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GLOBAL NONEXISTENCE OF THE SOLUTIONS FOR A NONLINEAR WAVE EQUATION WITH THE Q-LAPLACIAN OPERATOR*

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Dedicated to Prof. Boling Guo for his 70th Birthday

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Abstract We study the global nonexistence of the solutions of the nonlinear q-Laplacian wave equation

$$u_{tt} - \Delta_q u + (-\Delta)^\alpha u_t = |u|^{p-2}u,$$

where $0 < \alpha \leq 1$, $2 \leq q < p$. We obtain that the solution blows up in finite time if the initial energy is negative. Meanwhile, we also get the solution blows up in finite time with suitable positive initial energy under some conditions.

Key Words q-Laplacian operator; nonlinear wave equation; global nonexistence.
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1. Introduction

We study the initial boundary value problem

$$\begin{cases} u_{tt} - \Delta_q u + (-\Delta)^\alpha u_t = |u|^{p-2}u, & x \in \Omega, t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $2 \leq q < p$, $-\Delta_q u = -\sum\limits_{i=1}^{\infty} \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{q-2}\frac{\partial u}{\partial x_i})$, and Ω is a bounded domain in $R^n, n \geq 1$, with smooth boundary $\partial\Omega$. For this problem, H. Gao and T. F. Ma [1] had obtained the global existence of the solution when $q > p$ and with small initial data when $q \leq p$.

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When $q = 2$, with the linear damping term($\alpha = 0$), H. Levine([2, 3]) had proved the solution blows up in the finite time with negative initial energy. When $q = 2$, and the damping term is given by $|u_t|^r u_t$, here $r \geq 0$, many authors had studied the existence and uniqueness of the global solution and the blowup of the solution, see [4-6]. Our objective is to study the global nonexistence for this kind of equations with $q < p$ under a weaker damping term. For negative initial energy, we use the energy method with some modifications to [7] and [8], and obtain the global nonexistence for (1.1). For positive initial energy, we use the concavity technique developed by Levine [3] to get the global nonexistence for (1.1), this method can also be found in P. Pucci and J. Serrin [9].

The damping term we consider here is different from [10]. Since for an arbitrary $0 < \alpha \leq 1$, the condition (3d) in [10] does not always hold. For the model we consider here, by [10] we know $V = L^2(\Omega)$, $W = L^p(\Omega)$ correspondingly for our case, and $W' = L^{p'}(\Omega)$, here $\frac{1}{p'} = 1 - \frac{1}{p} > \frac{1}{2}$, and

$$Q(t, v) = (-\Delta)^\alpha v,$$
$$\mathcal{D}(t, v) = \int_\Omega (Q(t, v), v)dx = \|(-\Delta)^{\alpha/2} v\|_{L^2}^2.$$

By Sobolev imbedding $W^{2\alpha,p'}(\Omega) \hookrightarrow W^{\alpha,2}(\Omega)$(see [11]) with

$$\alpha \geq n(\frac{1}{p'} - \frac{1}{2}), \qquad (*)$$

we have

$$\|(-\Delta)^\alpha v\|_{L^{p'}} \leq C\|(-\Delta)^{\alpha/2} v\|_{L^2},$$

here C is a constant. The above inequality just is (3d) in [10] with $\delta(t)$ being constant and $m = m' = 2$. We know the condition $(*)$ does not always hold for any given $0 < \alpha \leq 1$ and for all p satisfying the condition (2.2) in the sequel, that is (3d) in [10] does not always hold for arbitrary $0 < \alpha \leq 1$. But our results hold for any $0 < \alpha \leq 1$ and all p satisfying the condition (2.2).

Here we use standard notations. We often write $u(t)$ instead $u(t, x)$ and $u'(t)$ instead $u_t(t, x)$. The norm in $L^q(\Omega)$ is denoted by $\|\cdot\|_q$ and in $W_0^{1,q}(\Omega)$ we use the norm $\|u\|_{1,q}^q = \sum_{i=1}^n \|u_{x_j}\|_q^q$.

For convenience, we recall some of the basic properties of the operators used here. The degenerate operator $-\Delta_q$ is unbounded, monotone and hemicontinuous from $W_0^{1,q}(\Omega)$ to $W_0^{-1,p}(\Omega)$, where $q^{-1} + p^{-1} = 1$. The power for the Laplacian operator is defined by $(-\Delta)^\alpha u = \sum_{j=1}^\infty \lambda_j^\alpha (u, \varphi_j)\varphi_j$, where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ....$ and $\varphi_1, \varphi_2, \varphi_3....$ are respectively the sequence of the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$. Then

$$\|u\|_{D((-\Delta)^\alpha)} = \|(-\Delta)^\alpha u\|_2, \quad \forall u \in D((-\Delta)^\alpha)$$

and for $q \geq 2, \quad 0 < \alpha \leq 1, W_0^{1,q}(\Omega) \hookrightarrow D((-\Delta)^{\frac{\alpha}{2}}) \hookrightarrow L^2(\Omega)$.

2. Global Nonexistence of the Solution for (1.1)

The purpose of this paper is to study the global nonexistence of the solution for the problem (1.1). First, we make some preparations.

Assume $0 < \alpha \leq 1$ and the p, q satisfy the condition

$$\begin{aligned} &2 \leq q < p < \tfrac{nq}{n-q}, \qquad \text{for} \quad n > q, \\ &2 \leq q < p, \qquad\qquad \text{for} \quad n \leq q. \end{aligned} \tag{2.1}$$

Lemma 1(Local existence) *Let the condition* (2.1) *hold, for any initial data* $(u_0, u_1) \in W_0^{1,q}(\Omega) \times L^2(\Omega)$, *if* T *is small enough, then there exists a weak solution* u *of* (1.1) *which satisfies*

$$u \in L^\infty((0,T); W_0^{1,q}(\Omega)), \qquad u' \in L^\infty((0,T); L^2(\Omega)) \bigcap L^2((0,T); D((-\Delta)^{\frac{\alpha}{2}})).$$

By the method of [1], using Galerkin method, we can get the proof.

For the weak solution, we define energy as following:

$$E(t) = \tfrac{1}{2}\|u_t\|_2^2 + \tfrac{1}{q}\|u\|_{1,q}^q - \tfrac{1}{p}\|u\|_p^p. \tag{2.2}$$

We suppose the following *weak conservation law* holds

$$E(t) + \int_0^t \|(-\Delta)^{\frac{\alpha}{2}} u_t\|_2^2 dt \leq E(0), \qquad t \in [0,T]. \tag{2.3}$$

Our main results are the following two theorems.

Theorem 1 *Let the conditions* (2.1) *and* (2.3) *hold, for any initial data* $(u_0, u_1) \in W_0^{1,q}(\Omega) \times L^2(\Omega)$, *if* $E(0) < 0$, *then the solution of* (1.1) *blows up at finite time* T_0 *(*T_0 *can be seen in the proof) in the* L^p *norm.*

Theorem 2 *Let the conditions* (2.1) *and* (2.3) *hold, for any initial data* $(u_0, u_1) \in W_0^{1,q}(\Omega) \times L^2(\Omega)$, *and with suitable positive initial energy, namely* $0 < E(0) < d$, $\|u_0\|_q > z_1$ *(*d *is the maximum of function* $Q(z) = \frac{1}{q}z^q - \frac{C_3^p}{p}z^p, Q(z_1) = d$ *and* C_3 *is the Sobolev constant in* $\|u\|_p \leq C_3\|u\|_{1,q}$*), if the solution of* (1.1) *which satisfies*

$$u \in C([0,T]; W_0^{1,q}(\Omega)), \qquad u' \in C([0,T]; L^2(\Omega)) \bigcap L^2((0,T); D((-\Delta)^{\frac{\alpha}{2}})),$$

then the solution of (1) *can not exist globally.*

Proof of Theorem 1 Using $E(0) < 0$ and (2.3), we have

$$E(t) \leq E(0) < 0. \tag{2.4}$$

Let

$$H(t) = q(-E(t)) + (\frac{q}{2}+1)\|u_t\|_2^2 + \frac{p-q}{2p}\|u\|_p^p. \tag{2.5}$$

Now, we introduce $F(t) = \frac{1}{2}\|u\|_2^2$ for any solution u, then differentiate F(t) with respect to t we have

$$F'(t) = \int_\Omega uu_t dx. \tag{2.6}$$

Since u is a solution of (1.1), by (2.5) and (2.6) we obtain

$$\begin{aligned} F''(t) &= \int_\Omega (u_{tt}u + u_t^2)dx \\ &= \int_\Omega (u_t^2 + (\Delta_q u + |u|^{p-2}u - (-\Delta)^\alpha u_t)u)dx \\ &= H(t) + \frac{p-q}{2p}\|u\|_p^p - \int_\Omega (-\Delta)^\alpha u_t u dx. \end{aligned} \tag{2.7}$$

By (2.1) and (2.4) we get

$$\frac{1}{q}\|u\|_{1,q}^q \le \frac{1}{p}\|u\|_p^p. \tag{2.8}$$

Using (2.7) and (2.8) we have

$$F''(t) \ge H(t) + \frac{p-q}{2q}\|u\|_{1,q}^q - \int_\Omega (-\Delta)^\alpha u_t u dx. \tag{2.9}$$

Before estimating $|\int_\Omega (-\Delta)^\alpha u_t u dx|$, we claim that there exists a constant C_1 satisfying

$$\|(-\Delta)^{\frac{\alpha}{2}}u\|_2 \le C_1\|u\|_{1,q}. \tag{2.10}$$

In fact the inequality above can be obtained from the imbedding $W_0^{1,q}(\Omega) \hookrightarrow D((-\Delta)^{\frac{\alpha}{2}})$ and $\|u\|_{D((-\Delta)^{\frac{\alpha}{2}})} = \|(-\Delta)^{\frac{\alpha}{2}}u\|_2$.

We claim that there exists a constant C_2 satisfying

$$\|u\|_{1,q}^{1-\frac{q}{2}} \le C_2(-E(t))^{-(1-\beta)}, \quad 0<\beta<1. \tag{2.11}$$

In fact, by (2.1) and (2.4) we have $\frac{1}{p}\|u\|_p^p \ge -E(t)$, namely

$$\|u\|_p^p \ge p(-E(t)). \tag{2.12}$$

Since $1 < p \le \frac{nq}{n-q}$, we have $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$, so that there exists a constant C_3 satisfying

$$\|u\|_p \le C_3\|u\|_{1,q}. \tag{2.13}$$

By (2.12) and (2.13) we get

$$(C_3\|u\|_{1,q})^p \ge p(-E(t)),$$

since $\frac{2-q}{2p} \le 0$, we obtain

$$C_3^{\frac{2-q}{2}} \|u\|_{1,q}^{\frac{2-q}{2}} \le (-E(t))^{\frac{2-q}{2p}}.$$

Let $C_2 = C_3^{\frac{q-2}{2}}$, $\beta = \frac{2p-q+2}{2p}$. It is easy to verify that $\frac{1}{2} < \beta < 1$, thus the claim above is proved.

Using (2.4), (2.10), (2.11) and Hölder inequality, we have

$$\begin{aligned}\left|\int_\Omega (-\Delta)^\alpha u_t u dt\right| &\le \|(-\Delta)^{\frac{\alpha}{2}} u_t\|_2 \|(-\Delta)^{\frac{\alpha}{2}} u\|_2 \\ &\le C_1 \|(-\Delta)^{\frac{\alpha}{2}} u_t\|_2 \|u\|_{1,q} \\ &= C_1 \|(-\Delta)^{\frac{\alpha}{2}} u_t\|_2 \|u\|_{1,q}^{1-\frac{q}{2}} \|u\|_{1,q}^{\frac{q}{2}} \\ &\le C_1 (\frac{1}{2\varepsilon} \|(-\Delta)^{\frac{\alpha}{2}} u_t\|_2^2 + 2\varepsilon \|u\|_{1,q}^q) \|u\|_{1,q}^{1-\frac{q}{2}} \\ &\le \frac{C_1 C_2}{2\varepsilon} (-E(t))^{-(1-\beta)} \|(-\Delta)^{\frac{\alpha}{2}} u_t\|_2^2 \\ &\quad + 2\varepsilon C_1 C_2 \|u\|_{1,q}^q (-E(0))^{-(1-\beta)}. \end{aligned} \tag{2.14}$$

Choose ε such that

$$2\varepsilon C_1 C_2 (-E(0))^{-(1-\beta)} = \tfrac{p-q}{2q}, \tag{2.15}$$

that is

$$\varepsilon = \frac{p-q}{4qC_1C_2} (-E(0))^{(1-\beta)} > 0,$$

and let

$$\theta_1 = \frac{C_1 C_2}{2\varepsilon}, \tag{2.16}$$

then (2.14) becomes

$$\left|\int_\Omega (-\Delta)^\alpha u_t u dt\right| \le \theta_1 (-E(t))^{-(1-\beta)} \|(-\Delta)^{\frac{\alpha}{2}} u_t\|_2^2 + \frac{p-q}{2q} \|u\|_{1,q}^q.$$

So we have

$$F''(t) \ge H(t) - \theta_1 (-E(t))^{-(1-\beta)} \|(-\Delta)^{\frac{\alpha}{2}} u_t\|_2^2. \tag{2.17}$$

Now we define

$$G(t) = (-E(t))^\beta + \beta \theta_1^{-1} F'(t), \tag{2.18}$$

then from (2.3), (2.8) and (2.18)

$$\begin{aligned} G'(t) &= \beta (-E(t))^{-(1-\beta)} (-E'(t)) + \beta \theta_1^{-1} F''(t) \\ &\ge \beta \theta_1^{-1} H(t) \ge (2\theta_1)^{-1} H(t) > 0 \end{aligned} \tag{2.19}$$

and there exists a $t_0 \ge 0$, such that $G(t) \ge G(t_0) > 0$, for $t \ge t_0$, where we can take

$$t_0 = 0, \qquad \text{if } \; G(0) \equiv (-E(0))^\beta + \beta \theta_1^{-1} \int_\Omega u_0 u_1 dx > 0. \tag{2.20}$$

Using Höder inequality, we get

$$\left|\int_\Omega uu_t dx\right| \le \|u_t\|_{p/(p-1)}^{(p-1)/p}\|u\|_p \le C_4\|u_t\|_2\|u\|_p,$$

where C_4 is the imbedding constant for $L^2(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$. By Young inequality we have

$$\begin{aligned} G(t)^{1/\beta} &\le 4\{(-E(t)) + (\theta_1^{-1}|F(t)'|^{1/\beta})\} \\ &\le 4\{(-E(t)) + (C_4\theta_1^{-1})^{1/\beta}\|u_t\|_2^{1/\beta}\|u\|_p^{1/\beta}\} \\ &\le 4\{(-E(t)) + \|u_t\|_2^2 + (C_4\theta_1^{-1})^{2/(2\beta-1)}\|u\|_p^{2/(2\beta-1)}\}. \qquad (2.21) \end{aligned}$$

In fact it is easy to verify that $\frac{2}{2\beta-1} < p$. By (2.12) we can get $\|u\|_p \ge [p(-E(0))]^{1/p} \ge (-E(0))^{1/p} > 0$, hence

$$(-E(0))^{-1/p}\|u\|_p \ge 1. \qquad (2.22)$$

By (2.21) and (2.22), we have

$$\begin{aligned} G(t)^{1/\beta} &\le 4\{(-E(t)) + \|u_t\|_2^2 + (C_4\theta_1^{-1})^{2/(2\beta-1)}\|u\|_p^{2/(2\beta-1)}[(-E(0))^{-1/p}\|u\|_p]^{p-2/(2\beta-1)}\} \\ &\le 4\{q(-E(t)) + (\frac{q}{2}+1)\|u_t\|_2^2 + (C_4\theta_1^{-1})^{2/(2\beta-1)}(-E(0))^{\frac{q-p}{2-q+p}}\|u\|_p^p\}. \qquad (2.23) \end{aligned}$$

Using (2.5) we get

$$H(t) \ge \frac{p-q}{2p}\|u\|_p^p.$$

Letting $\theta_2 = 4\max\{1, \frac{2p}{p-q}(C_4\theta_1^{-1})^{2/(2\beta-1)}(-E(0))^{\frac{q-p}{2-q+p}}\}$ we obtain

$$G(t)^{1/\beta} \le \theta_2 H(t). \qquad (2.24)$$

Hence

$$\partial_t\{G(t)^{1-1/\beta}\} = (1-1/\beta)G(t)^{-1/\beta}G'(t) \le -(1-\beta)(2\beta\theta_1\theta_2)^{-1}. \qquad (2.25)$$

Assume (2.20) holds, then

$$G(t)^{1-1/\beta} - G(0)^{1-1/\beta} = \partial_t\{G(t)^{1-1/\beta}\}t.$$

Using (2.25) we have

$$G(t) \ge \{G(0)^{1-1/\beta} - (1-\beta)(2\beta\theta_1\theta_2)^{-1}t\}^{-\beta/(1-\beta)}, \qquad (2.26)$$

hence there exists a $T > 0$, such that

$$T \le T_0 = 2\beta\theta_1\theta_2(1-\beta)^{-1}G(0)^{1-1/\beta}$$

and

$$\lim_{t\to T_0^-} G(t) = +\infty. \tag{2.27}$$

By (2.1) and (2.4), we get

$$-E(t) + \frac{1}{2}\|u_t\|_2^2 \le \frac{1}{p}\|u\|_p^p. \tag{2.28}$$

Combining (2.5), (2.23) with (2.28), we have there exists a constant C_5 satisfying

$$G(t)^{\frac{1}{\beta}} \le C_5\|u\|_p^p, \tag{2.29}$$

hence from (2.27) and (2.29), we obtain

$$\lim_{t\to T_0^-} \|u\|_p^p = +\infty.$$

Thus the proof of **Theorem 1** has completed.

Before the prove of **Theorem 2**, we make some preparations. We define the polynomial Q by

$$Q(z) = \frac{1}{q}z^q - \frac{C_3^p}{p}z^p, \tag{2.30}$$

here C_3 is the Sobolev imbedding constant for $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$. Then

$$Q'(z) = z^{q-1} - C_3^p z^{p-1}. \tag{2.31}$$

We know $Q'(z)$ have only one zero point, that is $z_1 = C_3^{p/(q-p)}$. Hence Q(z) is strictly increasing in $[0, z_1)$ and strictly decreasing in $(z_1, +\infty)$.
Let

$$d = Q(z_1) = (\frac{1}{q} - \frac{1}{p})C_3^{\frac{pq}{q-p}} > 0. \tag{2.32}$$

Lemma 2 *Suppose u is a local solution of the problem* (1.1) *and the condition of* **Theorem 2** *holds, then there exists* z_0, $z_0 > z_1$, *satisfies*

$$\|u(t)\|_{1,q} \ge z_0, \qquad \forall t \in [0, T). \tag{2.33}$$

Proof of Lemma 2 By (2.2), (2.30) and Sobolev inequality we get

$$E(t) \ge \frac{1}{q}\|u\|_{1,q}^q - \frac{1}{p}\|u\|_p^p \ge \frac{1}{q}\|u\|_{1,q}^q - \frac{C_3^p}{p}\|u\|_{1,q}^p = Q(\|u\|_{1,q}), \text{ for any } t \ge 0, \tag{2.34}$$

hence

$$Q(z_1) = d > E(0) \ge Q(\|u(0)\|_{1,q}). \tag{2.35}$$

By continuity of Q, $\|u_0\|_q > z_1$ and (2.35), there exists z_0, $z_1 < z_0 \le \|u(0)\|_{1,q}$, satisfying $Q(z_0) = E(0)$. We first prove that $\|u(t)\|_{1,q}$ can not be located in (z_1, z_0). Otherwise, we assume there exists $t_0 \in [0, T)$ satisfying $\|u(t_0)\|_{1,q} \in (z_1, z_0)$, then

$$E(t_0) > Q(\|u(0)\|_{1,q}) \ge Q(z_0) = E(0).$$

This is a contradiction with $E(t) \leq E(0)$. By $\|u(0)\|_{1,q} > z_1$ and the continuation of $E(t)$ and u in $W_0^{1,q}(\Omega)$ on [0,T), we have $\|u(t)\|_{1,q} > z_1$ for all $t > 0$. So by the above discussion, Lemma 2 has been proved.

Proof of Theorem 2 By contradiction arguments, we assume the solution satisfying the condition of Theorem 2 exists globally. Let

$$F(t) = (u,u)+\int_0^t((-\Delta)^{\frac{\alpha}{2}}u,(-\Delta)^{\frac{\alpha}{2}}u)d\tau+(T_0-t)((-\Delta)^{\frac{\alpha}{2}}u(0),(-\Delta)^{\frac{\alpha}{2}}u(0))+\beta_1(t+t_0)^2,$$

here t_0, T_0, β_1 are constants to be given later.

$$\begin{aligned} F'(t) &= 2(u,u_t) + ((-\Delta)^{\frac{\alpha}{2}}u,(-\Delta)^{\frac{\alpha}{2}}u) - ((-\Delta)^{\frac{\alpha}{2}}u(0),(-\Delta)^{\frac{\alpha}{2}}u(0)) + 2\beta_1(t+t_0) \\ &= 2(u,u_t) + 2\int_0^t((-\Delta)^{\frac{\alpha}{2}}u,(-\Delta)^{\frac{\alpha}{2}}u_\tau)d\tau + 2\beta_1(t+t_0), \end{aligned} \tag{2.36}$$

since u is a solution of the problem(1.1) and using (2.36), we have

$$\begin{aligned} \frac{1}{2}F''(t) &= (u_t,u_t) + (u,u_{tt}) + ((-\Delta)^{\frac{\alpha}{2}}u,(-\Delta)^{\frac{\alpha}{2}}u_t)) + \beta_1 \\ &= (u_t,u_t) + \|u\|_p^p - \|u\|_{1,q}^q + \beta_1 \\ &= (1+\frac{p}{2})(u_t,u_t) + p\int_0^t((-\Delta)^{\frac{\alpha}{2}}u_\tau,(-\Delta)^{\frac{\alpha}{2}}u_\tau)d\tau + (\frac{p}{q}-1)\|u\|_{1,q}^q - pE(0) + \beta_1 \\ &\geq (1+\frac{p}{2})[(u_t,u_t)+\beta_1] + p\int_0^t((-\Delta)^{\frac{\alpha}{2}}u_\tau,(-\Delta)^{\frac{\alpha}{2}}u_\tau)d\tau \\ &\quad +(\frac{p}{q}-1)z_0^q - pE(0) - \frac{p}{2}\beta_1, \end{aligned} \tag{2.37}$$

by choosing β_1 such that $\frac{p}{2}\beta_1 = (\frac{p}{q}-1)z_0^q - pE(0) > (\frac{p}{q}-1)z_1^q - pd = 0$, β has been fixed. Then

$$F''(t) \geq (p+2)[(u_t,u_t)+\beta_1] + 2p\int_0^t((-\Delta)^{\frac{\alpha}{2}}u_\tau,(-\Delta)^{\frac{\alpha}{2}}u_\tau)d\tau,$$

choose t_0 large enough such that $F'(0) = 2(u_0,u_1) + 2\beta_1 t_0 > 0$. Hence $\forall t \in [0,T_0]$, we have $F(0), F'(0), F''(0) > 0$. Set

$$A = (u,u) + \int_0^t((-\Delta)^{\frac{\alpha}{2}}u,(-\Delta)^{\frac{\alpha}{2}}u)d\tau + \beta_1(t+t_0)^2,$$

$$B = \frac{1}{2}F', \quad C = (u_t,u_t) + \int_0^t((-\Delta)^{\frac{\alpha}{2}}u_\tau,(-\Delta)^{\frac{\alpha}{2}}u_\tau)d\tau + \beta_1,$$

we have $A < F$, $C \leq \frac{F''}{p+2}$.

Hence $\forall(\xi,\eta) \in \mathbf{R}^2$, $t \in [0,T_0]$, we get

$$\begin{aligned} A\xi^2 + 2B\xi\eta + C\eta^2 = {} & (\xi u+\eta u_t, \xi u+\eta u_t) \\ & + \int_0^t(\xi(-\Delta)^{\frac{\alpha}{2}}u + \eta(-\Delta)^{\frac{\alpha}{2}}u_t, \xi(-\Delta)^{\frac{\alpha}{2}}u + \eta(-\Delta)^{\frac{\alpha}{2}}u_t)d\tau \\ & + \beta_1(\xi(t+t_0)+\eta)^2 \geq 0. \end{aligned}$$

Then $\Delta = (2B)^2 - 4AC \leq 0$, hence $(\frac{1}{2}F')^2 - F\frac{F''}{p+2} \leq 0$, namely

$$FF'' - (\alpha+1)(F')^2 \geq 0$$

here $\alpha = \frac{p-2}{4}$. So $[F^{-\alpha}(t)]'' \geq 0$ holds for every $t \in [0, T_0]$, that $F^{-\alpha}(t)$ is a concavity function. We could obtain the blowup time T_b in the standard way (see [3]), where $T_b \leq \frac{F(0)}{\alpha F'(0)}$. Then we reach a contradiction with our assumption, so **Theorem 2** has been proved.

Remark About $F''(t)$ in the proof, it is formal calculation, we can make it rigorous by the approach of H. A. Levine and J. Serrin [10].

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