J. Partial Diff. Eqs. **19**(2006), 16–25
 Vol.19 No.1

A HARNACK INEQUALITY APPROACH TO THE INTERIOR REGULARITY GRADIENT ESTIMATES OF GEOMETRIC EQUATIONS*

Luis A. Caffarelli
(Department of Mathematics, The University of Texas at Austin,
1 University Station C1200, Austin, TX 78712-0257, USA,)
(E-mail: caffarel@math.utexas.edu)
Wang Lihe
(Department of Mathematics, The University of Iowa
14 MacLean Hall, Iowa City, IA 52242-1419, U.S.A.)
(E-mail: lwang@math.uiowa.edu)

Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

(Received Nov. 19, 2005)

Abstract In this paper we prove the gradient estimates for fully nonlinear geometric equation using a normal perturbation techniques.

Key Words Fully nonlinear equation; geometric equation; gradient estimate.
2000 MR Subject Classification 35B65, 35K55.
Chinese Library Classification O175.25,O175.29.

1. Introduction

We study the interior Lipschitz regularity for equations of the type

$$F(II, \nu) = 0, \tag{1}$$

where $II = II(S)$ is the second fundamental form of a hypersurface S in R^{n+1} ($n \geq 2$) and $\nu = \nu(S)$ is its normal. We always assume that F is uniformly elliptic in the tangential direction of the surface and Lipschitz in ν with Lipschitz constant linear in $|II|$. See more precise definitions of these terms in the next section. The main goal of this paper is to show that any C^1 viscosity solution has Lipschitz apriori estimates in its interior.

The main theorem of this paper is the following.

Theorem 1 *Suppose F satisfies* (3), (6) *and* (7). *Assume S is, in the sense of* (3), *a C^1 solution of* (1) *in the cylinder $C_1 = B_1 \times [-K, K]$, where $K = [S]_{L^\infty(B_1)} +$*

*Both authors were supported by National Science Foundation.

$|X_nS(0)|+1$, in this coordinate system and $X_n(S)$ the X_n coordinate of S such that $(0, X_n(S)) \in S$. Then $S \in Lip(C_{\frac{1}{2}})$. Moreover $\|S\|_{Lip(C_{\frac{1}{2}})}$ has a upper bound depending only on $\|S\|_{L^\infty}$, ellipticity and the dimension. In general, we denote $C_r(x) = B_r(x) \times [-\|S\|_{L^\infty(B_r)} - X_nS(0) - 1, \|S\|_{Lip(B_1)} + X_nS(0) + 1]$.

By the theorems in [1], the Lipschitz regularity of (1) implies $C^{1,\alpha}$ regularity. This kind of estimates is not new for the classical solutions. However, we think our approach gives not only a regularity theory but also a much better geometric intuition of how regularity 'propagates' along a solution surface (see Lemma 5).

Our methods are related with the work of N. Korevaar [2] and its extensions by Korevaar [3], Y. Li [4] and B. Guan and J. Spruck [5], where Lipschitz estimates for C^3 solutions were obtained.

To the contrast, our proof is more natural and geometrical. It is along the line of the regularity theory for free boundaries developed by the first author [6].

In fact, it is one of our current objects to unify the theory of free boundaries and the theory of elliptic equations.

The main technical contribution of this paper is the construction of 'variable' parallel surfaces, which are still subsolutions to the equation of the surface.

Our method also applies to parabolic equations and equations of motions of surfaces by their curvatures.

2. Viscosity Solutions and Preliminary Considerations

Instead of considering the second fundamental form itself, we will deal with its representations in coordinate systems of R^{n+1}.

Definition 1 *Let M be an $(n+1)\times(n+1)$ matrix. If*

$$v^T\ Mv = II(v, v)$$

for

$$v \in TS = \{\nu \cdot v = 0\},$$

we say that M is a representation of II.

Clearly, the representations of a second fundamental form are not unique.

Equation (1) is equivalent to equations of the form,

$$F(M, \nu) = 0 \qquad (2)$$

with the condition

$$\mu(e \otimes \nu + \nu \otimes e) + b(\nu \otimes \nu), \nu) = F(M, \nu) \qquad (3)$$

where $e \cdot \nu = 0$ and μ, b are real numbers. That is to say that F depends only on the 'tangential' part of M.

In representations, its calculations are easier than that for the fundamental form itself. In particular, a change of coordinates for the representations is simply: $G^T MG$ for some $(n+1) \times (n+1)$ orthogonal matrix G.

Example 1 Minimal Surface Equation

$$\begin{aligned} F(M, \nu) &= \operatorname{tr} M - \nu^T M \nu \\ &= \text{ mean curvature.} \end{aligned}$$

In fact, this equation also satisfies the condition, which can be easily checked that

$$|F(M, \nu) - F(M, \mu)| \leq |M||\nu - \mu|^2, \tag{4}$$

which analytically says the equation is only tangential. Geometrically it can be viewed as the tangential part of the normal vectors to S^{n-1} is always second order.

Definition 2 *We call F a geometric equation if (3) holds.*

Now, let us recall the basic properties for the distance function of a surface:

$$d(Y, S) = \min\{d(Y, X) | X \in S\}.$$

Suppose $d(Y, S) = d(Y, X)$ for $X \in S$. If S is C^2 at X, then there is a convenient coordinate system $e_1, e_2, \cdots, e_{n+1}$ at X such that the second fundamental form of S at X is diagonal

$$-D^2 d = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \\ & & 0 \end{bmatrix},$$

where μ_i are the principal curvatures of S and e_{n+1} is the normal at X. The normal for S^g at Y is $\frac{e_{n+1} - Dg}{|e_{n+1} - Dg|}$. Moreover if $1 - d\nu_i > 0$ for $i = 1, \cdots, n$, $-D^2 d$ at Y is also diagonal

$$\begin{bmatrix} \frac{\mu_1}{1-d\mu_1} & & 0 \\ & \ddots & \\ & \frac{\mu_n}{1-d\mu_n} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Lemma 2 *Let G be a smooth function. Suppose $G(0) = 0$, $DG(0) = \alpha e_{n+1}$ for some $\alpha \neq 0$. Then the second fundamental form of $G(X) = 0$ at 0 is*

$$-\frac{1}{\alpha} J(D^2 G),$$

where J is the projection to the left up corner. In particular, we may take $-\frac{1}{\alpha} D^2 G$ as its representation.

Proof Since $DG(0) = \alpha e_{n+1}$, we have

$$x_{n+1} = \frac{1}{2} x^T IIx + \text{ higher order terms.}$$

At the same time

$$\begin{aligned} 0 = G(X) &= \alpha x_{n+1} + \frac{1}{2}(x, x_{n+1}) D^2 G \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} + \text{ h.o.t.} \\ &= \alpha x_{n+1} + \frac{1}{2} x^T (J D^2 G) x + \text{ h.o.t..} \end{aligned}$$

The lemma follows immediately.

Definition 3 *Let D be an open domain in R^{n+1}. We say that $S \subset \partial D$ is a subsolution (supersolution) of* (1) *if the distance function $d = d(X, D)(d(x) = d(x, C(D)))$ is a viscosity supersolution of*

$$-F(-D^2 d, Dd) \leq 0, \tag{5}$$

in a neighborhood of S (exclude S). S is a solution if it is both a supersolution and a subsolution.

Remark The neighborhood mentioned in the above definition can be specified as the points whose distance to S is obtained in the 'interior' of the surface S. In the case that S is a Lipschitz graph in some coordinate system, we always take D as the domain below S in that coordinate system. Correspondingly, $\nu = Dd$ is always the upward normal.

Definition 4 *F is called uniformly elliptic if for $\mu > 0, e \perp \nu, |e| = 1$,*

$$\Lambda\mu \geq F(M + \mu(e \otimes e), \nu) - F(M, \nu) \geq \lambda\mu, \tag{6}$$

where Λ and λ are positive real constants.

Definition 5 *Let M be a representation of a second fundamental form. Let ν be its normal. Let $(e_1, \cdots, e_n)$ be a tangential orthogonal coordinate system. Let the representations of M in $(e_1, \cdots, e_n, \nu)$ be*

$$\begin{bmatrix} M_{1,1} & \cdots & M_{1,n} & M_{1,n+1} \\ & \cdots & & \\ M_{n,1} & \cdots & M_{n,n} & M_{n,n+1} \\ M_{n+1,1} & \cdots & M_{n+1,n} & M_{n+1,n+1} \end{bmatrix}$$

Denote

$$\begin{bmatrix} M_{1,1} & \cdots & M_{1,n} \\ & \cdots & \\ M_{n,1} & \cdots & M_{n,n} \end{bmatrix}$$

by M^ν.

Then M^ν depends on $e_1, \cdots, e_n$. However, $|M^\nu|$ is independent of $e_1, \cdots, e_n$.

Definition 6 *We say that F is geometrically Lipschitz in ν if*

$$|F(M,\nu_1) - F(M,\nu_2)| \le C|\nu_1 - \nu_2|\left(1 + |\nu_1 - \nu_2|(|M^{\nu_1}| + |M^{\nu_2}|)\right). \tag{7}$$

Now, we remark that we can extend the domain of F from $S(n+1) \times S^{n+1}$, where $S(n+1)$ are the $n+1$ by $n+1$ symmetric matrices, to $S(n+1) \times R^{n+1}$. We define

$$F(M, \alpha\nu) = \alpha F(\frac{1}{\alpha} M, \nu) \quad \text{if } \alpha \ne 0 \tag{8}$$

$$F(M, 0) = 0. \tag{9}$$

Clearly we have the following identity for any α,

$$F(\alpha M, \alpha\nu) = \alpha F(M, \nu). \tag{10}$$

We also have that F is uniformly elliptic and satisfies (7).

We point out that there are many different ways to extent the function F. One way is to extend F such that it is uniformly elliptic in $n+1$ directions. This can be achieved by defining

$$\widetilde{F}(M,n) = F(M,n) + |P(M)n|,$$

where P(M) is the projection of M to the direction n. However, for the simplicity of the computation in this paper, we keep the extension as in (8). The advantage of our extension is reflected by the following lemma.

Lemma 3 *S is a subsolution if and only if there is a defining function G of S with condition that $DG \ne 0$ and such that G satisfies the following in a neighborhood above S,*

$$-F(-D^2G, DG) \le 0. \tag{11}$$

3. Construction of Parallel Surfaces

Now, we start to prove our theorem. Let us start out to construct parallel surfaces of the solution surfaces.

Let S be a surface satisfying equation (1).

Let $g(X)$ be a function defined on the whole space $X = (x, x_{n+1})$. We will choose g a small smooth function.

Let us consider the surface S^g, which is above S, defined by

$$S^g = \{X : d(X,S) = g(X)\}. \tag{12}$$

Under additional conditions on g, we will prove that S^g is a subsolution of (1).

For any $Y \in S^g$, there is an $X \in S$, such that

$$d(Y,X) = d(Y,S) = g(Y).$$

Since the computation in this chapter has nothing to do with the orientation of the graph, we may assume that $\frac{Y-X}{|Y-X|} = e_{n+1}$ and $Y = 0$.

Before the detail computation for the equation of S^g, let us make an observation on the geometry between S and S^g. Although S^g is a surface inherited from S, S can be reconstructed from S^g under the condition of curvature bounds on S. If S is smooth and g is small, S is the lower envelope of balls with center $Y \in S^g$ and radius $g(Y)$.

In the case S is not smooth, the envelope will touch the points on S with curvatures bounded by $1/g(Y)$.

Let us calculate the equation for S^g.

First we calculate it in the case of classical solution. We will reduce the calculation for viscosity solutions to this case.

Definition 7 *For fixed* $0 < \lambda \le \Lambda < \infty$*. The geometric Pucci maximal operator is defined by:*

$$M_\nu(B) = M_{\lambda,\Lambda}(B) = \inf_A trA^\nu B,$$

where the infimum is taken over matrices such that when $x \perp \nu$ *and* $\lambda|x|^2 \le x^T Ax \le \Lambda|x|^2$*.*

Notice that $D(d-g)(0) = e_{n+1} - Dg$ and $d\mu_i \le 1$.

We have

$$\begin{aligned} F(-D^2(d-g), D(d-g)) &= F(D^2 g - D^2 d, e_{n+1} - Dg) \\ &\ge CM_n(D^2 g) + F(-D^2 d, e_{n+1} - Dg). \end{aligned}$$

Moreover,

$$\begin{aligned} F(-D^2 d, e_{n+1} - Dg) &= F\left(\cdots, \frac{\mu_i}{1-d\mu_i}, \cdots, e_{n+1} - Dg\right) \\ &\ge F\left(\cdots, \frac{\mu_i}{1-d\mu_i}, \cdots, e_{n+1}\right) - C|Dg|(|Dg|(\sum_1^n \frac{|\mu_i|}{1-d\mu_i} + 1)) \\ &\ge F\left(\cdots, \mu_i + \frac{d\mu_i^2}{1-d\mu_i}, \cdots, e_{n+1}\right) \\ &\quad -C|Dg|(|Dg|(\sum_1^n \frac{|\mu_i|}{1-d\mu_i} + 1)) \\ &\ge \lambda d \Sigma \frac{\mu_i^2}{1-d\mu_i} + F\left(\cdots, \mu_i, \cdots, e_{n+1}\right) \\ &\quad -C|Dg|(|Dg| \sum_1^n \frac{|\mu_i|}{1-d\mu_i} + 1) \\ &\ge \lambda d \Sigma \frac{\mu_i^2}{1-d\mu_i} - C|Dg|(|Dg| \sum_1^n \frac{|\mu_i|}{1-d\mu_i} + 1). \end{aligned}$$

Combining these inequalities, we have

$$
\begin{aligned}
F(-D^2(d-g), e_{n+1} - Dg) \geq & \, C \sum_{i=1}^{n} \left(\frac{M_n(D^2 g)}{n} + \frac{\lambda d \mu_i^2}{1 - d\mu_i} + \right. \\
& \left. + |Dg| \; (\pm C|Dg| + Cg) \frac{\mu_i}{1 - g\mu_i} - C \frac{|Dg|}{n} \right) \\
= & \sum_{i=1}^{n} \frac{C}{1 - g\mu_i} (a\mu_i^2 + b\mu_i + c)
\end{aligned}
$$

where $a = \lambda g, b = -\frac{M_n(D^2 g)}{n} d + (\pm|Dg| + Cg)|Dg|$ and $c = \frac{M_n(D^2 g)}{n} - C|Dg|$. Looking at the discriminant of the quadratic polynomial in μ_i, we have that S^g is a subsolution provided

$$
\left| \pm |Dg|^2 - C|Dg|g - \frac{M_n(D^2 g)}{n} g \right|^2 - 4\lambda g \left(\frac{M_n(D^2 g) - Cn|Dg|}{n} \right) < 0. \tag{13}
$$

We will take g so that $g = 0$ near the boundary. Now the above inequality is possible when it has strictly bigger than 1 of the order of vanishing of g near its zeros. In deed, it is a consequence of the following conditions, in the range of $g(x) > 0$,

$$
\frac{M_n(D^2 g)}{2nC} \geq |Dg|, \tag{14}
$$

$$
|D^2 g| + |Dg| + |g| \leq \frac{1}{C}, \tag{15}
$$

$$
|Dg|^4 < g M_n(D^2 g). \tag{16}
$$

This shows that $S^{\epsilon g}$ for ϵ small is a subsolution provided the conditions (14)-(16) on g hold.

We remark the geometric condition on the equation plays key role for the inhomogeneity of (16).

Now, we begin to show the same results in the sense of viscosity. Let us prove a lemma first.

Lemma 4 *Let S be a continuous graph. We assume that it is semiconvex (and then almost every where second order differentiable). Then S is a subsolution of the equation if and only if it satisfies the equation almost everywhere.*

Proof We only prove the 'if' part. The other part follows from the general properties of viscosity solutions and the fact that semiconvex functions are second order differentiable almost everywhere.

Suppose $S(x)$ is below $S(x_0) + p \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T A(x - x_0)$. We may assume $p = 0$ by taking appropriate coordinate system. We will show $F(A, 0) \geq 0$.

Let $u(x)$ be the representation of S in this coordinate system. Let $w(x) = [u(x_0) + \frac{1}{2}(x-x_0)(A+\epsilon)(x-x_0)] - u$, which has a strict local maximum at x_0. Let $\Gamma(w)$ be the convex envelope of w. Clearly, $\Gamma(w)$ is $C^{1,1}$.

By the Aleksandrov-Bakel′man-Pucci maximum principle,

$$0 \leq \int_{\{x:|x|\leq\delta, |D\Gamma w(x)|\leq\delta, w(x)=\Gamma w(x)\}} \det(D^2\Gamma(w))dx$$

for $\delta > 0$ sufficient small.

Hence we can find $x_k \to 0$ such that $p_k = Dw(x_k) \to 0$ and $D^2 w(x_k) \geq 0$ and S is second order differentiable and its differentials satisfies the equation at x_k. Consequently $F(II(S)(x_k), \nu(S)(x_k)) \geq 0$. Noting that $A + \epsilon \geq D^2 u(x_k)$, we have

$$F(II(S(\epsilon)(x_k)), \nu(S)(\epsilon)(x_k)) \geq F(II(S)(x_k), \nu(S)(x_k)) - C|\nu(S)(x_k) - \nu(S(\epsilon))(x_k)|,$$

where $S(\epsilon)$ is the graph of the function $u = \frac{1}{2}x^T(A+\epsilon)x$. Letting $k \to \infty$ and then $\epsilon \to 0$, we have the lemma. We remark that S^g is semiconvex (i.e. $C^{1,1}$ from below). At each point $Y_0 \in S^g$, there is an $X_0 \in S$ such that $d(X_0, Y_0) = g(y_0)$. Then S^g is above

$$\{Y : d(Y, X_0) = g(Y)\}$$

which is smooth at Y_0, if $|Dg| < 1$. Hence S^g is semiconvex.

Hence, we need only to check that S^g is a subsolution at the points where S^g is second order differentiable. Let Y_0 be such a point. Hence $d - g$ is second order differentiable at Y_0. Then there is an X_0 on the surface S such that $d(X_0, Y_0) = g(Y_0)$. It is not hard to see, by considering envelope of balls, that for any $0 < r < g(Y_0)$, $\{d = r\}$ defines a set which is second order differentiable along the line passing X_0 and Y_0. From the definition of viscosity solution for S, it follows that $\{d = r\}$ has a smooth approximation which is a classical subsolution on the point $X_0 + r\frac{Y_0 - X_0}{|Y_0 - X_0|}$.

Now, we are in the situation of classical solutions for that surface. Hence its parallel surface, which has the same curvature and normal at Y_0 with the defining function $d - g + r$, is a subsolution at Y_0:

$$F(-D^2(d-g), D(d-g)) \geq 0,$$

provided the conditions on g holds on the function $g - r$, which is true for r small.

4. Construction of g

Lemma 5 *For any $\epsilon, \delta > 0$. Let $x_0 \in B_{\frac{1}{4}}(0), 0 < r_0 < \frac{1}{4}$. Then there exists a smooth function defined on $(B_1 - B_{r_0}(x_0)) \times (-K, K)$ such that* (14)-(16) *hold.*

Proof We only to have check (14) and the other two are evident. Similar we only have to check when the tangent is vertical by a perturbation.

Remind that $X = (x, x_{n+1})$. We will take $\phi(X) = (1 - |x|^2)^p e^{Mx_{n+1}}$ for some large p and M.

We will compute when the maximal operator is defined on a vertical plane. Clearly that $D_{n+1,n+1}\phi = M^2\phi$ and $D_{\tau\tau}\phi = 4p(p+1)(1-|x|^2)^{p-2}\tau^2 e^{Mx_{n+1}} + 2p(1-|x|^2)^{p-1}e^{Mx_{n+1}}$. It is clear that $D_{n+1}\phi = M\phi$ and $D_\tau\phi = p(1 - |x|^2)^{p-1}e^{Mx_{n+1}}$ and that $D_\tau\phi = p\tau_k(x_{n+1} + K + |x|^2)^{p-1}e^{Mx_{n+1}}$.

Hence for $|x| \leq \frac{1}{2}$, one can take M large and then p large for $|x| \geq \frac{1}{2}$ so that

$$\mathcal{M}_{e_{n+1}}\phi \geq C|D\phi|.$$

The lemma follows.

Proof of Theorem 1 Suppose S is a C^1 solution and suppose S is the graph of the function $x_{n+1} = u(x)$. Take g that is constructed in Lemma 5. Consider the comparison surface: $S^{\epsilon g}$ and consider the vertical distance between $S^{\epsilon g}$ and S. Clearly S^ϵ is above S and their vertical distance is controlled by $\epsilon g(\sqrt{\text{Lip}(u)^2 + 1} + o(1))$, where $o(1) \to 0$ as $\epsilon \to 0$ and the convergence depends on the C^1 property the solution S.

We claim the maximal vertical distance between S and $S^{\epsilon g}$ is less of $C\epsilon g$ if C is large enough.

At the point where the maximum vertical is achieved, $S^{\epsilon g}$ is a supersolution. However if C is large, the tangent plane is more and more vertical. By Lemma 5, $S^{\epsilon g}$ is actually a subsolution there, which is a contradiction.

References

[1] Caffarelli L A, Wang Lihe. A Harnack inequality approach to the interior regularity of parabolic equations. *Indiana Univ. Math. J.*, 1993, **42**(1): 159–165.

[2] Korevaar N. An easy proof of the interior gradient bound for solutions to the Prescribed Mean Curvature Equation. Proceedings of Symposia in Pure Mathematics, 1986, **45**(Part II): 81-89.

[3] Korevaar N. A priori interior gradient bounds for solutions to elliptic Weingarten equations. *Ann. Inst. Henri Poincare*, 1987, **4**: 405-421.

[4] Li Y Y. Interior gradient estimates for solutions of certain fully nonlinear elliptic equations. *J. of Diff. Equation*, 1991, **90**(1): 172-185.

[5] Guan B, Spruck J. Interior gradient estimates for solutions of prescribed curvature equations of parabolic type. to appear in *Indiana Univ. Math. J.*

[6] Caffarelli L A. A Harnack inequality approach to the regularity of free boundaries, Part I: Lipschitz free boundaries are $C^{1,\alpha}$. *Revista Matematica Iberoamericana*, 1907, **3**(2): 139-161.

[7] Athanasopoulos I, Caffarelli L A. A theorem of real analysis and its applications to free boundary problems. *Comm. on Pure and Applied Math.*, 1985, **38**: 499-502

[8] Caffarelli L A. Interior estimates for fully nonlinear equations. *Ann. of Math.*, 1989, **130**: 189-213.

[9] Jensen R. The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations. *Arch. Rational Mech. Anal.*, 1988, **101**(1): 1-27.

[10] Krylov N V. On derivative bounds for solutions of nonlinear parabolic equations. *Dokl. Akad. Nauk. SSSR*, 1983, **274**: 21-25.

[11] Ladyzhenskaya O A, Ural'tseva N N. Quasilinear elliptic equations and variational problems with several independent variables. *English Translation in Russian Math. Surveys*, 1961, **16**(1): 17-92.

[12] Ladyzhenskaya O A, Ural'tseva N N. On Hölder continuity of solution and their derivatives of linear and quasilinear elliptic and parabolic equations. *English Translation in Amer. Math. Soc. Transl.*, 1967, **61**(2): 207-269

[13] Wang L. On the regularity theory of fully nonlinear parabolic equations I. *Comm. Pure Appl. Math.*, 1992, **45**: 27-76.

[14] Wang L. On the regularity theory of fully nonlinear parabolic equations II. *Comm. Pure Appl. Math.*, 1992, **45**: 141-178.