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COMPLETE REDUCIBILITY FOR QUASILINEAR HYPERBOLIC SYSTEMS*

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Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

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Abstract In this paper we present a necessary and sufficient condition to guarantee the complete reducibility for quasilinear hyperbolic systems and give some examples.

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1. Introduction

In [1] we have introduced the concept of the completely reducible quasilinear hyperbolic system and discussed the singularity caused by eigenvectors for this kind of system in the case of constant eigenvalues. In this paper we will present a method for checking if a given quasilinear strictly hyperbolic system is completely reducible or not, and give some examples.

2. A Necessary and Sufficient Condition for a Quasilinear Strictly Hyperbolic System Being Completely Reducible

Consider the following first order quasilinear strictly hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{2.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth entries $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

By strict hyperbolicity, on the domain under consideration $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u),\ \lambda_2(u),\ \cdots,\ \lambda_n(u). \tag{2.2}$$

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For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{2.3}$$

All $\lambda_i(u)$, $l_i(u)$ and $r_i(u)$ $(i = 1, \cdots, n)$ have the same regularity as $A(u)$. Without loss of generality, we assume that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n), \tag{2.4}$$

where δ_{ij} denotes the Kronecker's symbol.

System (1.1) can be equivalently reduced to the following characteristic form

$$l_i(u)\Big(\frac{\partial u}{\partial t} + \lambda_i(u)\frac{\partial u}{\partial x}\Big) = 0 \quad (i = 1, \cdots, n). \tag{2.5}$$

For $i = 1, \cdots, n$, the i-th equation in (2.5) contains only the directional derivative of u with respect to t along the i-th characteristic direction $\dfrac{dx}{dt} = \lambda_i(u)$.

By the definition given in [1], system (2.1) is m-step (globally) completely reducible, if there is a global diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$

$$u = u(\tilde{u}) \tag{2.6}$$

such that the corresponding system for $\tilde{u}$

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{A}(\tilde{u})\frac{\partial \tilde{u}}{\partial x} = 0 \tag{2.7}$$

has the following standard form:

$$\tilde{A}(\tilde{u}) = \begin{pmatrix} \tilde{\Lambda}^{(1)}(\tilde{u}) & & & \\ \tilde{A}_{21}(\tilde{u}) & \tilde{\Lambda}^{(2)}(\tilde{u}) & & \\ \vdots & & \ddots & \\ \tilde{A}_{m1}(\tilde{u}) & \cdots \tilde{A}_{m,m-1}(\tilde{u}) & & \tilde{\Lambda}^{(m)}(\tilde{u}) \end{pmatrix}, \tag{2.8}$$

where $\tilde{\Lambda}^{(a)}(\tilde{u})$ $(a = 1, \cdots, m)$ are diagonal matrices, the entries of which are given by $\tilde{\lambda}_i(\tilde{u}) = \lambda_i(u(\tilde{u}))$ $(i = 1, \cdots, n)$ respectively. If this diffeomorphism (2.6) is only valid in a local domain, system (2.1) is called to be m-step locally completely reducible. If there is no such diffeomorphism (2.6) even in the local sense, system (2.1) is non-completely reducible.

Without loss of generality, in what follows we consider only the 2-step completely reducible case.

By definition, under diffeomorphism (2.6), a 2-step completely reducible quasilinear strictly hyperbolic system (2.1) can be reduced to the following standard form

$$\begin{cases} \dfrac{\partial \tilde{u}^{(1)}}{\partial t} + \tilde{\Lambda}^{(1)}(\tilde{u})\dfrac{\partial \tilde{u}^{(1)}}{\partial x} = 0, \\ \dfrac{\partial \tilde{u}^{(2)}}{\partial t} + \tilde{\Lambda}^{(2)}(\tilde{u})\dfrac{\partial \tilde{u}^{(2)}}{\partial x} + \tilde{A}_{21}(\tilde{u})\dfrac{\partial \tilde{u}^{(1)}}{\partial x} = 0, \end{cases} \tag{2.9}$$

where

$$\tilde{u}^{(1)} = (\tilde{u}_1, \cdots, \tilde{u}_k)^T, \quad \tilde{u}^{(2)} = (\tilde{u}_{k+1}, \cdots, \tilde{u}_n)^T \tag{2.10}$$

and without loss of generality

$$\tilde{\Lambda}^{(1)}(\tilde{u}) = \mathrm{diag}\{\tilde{\lambda}_1(\tilde{u}), \cdots, \tilde{\lambda}_k(\tilde{u})\}, \quad \tilde{\Lambda}^{(2)}(\tilde{u}) = \mathrm{diag}\{\tilde{\lambda}_{k+1}(\tilde{u}), \cdots, \tilde{\lambda}_n(\tilde{u})\}. \tag{2.11}$$

This shows that the first k equations corresponding to $\lambda_r(u)$ $(r = 1, \cdots, k)$ in the system (2.5) of characteristic form must be diagonalizable.

Let

$$L_i(u) = l_i(u)du. \tag{2.12}$$

By Frobenius theorem (see [2, 3]) the i-th equation in system (2.5) is diagonalizable in the local sense if and only if

$$L_i(u)\Lambda dL_i(u) \equiv 0, \tag{2.13}$$

namely,

$$\sum_{\substack{p,\ q,\ r \\ \text{circular summation}}} l_{ir}(u)\Big(\frac{\partial l_{ip}(u)}{\partial u_q} - \frac{\partial l_{iq}(u)}{\partial u_p}\Big) \equiv 0,$$
$$\forall p,\ q,\ r \in \{1, \cdots, n\} \quad \text{different from each other.} \tag{2.14}$$

Hence, when (2.14) hold for $i = 1, \cdots, k$, there exists at least a local diffeomorphism

$$u = u(\bar{u}) \tag{2.15}$$

such that system (2.1) reduces to

$$\frac{\partial \bar{u}}{\partial t} + \bar{A}(\bar{u})\frac{\partial \bar{u}}{\partial x} = 0, \tag{2.16}$$

where

$$\bar{A}(\bar{u}) = \begin{pmatrix} \bar{\Lambda}^{(1)}(\bar{u}) & 0 \\ \bar{A}_{21}(\bar{u}) & \bar{A}_{22}(\bar{u}) \end{pmatrix}, \tag{2.17}$$

then the corresponding matrix composed of the left eigenvectors is

$$\bar{L}(\bar{u}) = \begin{pmatrix} I_k & 0 \\ \bar{L}_{21}(\bar{u}) & \bar{L}_{22}(\bar{u}) \end{pmatrix}, \tag{2.18}$$

in which I_k denotes the unit matrix of order k; moreover,

$$\bar{\Lambda}^{(1)}(\bar{u}) = \mathrm{diag}\{\bar{\lambda}_1(\bar{u}), \cdots, \bar{\lambda}_k(\bar{u})\} = \mathrm{diag}\{\lambda_1(u(\bar{u})), \cdots, \lambda_k(u(\bar{u}))\}. \tag{2.19}$$

Set

$$\bar{\Lambda}(\bar{u}) = \mathrm{diag}\{\bar{\lambda}_1(\bar{u}), \cdots, \bar{\lambda}_n(\bar{u})\} = \mathrm{diag}\{\lambda_1(u(\bar{u})), \cdots, \lambda_n(u(\bar{u}))\} \tag{2.20}$$

and

$$\bar{\Lambda}^{(2)}(\bar{u}) = \mathrm{diag}\{\bar{\lambda}_{k+1}(\bar{u}), \cdots, \bar{\lambda}_n(\bar{u})\} = \mathrm{diag}\{\lambda_{k+1}(u(\bar{u})), \cdots, \lambda_n(u(\bar{u}))\}. \tag{2.21}$$

By

$$\bar{A}(\bar{u}) = \bar{L}^{-1}(\bar{u})\bar{\Lambda}(\bar{u})\bar{L}(\bar{u}), \tag{2.22}$$

we have

$$\bar{A}_{22}(\bar{u}) = \bar{L}_{22}^{-1}(\bar{u})\bar{\Lambda}^{(2)}(\bar{u})\bar{L}_{22}(\bar{u}). \tag{2.23}$$

Hence, the eigenvalues of $\bar{A}_{22}(\bar{u})$ are given by (2.21) and the matrix composed of the left eigenvectors of $\bar{A}_{22}(\bar{u})$ is just $\bar{L}_{22}(\bar{u})$.

Noting that the first k equations in system (2.16) are of the diagonal form, it is easy to see that system (2.16) is 2-step completely reducible if and only if the left eigenvectors of $\bar{A}_{22}(\bar{u})$ are all diagonalizable, namely, the reduced system, $\bar{u}^{(1)} = (\bar{u}_1, \cdots, \bar{u}_k)$ being regarded as parameters,

$$\frac{\partial \bar{u}^{(2)}}{\partial t} + \bar{A}_{22}(\bar{u})\frac{\partial \bar{u}^{(2)}}{\partial x} = 0 \tag{2.24}$$

can be rewritten in a diagonal form, where $\bar{u}^{(2)} = (\bar{u}_{k+1}, \cdots, \bar{u}_n)$.

Let $\bar{l}_i(\bar{u})$ be a row vector of $\bar{L}_{22}(\bar{u})$, i.e., a left eigenvector of $\bar{A}_{22}(\bar{u})$, and

$$\bar{L}_i(\bar{u}) = \bar{l}_i(\bar{u})d\bar{u}^{(2)} \quad (i = k+1, \cdots, n). \tag{2.25}$$

As before, the left eigenvectors of $\bar{A}_{22}(\bar{u})$ are all diagonalizable in the local sense if and only if

$$\bar{L}_i(\bar{u})\Lambda d\bar{L}_i(\bar{u}) \equiv 0 \quad (i = k+1, \cdots, n), \tag{2.26}$$

in which $\bar{u}^{(1)} = (\bar{u}_1, \cdots, \bar{u}_k)$ are still regarded as parameters, namely, for $i = k + 1, \cdots, n$,

$$\sum_{\substack{p,\ q,\ r \\ \text{circular summation}}} \bar{l}_{ir}(\bar{u})\Big(\frac{\partial \bar{l}_{ip}(\bar{u})}{\partial \bar{u}_q} - \frac{\partial \bar{l}_{iq}(\bar{u})}{\partial \bar{u}_p}\Big) \equiv 0,$$
$$\forall\, p,\ q,\ r \in \{k+1, \cdots, n\} \quad \text{different from each other.} \tag{2.27}$$

Under assumption (2.26) or (2.27), there exists at least a local diffeomorphism

$$\bar{u} = \bar{u}(\tilde{u}) \tag{2.28}$$

such that system (2.16) can be further reduced to the standard form (2.9) of the 2-step completely reducible system.

The m-step completely reducible system can be considered in an entirely similar manner.

Thus, system (2.1) is locally completely reducible if and only if the system can be successively diagonalizable, namely, in each step some equations in the corresponding reduced system are diagonalizable. When the diffeomorphisms used in the previous procedure are valid all in the global sense, system (2.1) is then (globally) completely reducible. When the previous procedure fails in a step (particularly in the first step, namely, there is no equation in system (2.5), which can be diagonalizable even in the local sense), system (2.1) is non-completely reducible.

3. Examples

Example 1 In system (2.1), suppose that $n = 3$ and

$$A(u) = \begin{pmatrix} 0 & 0 & e^{u_2} \\ e^{-u_2} & 0 & e^{u_2} \\ e^{-u_2} & 0 & 0 \end{pmatrix}. \tag{3.1}$$

The eigenvalues are

$$\lambda_1(u) \equiv -1 < \lambda_2(u) \equiv 0 < \lambda_3(u) \equiv 1 \tag{3.2}$$

and the corresponding left eigenvectors can be taken as

$$\begin{cases} l_1(u) = (-e^{-u_2}, 0, 1), \\ l_2(u) = (1, -1, 1), \\ l_3(u) = (e^{-u_2}, 0, 1) \end{cases} \tag{3.3}$$

(cf. Example 4 in [1]).

Obviously, the equation corresponding to $l_2(u)$ in system (2.5) can be globally diagonalizable. Moreover, it is easily seen that (2.14) fail for $i = 1$ and 3. Hence, in the first step only one equation can be reduced to a diagonal form. Correspondingly, setting

$$\bar{u}_1 = u_1, \quad \bar{u}_2 = u_2 - u_1 - u_3, \quad \bar{u}_3 = u_3 \tag{3.4}$$

and noting

$$\bar{l}_i(\bar{u}) = l_i(u) \frac{\partial u}{\partial \bar{u}}, \tag{3.5}$$

we have

$$\begin{cases} \bar{l}_1(\bar{u}) = (-e^{-(\bar{u}_1+\bar{u}_2+\bar{u}_3)}, 0, 1), \\ \bar{l}_3(\bar{u}) = (e^{-(\bar{u}_1+\bar{u}_2+\bar{u}_3)}, 0, 1), \end{cases} \tag{3.6}$$

hence the corresponding system of characteristic form can be rewritten as

$$\begin{cases} e^{-\bar{u}_1}\Big(\dfrac{\partial \bar{u}_1}{\partial t} - \dfrac{\partial \bar{u}_1}{\partial x}\Big) - e^{\bar{u}_2+\bar{u}_3}\Big(\dfrac{\partial \bar{u}_3}{\partial t} - \dfrac{\partial \bar{u}_3}{\partial x}\Big) = 0, \\ \dfrac{\partial \bar{u}_2}{\partial t} = 0, \\ e^{-\bar{u}_1}\Big(\dfrac{\partial \bar{u}_1}{\partial t} + \dfrac{\partial \bar{u}_1}{\partial x}\Big) + e^{\bar{u}_2+\bar{u}_3}\Big(\dfrac{\partial \bar{u}_3}{\partial t} + \dfrac{\partial \bar{u}_3}{\partial x}\Big) = 0. \end{cases} \tag{3.7}$$

The reduced system composed of the first and third equations of (3.7), in which $\bar{u}_2$ is regarded as a parameter, can be reduced to a diagonal form by the following transformation

$$\begin{cases} \tilde{u}_1 = e^{-\bar{u}_1} + e^{\bar{u}_2+\bar{u}_3}, \\ \tilde{u}_3 = e^{-\bar{u}_1} - e^{\bar{u}_2+\bar{u}_3}. \end{cases} \tag{3.8}$$

Hence, with the transformation (3.8) and $\tilde{u}_2 = \bar{u}_2$, the whole system (3.7) reduces to

$$\begin{cases} \dfrac{\partial \tilde{u}_1}{\partial t} - \dfrac{\partial \tilde{u}_1}{\partial x} + a(\tilde{u})\dfrac{\partial \tilde{u}_2}{\partial x} = 0, \\ \dfrac{\partial \tilde{u}_2}{\partial t} = 0, \\ \dfrac{\partial \tilde{u}_3}{\partial t} + \dfrac{\partial \tilde{u}_3}{\partial x} + a(\tilde{u})\dfrac{\partial \tilde{u}_2}{\partial x} = 0, \end{cases} \tag{3.9}$$

where

$$a(\tilde{u}) = \frac{\tilde{u}_1 - \tilde{u}_3}{2}. \tag{3.10}$$

(3.9) is of the standard form of the 2-step completely reducible system. However, since (3.8) with $\tilde{u}_2 = \bar{u}_2$ is only a local diffeomorphism, the system under consideration is only locally completely reducible.

Example 2 For the system given by A. Jeffrey [4], n=3 and

$$A(u) = \begin{pmatrix} -\mathrm{ch}2u_2 & 0 & -\mathrm{sh}2u_2 \\ \mathrm{ch}u_2 & 0 & \mathrm{sh}u_2 \\ \mathrm{sh}2u_2 & 0 & \mathrm{ch}2u_2 \end{pmatrix}. \tag{3.11}$$

The eigenvalues are still given by (3.2) and the corresponding left eigenvectors can be taken as

$$\begin{cases} l_1(u) = (\mathrm{ch}u_2, 0, \mathrm{sh}u_2), \\ l_2(u) = (\mathrm{ch}u_2, 1, \mathrm{sh}u_2), \\ l_3(u) = (\mathrm{sh}u_2, 0, \mathrm{ch}u_2). \end{cases} \tag{3.12}$$

It can be easily checked that (2.14) fail for all $i = 1,\ 2,\ 3$. Hence, this system is non-completely reducible and there is no equation in the corresponding characteristic form (2.5), which can be diagonalizable.

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