1.1) *and* (1.2)*. Then we have*

(1) *For any $t \in (0,T)$*

$$E(u)(t) + \int_0^t \int_M |u_t|^2 dxdt \le E(u)(0) + \int_0^t \int_M (u \times h(u) + h(u))u_t. \tag{2.1}$$

(2) *For any $x_0 \in M$, any $0 < r < R$, and any $0 < t_1 < t_2 < T$, there exists a constant $C_0 > 0$ such that*

$$\begin{aligned} E(u, B_r(x_0))(t_2) &+ \int_{t_1}^{t_2} \int_\Omega |u_t|^2 dxdt \\ &\le E(u, B_r(x_0))(t_1) + \frac{C_0}{(R-r)^2} \int_{t_1}^{t_2} \int_{B_R(x_0)} |\nabla u|^2 dxdt \\ &\quad + \int_{t_1}^{t_2} \int_{B_R(x_0)} \phi^2 |u \times h(u) + h(u)|^2, \end{aligned} \tag{2.2}$$

where $\phi \in C_0^\infty(B_R(x_0))$ is a cutoff function such that $0 \le \phi \le 1$, $\phi \equiv 1$ in $B_r(x_0)$ ($0 < r < R$), $\phi \equiv 0$ outside $B_R(x_0)$ and $|\nabla \phi| \le \frac{C}{R-r}$.

Lemma 2.2 *Let $u \in C^\infty(P_R, S^2)$ be a solution of (1.3) and (1.2). Then for any $0 < r' < r < R$ and for any $p > 4$, $\alpha = \frac{1}{2}(1 - \frac{4}{p})$ there holds*

$$\|\nabla u\|_{C^{\alpha,2\alpha}(P_r)} \le C\Big\{(R-r)^{-2+\frac{4}{p}} + R^{\frac{4}{p}} \|\nabla u\|^2_{L^\infty(P_R)}\Big\} \tag{2.3}$$

$$\|u\|_{W_p^{1,2}(P_r)} \le C\Big\{(R-r)^{-2+\frac{4}{p}} + R^{\frac{4}{p}} \|\nabla u\|^2_{L^\infty(P_R)}\Big\} \tag{2.4}$$

$$\begin{aligned} \|D^2 u\|_{L^\infty(P_{r'})} \le C\Big\{ & r^{\frac{4}{p}} \|\nabla u\|^3_{L^\infty(P_r)} + \|\nabla u\|_{L^\infty(P_r)} \|D^2 u\|_{L^p(P_r)} \\ & + (r-r')^{-2+\frac{4}{p}}(1 + \|\nabla u\|_{L^\infty}) + (r-r')^{-1} \|D^2 u\|^2_{L^p(P_r)}\Big\} \end{aligned} \tag{2.5}$$

where C is independent of r, r', R.

Lemma 2.3 (Small Energy Regularity) *There exist $\varepsilon_0 > 0$ and $C > 0$ such that for any solution $u \in C^\infty(P_R(z_0), S^2)$ of (1.1) and (1.2), if*

$$\sup_{t \in [t_0 - R^2, t_0]} \int_{B_R(x_0)} |\nabla u|^2(x,t) dx \le \varepsilon_0 \tag{2.6}$$

then

$$\|\nabla u\|_{L^\infty(P_{\frac{R}{3}}(z_0))} \le CR^{-1} \tag{2.7}$$

$$\|\nabla u\|_{C^{\alpha,2\alpha}P_{\frac{R}{4}}(z_0)} \le C(\alpha)R^{-1-2\alpha},\ 0<\alpha<1/2 \tag{2.8}$$

and

$$\|D^2u\|_{L^\infty P_{\frac{R}{5}}(z_0)} \le CR^{-2}. \tag{2.9}$$

3. Bubbles

In this section we shall study the behavior of the solutions of (1.1) near their singularities.

Let $u \in C^\infty(M\times(0,T),S^2)$ be a solution of (1.1) with $u_0 \in H^1(M,S^2)$ which develops singularities at $t=T$ (see [3]). By Lemma 2.3 it is easy to see that the singularities (x,T) are characterized by

$$\limsup_{t\uparrow T}\int_{B_R(x)}|\nabla u|^2(y,t)dydt \ge \varepsilon_0,\ \forall R>0 \tag{3.1}$$

where $\varepsilon_0>0$ is determined by Lemma 2.3.

Let $(x_1,T),\cdots,(x_k,T)$ be the singular points at $t=T$. Choose $R>0$ such that $B_{2R}(x_i)\cap B_{2R}(x_j)=\emptyset$ for $i\ne j$. Then it follows from the energy inequality (2.2) that for $\tau = T-\dfrac{\varepsilon_0R^2}{2C_0E(u)(0)}$

$$\begin{aligned} E(u,B_R(x_i))(t) &+ \int_\tau^t\int_{B_R(x_i)}|u_t|^2dxdt \\ &\le CR^{-2}\int_\tau^t\int_{B_{2R}(x_i)}|\nabla u|^2dxdt + E(u,B_{2R}(x_i))(\tau) \\ &\quad + \int_\tau^t\int_{B_{2R}(x_i)}|u\times h(u)+h(u)|^2dxdt. \end{aligned} \tag{3.2}$$

Here it follows from (2.1) that

$$\begin{aligned} E(u)(\tau) &= \int_M|\nabla u|^2dx \\ &\le E(u)(0)+\int_0^\tau\int_M|u\times h(u)+h(u)|^2. \end{aligned} \tag{3.3}$$

Therefore

$$\sum_{i=1}^k E(u,B_R(x_i))(t) + \sum_{i=1}^k\int_\tau^t\int_{B_R(x_i)}|u_t|^2$$

$$\leq CR^{-2}\sum_{i=1}^{k}\int_{\tau}^{t}\int_{B_{2R}(x_i)}|\nabla u|^2+\sum_{i=1}^{k}E(u,B_{2R}(x_i))(\tau).$$
$$+\sum_{i=1}^{k}\int_{\tau}^{t}\int_{B_{2R}(x_i)}|u\times h(u)+h(u)|^2. \tag{3.4}$$

The equation (3.3) and (3.4) yields

$$\begin{aligned}k\varepsilon_0 &\leq \sum_{i=1}^{k}\limsup_{t\uparrow T}\int_{B_R(x_i)\cap M}|\nabla u|^2(x,t)dx\\ &\leq C_0R^{-2}\frac{\varepsilon_0R^2}{2C_0E(u)(0)}\left[kE(u)(0)+\int_0^{\tau}\int_M|h(u)|^2\right]\\ &\quad+\limsup_{t\uparrow T}\int_{\tau}^{t}\int_M|h(u)|^2dxdt+\sum_{i=1}^{k}\int_{B_{2R}(x_i)}|\nabla u|^2(x,\tau)dx.\end{aligned} \tag{3.5}$$

It follows from (3.5) and the continuity of u that

$$k\leq C_1 \tag{3.6}$$

where C_1 is a finite number (One may compare this result with the one in [3]). Hence we may choose a number $\delta>0$ such that $B_\delta(x_i)\cap B_\delta(x_j)=\emptyset$ if $i\neq j$. For simplicity, we may assume that there is only one singular point, x_0, on $t=T$ and consider the only one ball $B_\delta(x_0)$.

In this case we have the following theorem describing the bubble which is just the same as in [1]. In other word, the applied fields do not affect the bubbles.

Theorem 3.1 *There exist sequence $t_m\uparrow T$, $x_m\to x_0$, $R_m\downarrow 0$ and a smooth non-constant harmonic mapping $\phi: R^2\to S^2$ such that as $m\to\infty$.*

(1) *The rescaling sequence $v_m(x)=u(R_mx+x_m,t_m)\to\phi(x)$ strongly in $H^1_{\rm loc}\cap C^1_{\rm loc}(R^2,S^2)$. Moreover, ϕ has finite energy and extends to a harmonic mapping: $S^2\to S^2$, referred as bubble.*

(2) *The sequence $u(\cdot,t_m)\to u(\cdot,T)$ strongly in $H^1_{\rm loc}\cap C^1_{\rm loc}(B_\delta(x_0)\setminus\{x_0\})$ but not in $H^1(B_\delta(x_0))$.*

(3) *If $T=+\infty$, we have, in addition, that $u_t(\cdot,t_m)\to 0$ strongly in $L^2(B_\delta(x_0))$.*

Proof It follows from (2.1) that for $u_0\in H^1(M,S^2)$ and $0<t<T$

$$E(u)(t)+\frac{1}{2}\int_0^t\int_M|u_t|^2dxdt\leq M^* \tag{3.7}$$

for some positive constant M^*. Therefore we may choose a sequence $\tau_m\uparrow T$ such that

$$\nabla u(\cdot,\tau_m)\rightharpoonup\nabla u(\cdot,T),\ \text{weakly in}\ \ L^2(M).$$

If $T=+\infty$, then

$$u_t(\cdot,\tau_m)\to 0,\ \text{strongly in}\ \ L^2(M).$$

From Lemma 2.3, if

$$\varepsilon_0\geq \sup_{[t_0-R^2,t_0]}\int_{B_R(x_0)}|\nabla u|^2dx$$

then (x_0,t_0) is a regular point. Now (x_0,T) is a singular point, so we have

$$\varepsilon_0\leq \sup_{[T-R^2,T]}\int_{B_R(x_0)}|\nabla u|^2dx.$$

Take $x_m\to x_0$, $R_m\downarrow 0$, such that

$$\begin{aligned}\varepsilon_0 &= \sup_{[T-\delta^2,\tau_m]}\sup_{x\in B_\delta(x_0)}\int_{B_{Rm}(x)}|\nabla u|^2(y,t)dy\\ &= \int_{B_{Rm}(x_m)}|\nabla u|^2(x,\tau_m)dx. \end{aligned}\tag{3.8}$$

Let C_0 be as before and

$$0<C_1^2\leq \frac{\varepsilon^2}{4C_0M^*}$$

where M^* is determined by (3.7). For any $t\in[\tau_m-C_1^2R_m^2,\tau_m]$ it follows from the integral inequality (for $r=R_m$, $R=2R_m$, $x_0=x_m$, $t_1=t$, $t_2=\tau_m$) that

$$\begin{aligned}\varepsilon_0 &= \int_{B_{Rm}(x_m)}|\nabla u|^2(x,\tau_m)dx\\ &\leq E(u,B_{2R_m}(x_m)(t))+C_0R_m^{-2}\int_t^{\tau_m}\int_{B_{2Rm}(x_m)}|\nabla u|^2\\ &\quad+\int_t^{\tau_m}\int_{B_{2Rm}(x_m)}|h(u)|^2. \end{aligned}\tag{3.9}$$

Applying (2.2) (since $0\leq\tau_m-t\leq C_1^2R_m^2\leq \frac{\varepsilon_0}{4C_0E(u)(0)}R_m^2$), we have

$$\begin{aligned}C_0R_m^{-2}\int_t^{\tau_m}\int_{B_{2Rm}(x_m)}|\nabla u|^2 &\leq C_0R_m^{-2}(\tau_m-t)\int_M|\nabla u|^2\\ &\leq C_0R_m^{-2}[E(u)(0)+\int_0^{\tau_m}\int_M|h(u)|^2]\\ &\leq \frac{\varepsilon_0}{4}. \end{aligned}\tag{3.10}$$

Hence we have

$$\begin{aligned}\varepsilon_0 &= \int_{B_{Rm}(x_m)}|\nabla u|^2(x,\tau_m)dx\\ &\leq E(u,B_{2R_m}(x_m))(t)+\frac{\varepsilon_0}{4}+\int_t^{\tau_m}\int_{B_{2Rm}(x_m)}|h(u)|^2\\ &\leq \int_{B_{2Rm}(x_m)}|\nabla u|^2+\frac{\varepsilon_0}{4}+\frac{\varepsilon_0}{4} \end{aligned}\tag{3.11}$$

under the assumptions on h.

Finally we have

$$\int_{B_{2Rm}(x_m)} |\nabla u|^2(x,t)dxdt \geq \frac{\varepsilon_0}{2}. \tag{3.12}$$

Denote $D_m = \{x \in \mathbb{R}^2 : R_m x + x_m \in B_\delta(x_0)\}$ and let $w_m(x,t) = u(R_m x + x_m, R_m^2 t + \tau_m)$, then $w_m : D_m \times [-C_1^2, 0] \to S^2$ and solves

$$\partial_t w_m = \Delta w_m + |\nabla w_m|^2 w_m + w_m \times \Delta w_m + R_m^2 w_m \times h(w_m) \tag{3.13}$$

and as $m \to \infty$

$$\int_{-C_1^2}^{0} \int_{D_m} |\partial_t w_m|^2 \leq \int_{\tau_m - C_1^2 R_m^2}^{\tau_m} \int_M |\partial_t u|^2 \to 0. \tag{3.14}$$

It follows from (3.12) that for all $t \in [-C_1^2, 0]$

$$\int_{B_2} |\nabla w_m|^2(x,t)dxdt = \int_{B_{2Rm}(x_m)} |\nabla u|^2(x, R_m^2 t + \tau_m)dx \geq \frac{\varepsilon_0}{2}. \tag{3.15}$$

Using (3.8) we know that for m large enough

$$\begin{aligned} \varepsilon_0 &= \sup_{[\tau_m - C_1^2, \tau_m]} \sup_{x \in B_\delta(x_0)} \int_{B_{Rm}(x)} |\nabla u|^2(y,t)dy \\ &\geq \sup_{[-C_1^2, 0]} \sup_{x \in D_m} \int_{B_1(x)} |\nabla w_m|^2(y,t)dy \end{aligned} \tag{3.16}$$

where we have used $R_m y + x_m \to x_0$.

Now applying Lemma 2.3 to the equation (3.13) we get

$$\sup_{t \in [\frac{-C_1^2}{4}, 0]} \|\nabla w_m\|_{C^{2\alpha}_{\mathrm{loc}}(\mathbb{R}^2)} \leq C. \tag{3.17}$$

Combining (3.14), (3.15), (3.16) with (3.17) we may choose a sequence $\eta_m \in (\frac{-C_1^2}{4}, 0)$ such that as $\eta_m \to \infty$

$$\int_{D_m} |\partial_t w_m|^2(x, \eta_m)dx \to 0, \tag{3.18}$$

$$\int_{B_2} |\nabla w_m|^2(x, \eta_m)dx \geq \frac{\varepsilon_0}{2}, \tag{3.19}$$

$$\|\nabla w_m(\cdot, \eta_m)\|_{C^{2\alpha}_{\mathrm{loc}}(\mathbb{R}^2)} \leq C. \tag{3.20}$$

Hence there exists a subsequence of $\{w_m(x, \eta_m)\}$ (still denote it by $\{w_m(x, \eta_m)\}$) and a mapping $\phi : \mathbb{R}^2 \to S^2$ such that

$$w_m(\cdot, \eta_m) \to \phi, \text{ strongly in } H^1_{\mathrm{loc}} \cap C^1_{\mathrm{loc}}(\mathbb{R}^2, S^2). \tag{3.21}$$

Let $t = \eta_m$ in (3.13) and then let $m \to \infty$. We get from (3.18), (3.20) and $R_m \to 0$ that

$$\Delta \phi + \phi \times \Delta \phi + |\nabla \phi|^2 \phi = 0, \tag{3.22}$$

therefore

$$\phi \times [\Delta\phi + \phi \times \Delta\phi + |\nabla\phi|^2\phi] = 0,$$

that is

$$\phi \times \Delta\phi + \phi \times (\phi \times \Delta\phi) = 0.$$

Hence

$$\phi \times \Delta\phi = -\phi \times (\phi \times \Delta\phi) = \Delta\phi + |\nabla\phi|^2\phi.$$

This implies

$$\Delta\phi + |\nabla\phi|^2\phi = 0. \tag{3.23}$$

On the other hand $\phi \in H^1_{\rm loc} \cap C^1_{\rm loc}(\mathbb{R}^2, S^2)$, then we have from (3.23) that ϕ is a smooth harmonic mapping which is not a constant since $\int_{B_2} |\nabla\phi|^2 dx \geq \frac{\varepsilon_0}{2}$.

It follows from (3.21) and (2.1) that

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla\phi|^2 dx &\leq \lim_{m\to\infty} \sup \int_{D_m} |\nabla w_m|^2(x, \eta_m) dx \\ &\leq \lim_{m\to\infty} \sup \int_M |\nabla u|^2(x, R_m^2\eta_m + \tau_m) dx \\ &\leq \lim_{m\to\infty} \sup [E(u)(0) + \int_0^{R_m^2\eta_m+\tau_m} \int_M |h(u)|^2] \\ &\leq M^*. \end{aligned} \tag{3.24}$$

This implies that ϕ has finite energy.

From the conformal equivalence we know $\mathbb{R}^2 = S^2 \backslash \{p\}$, therefore ϕ may be extended to a smooth harmonic mapping from S^2 to S^2. Now let $t_m = R_m^2\eta_m + \tau_m$, then $v_m(x) = w_m(x, \eta_m) = u(R_m x + x_m, t_m)$ is the desired re-scaling sequence in conclusion (1) of Theorem 3.1.

If $T = +\infty$, then it follows from

$$u_t(\cdot, \tau_m) \to 0, \text{ strongly } \text{ in } L^2(M)$$

that

$$\int_{D_m} |\partial_t w_m|^2(x, 0) \leq R_m^2 \int_{B_\delta(x_0)} |\partial_t u|^2(x, t_m) dx \to 0.$$

This implies that (3.18), (3.19) and (3.20) hold for $\eta_m = 0$ since (3.15) and (3.16) hold for $t \in [-C_1^2, 0]$.

So if $T = +\infty$, by letting $t_m = \tau_m$, we may get conclusion (1) and (3) of Theorem 3.1. Now we prove that (2) of the theorem holds.

It follows from the characterizations of the singularities and Lemma 2.3 that $\forall x \in B_\delta(x_0) \setminus \{x_0\}$ there holds

$$\|\nabla u\|_{C^{\alpha,2\alpha}(P_{R/4}(x,T))} \leq C(R)$$

for some $R > 0$ and all $0 < \alpha < 1/2$. Especially, we have

$$\|\nabla u(\cdot, t_m)\|_{C^{2\alpha}(B_{R/4}(x))} \leq C.$$

This implies that there exists a subsequence of $u(\cdot, t_m)$ (still denote it by $u(\cdot, t_m)$) such that

$$\nabla u(\cdot, t_m) \to \nabla u(\cdot, T), \text{ strongly in } C^{2\alpha'}(B_{R/4}(x))$$

for some $0 < \alpha' < \alpha$. So the conclusion (2) of Theorem 3.1 follows.

Finally we prove that $\nabla u(\cdot, t_m)$ can not converge to $\nabla u(\cdot, T)$ in $H^1(B_\delta(x_0))$. If on the contrary, we assume

$$\nabla u(\cdot, t_m) \to \nabla u(\cdot, T), \text{ strongly in } H^1(B_\delta(x_0)),$$

then we should have from $E(u)(T) \leq M^*$ that there exists $r > 0$ such that

$$\int_{B_r(x_0)} |\nabla u|^2(x, T)dx \leq \frac{\varepsilon_0}{4}.$$

Thus there exists $\overline{m}$ such that for $m > \overline{m}$ we have

$$\int_{B_r(x_0)} |\nabla u|^2(x, t_m)dx \leq \frac{\varepsilon_0}{2}.$$

Choose t_J in such set and $m > \overline{m}$ such that $T - t_J < \dfrac{r^2}{8C_0 M^*}$ then as before we have

$$\sup_{t\in[t_J,T]} \int_{B_{\frac{r}{2}}(x_0)} |\nabla u|^2(x, T)dx \leq \varepsilon_0.$$

This combined with the small energy regularity theorem yields

$$\sup_{t\in[T-\frac{r}{8},T]} \|\nabla u\|_{C^{2\alpha}(B_{\frac{r}{8}}(x_0))} \leq C(r).$$

This implies that (x_0, T) is not a singular point. This leads to a contradiction.

The theorem is proved. Q.E.D

4. Energy Identity

The conclusion (1) of Theorem 3.1 in above section describes the bubbles. In fact, if we choose different subsequence of $\{w_m(\cdot, \eta_m)\} = \{u(R_m x + x_m, t_m)\}$, then we get different bubbles at each singular point. Noticing that u has only finite energy and since each bubble ϕ is not a constant and each bubble cut off some energy from the singularity, we know that at each singular point, there are only finitely many bubbles.

In this section we shall prove that the energy concentrated at each singular point is consumed by such bubbles. That is, we want to prove:

Theorem 4.1 *Let* $u \in C^\infty(M\times(0,T),S^2)$ *be a solution of (1.1) with* $u_0 \in H^1(M,S^2)$ *and assume that* u *develops singular points at* $t = T$, (x_i,T) $(i = 1,2,\cdots,p)$. *Let* ϕ_j $(j = 1,2,\cdots,q)$ *be the bubbles of these singular points in the sense of Theorem 1.1* $(q \geq p)$. *Then one has*

$$\lim_{t\uparrow T} E(u)(t) = E(u)(T) + \sum_{j=1}^{q} E(\phi_j, \mathbb{R}^2). \tag{4.1}$$

To prove this theorem, we first recall two lemmas from [7]

Lemma 4.1 [7] *There exists* $\varepsilon_1 > 0$ *such that if* $u \in C^\infty([R_1,R_2]\times S^1, S^2)$ *satisfies*

$$u_{rr} + u_{\theta\theta} = |\nabla u|^2 u + F \tag{4.2}$$

and $\sup\limits_{[R_1,R_2]\times S^1} |\nabla u| \leq \varepsilon_1$, *then there is a constant* $C > 0$ *such that*

$$\begin{aligned}\int_{R_1}^{R_2}\left(\int_{S^1\times\{r\}}|u_\theta|^2 d\theta\right)^{1/2} dr \\ \leq C\Bigg\{\left(\int_{S^1\times\{R_1\}}|u_\theta|^2 d\theta\right)^{1/2} + \left(\int_{S^1\times\{R_2\}}|u_\theta|^2 d\theta\right)^{1/2} \\ + \left(\int_{R_1}^{R_2}(e^{2r}\int_{S^1\times\{r\}}|F|^2 d\theta)dr\right)^{1/2}\Bigg\}.\end{aligned} \tag{4.3}$$

Lemma 4.2 [7] *Let* $u \in C^\infty(B_1,S^2)$ *be a solution of (4.2). If* $F \in L^2(B_1)$, *then for any* $0 < R < 1$

$$\int_{\partial B_R}|u_r|^2 ds \leq R^{-2}\int_{\partial B_R}|u_\theta|^2 ds + 2\int_{B_R}|F||\nabla u| dx. \tag{4.4}$$

Using these two lemmas we first prove:

Lemma 4.3 *Let* $u \in C^\infty(M\times(0,T),S^2)$ *be a solution of (1.1) with* $u_0 \in H^1(M,S^2)$ *and assume that* (x_0,T) *is the unique singular point of* u *on* $B_\delta(x_0)\times\{t = T\}$. *Then there exists a positive constant* $L > 0$ *such that*

$$\lim_{t\uparrow T} E(u,B_\delta(x_0))(t) = E(u,B_\delta(x_0))(T) + L. \tag{4.5}$$

Proof There exists a subsequence $\{t_m\}$: $t_m \uparrow T$ such that

$$E(u,B_\delta(x_0))(t_m) = E(u,B_\delta(x_0)\setminus B_\eta(x_0))(t_m) + E(u,B_\eta(x_0))(t_m). \tag{4.6}$$

Choose $t_m \uparrow T$ such that $\lim\limits_{t\uparrow T} E(u,B_\delta(x_0))(t_m)$ exists. We have $\lim\limits_{t\uparrow T} E(u,B_\eta(x_0))(t_m)$ exists. Sending $\eta_m \to 0$ we have

$$\lim_{m\to\infty} E(u,B_\delta(x_0))(t_m) = E(u,B_\delta(x_0))(T) + L,\ L > 0. \tag{4.7}$$

Suppose $\{s_m\}$ is a sequence such that $s_m \uparrow T$ and

$$\lim_{m\to\infty} E(u, B_\delta(x_0))(s_m) = E(u, B_\delta(x_0))(T) + S,\ \ S > 0 \tag{4.8}$$

we want to prove $S = T$.

Taking a subsequence of $\{s_m\}$ (still denote by $\{s_m\}$) such that $s_m \le t_m \le s_{m+1} \le t_{m+1}$ and noting that $t_m \uparrow T$, $s_m \uparrow T$, $T < +\infty$, we get $s_{m+1} - s_m \to 0$, $t_m - s_m \to 0$. Using (2.2), for $0 < \eta < \delta$, $\frac{1}{j} < \eta$, $t_1 = s_m, t_2 = t_m$, we have

$$\begin{aligned} E(u, B_{\frac{1}{j}}(x_0))(t_m) + \int_{t_1}^{t_2}\int_M |u_t|^2 & \\ \le C_0(\eta - \frac{1}{j})^{-2} \int_{s_m}^{t_m}\int_{B_\eta(x_0)} |\nabla u|^2 &+ E(u, B_\eta(x_0))(s_m) \\ + \int_{s_m}^{t_m}\int_{B_\eta(x_0)} |h(u)|^2. & \end{aligned} \tag{4.9}$$

Therefore

$$E(u, B_{\frac{1}{j}}(x_0))(t_m) - C_0(\eta - \frac{1}{j})^{-2}(t_m - s_m)E(u)(0) - (t_m - s_m)C_1 \le E(u, B_\eta(x_0))(s_m). \tag{4.10}$$

And hence

$$\begin{aligned} E(u, B_\delta(x_0))(T) + S &= \lim_{m\to\infty}\{E(u, B_\delta(x_0)\setminus B_\eta(x_0))(s_m) + E(u, B_\eta(x_0))(s_m)\} \\ &\ge E(u, B_\delta(x_0)\setminus B_\eta(x_0))(T) + \lim_{m\to\infty} E(u, B_{\frac{1}{j}}(x_0))(t_m) \\ &= E(u, B_\delta(x_0)\setminus B_\eta(x_0))(T) + E(u, B_{\frac{1}{j}}(x_0))(T) + L. \end{aligned} \tag{4.11}$$

Letting $\eta \to 0$ and $j \to \infty$ in (4.11) we are led to $S \ge L$. Similarly we can prove $S \le L$. Therefore $S = L$. The lemma is proved.

The proof shows that $\lim\limits_{t\uparrow T} E(u)(t)$ exists.

Now we are in the position to prove Theorem 4.1.

Proof of Theorem 4.1 We assume that there is only one singular point $(0, T)$ at $t = T$, and there is only one bubble ϕ separated at $(0, T)$ in the sense of above section.

First we assume $T = +\infty$.

Let t_m, x_m, R_m be the sequences in Theorem 3.1. Then the sequence $u(\cdot, t_m)$ satisfies (3.3) and $u_t(\cdot, t_m)$ solves (3.4). Denote $u_m = u(\cdot, t_m)$, then $v_m = u_m(R_m x + x_m)$ satisfies (3.2). For small $\delta > 0$ and large $R > 0$, set $I_A = [R_m R, \delta]$, $I_B = [|\ln\delta|, |\ln R_m R|]$, $A_m(\delta, R) = I_A \times S^1$ and $B_m(\delta, R) = I_B \times S^1$, (r_m, θ_m) is the coordinates centered at x_m. Define a mapping $f(r, \theta)$ for $(r, \theta) \in \mathbb{R}^1 \times S^1$ by $f(r, \theta) = (e^{-r}, \theta)$. Let $\mathbb{R}^1 \times S^1$ be given the flat metric $dr^2 + d\theta^2$. Then f is a conform mapping from $B_m(\delta, R)$ to $A_m(\delta, R)$. Let $w_m(r, \theta) = u_m(f(r, \theta)) = u_m(e^{-r}, \theta)$. Then it follows from (1.3) that

$$\Delta w_m + w_m \times \Delta w_m + |\nabla w_m|^2 w_m = g_m,\ \text{ in }\ [|\ln\delta|, \infty] \times S^1 \tag{4.12}$$

where

$$g_m = e^{-2r} u_t(e^{-r}, \theta, t_m) - e^{-2r} u \times h(u). \tag{4.13}$$

Taking cross product with w_m on both sides of (4.12) and using

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c,$$

we have

$$\Delta w_m + |\nabla w_m|^2 w_m = \frac{1}{2}(g_m - w_m \times g_m), \text{ in } [|\ln \delta|, \infty] \times S^1 \tag{4.14}$$

and there also holds

$$E(w_m, B_m(\delta, R)) = E(u_m, A_m(\delta, R)). \tag{4.15}$$

Now we are in the position to use the same method as in [1] to finish the proof of the equality

$$\lim_{t \uparrow T} E(u)(t) = E(\phi, \mathbb{R}^2) + E(u)(T). \tag{4.16}$$

We omit the details. This proves (4.1).

Similar equality holds for $T < \infty$. Theorem 4.1 is proved.

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