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# GLOBAL SOLUTIONS TO AN INITIAL BOUNDARY VALUE PROBLEM FOR THE MULLINS EQUATION

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Dedicated to Prof. Boling Guo for his 70th Birthday

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**Abstract** In this article we study the global existence of solutions to an initial boundary value problem for the Mullins equation which describes the groove development at the grain boundaries of a heated polycrystal, both the Dirichlet and the Neumann boundary conditions are considered. For the classical solution we also investigate the large time behavior, it is proved that the solution converges to a constant, in the $L^\infty(\Omega)-$norm, as time tends to infinity.

**Key Words** Mullins equation; initial boundary value problem; global solutions.
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## 1. Introduction

In the present article we are interested in the global existence of solutions to an initial boundary value problem for the Mullins equation which describes the groove development at the grain boundaries of a heated polycrystal. When the weak solution happens to be classical we also investigate the large time behavior of the solution. This model was proposed by Mullins in 1957, see [1]. In the classical theory of thermal grooving, two principal mechanisms for mass transport on a metal surface can be distinguished, the evaporation-condensation and the surface diffusion. For some metals like magnesium, the first mechanism plays a dominated role after a very short time. While for some other metals, such as gold, the second mechanism dominates the process for a very long time. We refer to [1] for more details. The initial boundary value problem reads

$$u_t = D\left(1+u_x^2\right)^{-1} u_{xx}, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

$$u|_{t=0} = u_0, \tag{1.3}$$

where the equation (1.1) must be satisfied in $Q_T = (0,T) \times \Omega$, $T$ is a certain real number, $\Omega = (a,b)$, and $\partial\Omega = \{a,b\}$ with $a, b$ being real numbers such that $a < b$. We also consider the Neumann boundary condition, namely (1.2) is replaced by

$$u_x|_{\partial\Omega} = 0. \tag{1.4}$$

Here $u = u(t,x)$ is the unknown, $u_0$ is the initial data which is given, where $x, u$ are Cartesian coordinates and $t$ is the time. $D$ is a constant defined by

$$D = \frac{p_0 \gamma \omega^2}{(2\pi M)^{\frac{1}{2}} (kT)^{\frac{3}{2}}},$$

where $p_0$ is the vapor pressure in equilibrium with a plane surface (the curvature $K = 0$), $\gamma$ is the surface-free energy per unit area, $\omega$ is the molecular volume, $M$ is the weight of a molecule, and $k$ is the Boltzmann constant and $T$ is the absolute temperature, respectively. For simplicity we assume that $D = 1$. As we shall see later on, we state the existence theorem of solutions to the problems for both the Dirichlet and the Neumann boundary conditions, we investigate mainly the problem with Dirichlet boundary condition since many parts of the proofs for the two problems are similar, however, we still state the key ingredients in the proof of the theorem for the problem with the Neumann boundary condition, which is crucially different from those arising in the Dirichlet problem.

Equation (1.1) is a model for thermal grooving of the first mechanism. We choose the free energy function as

$$f(u_x) = \frac{\nu}{2}|u_x|^2,$$

suppose that $u$ is a classical solution to (1.1) – (1.3), then one has

$$\begin{aligned} \frac{d}{dt}\int_\Omega f(u_x(t,x))dx &= \nu \int_\Omega u_x u_{xt} = -\nu \int_\Omega u_{xx} u_t dx \\ &= -\nu \int_\Omega \left(1+u_x^2\right)^{-1} u_{xx}^2 dx \\ &\le 0. \end{aligned} \tag{1.5}$$

Therefore, the second law of thermodynamics is satisfied. If we define

$$J = \int^{u_x} \frac{dy}{1+y^2},$$

we find that (1.1) become $u_t = J_x$, so $J$ is a flux, (1.1) defines a gradient flow.

On the other hand, we can easily see that (1.1) is non-uniformly parabolic since the coefficient of its leading term may decay to zero as $u_x$ tends to infinity. Thus, we modify the equation to a uniformly parabolic one, to solve this approximate problem we employ an existence theorem for quasilinear parabolic equations, see e.g. Ladyzenskaya, Solonnikov and Uralceva [2]. Then we pass the approximate solutions to the limit which is just a weak solution to the original problem.

In what follows, we shall prove the global existence of weak solution and classical solution, for the second case we also investigate the large time behavior. To do so, we need to introduce some notations. We denote the scalar product over $Q_T$ by $(\cdot,\cdot)_T$, its corresponding norm by $\|\cdot\|_T$, and the scalar product over $\Omega$ by $(\cdot,\cdot)$, its corresponding norm by $\|\cdot\|$, respectively. We use the notation $\int^x = \int_A^x$ for any fixed $A \in \mathbb{R}$. The standard notations for the Sobolev and Hölder spaces are adopted.

**Definitions 1) Weak solutions** *For any $u_0 \in L^2(\Omega)$. A function $u$ which satisfies*

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;H_0^1(\Omega))$$

*(or $u \in L^\infty(0,T;L^2(\Omega)) \cap L^1(0,T;H^1(\Omega))$ for the Neumann boundary conditions), is called a weak solution to the problem* (1.1) – (1.3) *if it satisfies*

$$(u,\varphi_t)_T = (\mathrm{arctg}(u_x),\varphi_x)_T - (u_0,\varphi) \tag{1.6}$$

*for any test function $\varphi \in C_0^\infty((-\infty,T)\times\Omega)$ ( or $\varphi \in C_0^\infty((-\infty,T)\times\mathbb{R})$ for the Neumann boundary conditions ). Here* $\mathrm{arctg}(u_x) = \int_0^{u_x} \frac{dy}{1+y^2}$.

**2) Strong solutions** *A function $u$ is called a strong solution to the problem* (1.1) – (1.3) *if $u$ is a weak solution and satisfies the semi-regularity properties*

$$u \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \quad u_t \in L^2(Q_T).$$

(*Remark: for the Neumann condition, the properties are $u_t \in L^2(Q_T)$ and $u \in L^\infty(0,T;$ $H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H^1(\Omega))$.*)

We can state the main results as follows.

A) For the Dirichlet boundary, we have

**Theorem 1.1** I) *Assume that $u_0 \in H_0^1(\Omega)$, and that the compatibility condition $u_0(a) = u_0(b) = 0$ is satisfied.*

*Then there exists a unique global weak solution $u$ such that*

$$u \in L^\infty(0,T;H_0^1(\Omega)), \quad u_t \in L^2(Q_T).$$

II) *Suppose that $u_0$ satisfies the conditions in* I)*, moreover assume that $u_0 \in W^{1,\infty}(\Omega)$, then there exists a unique strong solution $u$ which satisfies furthermore that*

$$\|u_x\|_{L^\infty(Q_T)} \le C.$$

III) *Further, if we assume that $u_0 \in C^{2+\alpha}(\bar{\Omega})$ (here and hereafter we assume that $0<\alpha<1$ is a constant ) and one more compatibility condition*

$$u_{0xx}|_{x=a,b} = 0$$

*is met. Then there exists a unique classical solution u such that*

$$u_{tx}, u_{xxx} \in L^2(Q_T).$$

*Moreover*

$$\|u(t)\|_{L^\infty(\Omega)} \to 0$$

*as* $t \to \infty$.

B) For the Neumann boundary, we have

**Theorem 1.2** I) *Assume that* $u_0 \in H^1(\Omega)$. *Then there exists a global weak solution u such that*

$$u \in L^\infty(0,T;H^1(\Omega)), \quad u_t \in L^2(Q_T).$$

II) *Besides the conditions for* $u_0$ *in* I), *we assume that* $u_0 \in W^{1,\infty}(\Omega)$, *then there exists a unique strong solution u which satisfies furthermore that*

$$\|u_x\|_{L^\infty(Q_T)} \le C.$$

III) *Assume that* $u_0 \in C^{2+\alpha}(\bar\Omega)$ *and the compatibility conditions*

$$u_{0x}|_{x=a,b} = 0, \quad u_t|_{t=0,\,x=a,b} = u_{0xx}|_{x=a,b}$$

*are satisfied. Then there exists a unique classical solution u such that*

$$\int_\Omega u(t,x)dx = \int_\Omega u_0(x)dx, \quad u_{tx}, u_{xxx} \in L^2(Q_T).$$

*Moreover*

$$\|u(t) - \bar u_0\|_{L^\infty(\Omega)} \to 0$$

*as* $t \to \infty$. *Here we used the notation* $\bar f(t) = \frac{1}{\text{meas }(\Omega)} \int_\Omega f(t,x)dx$.

The main difficulty in the proof of Theorem 1.1 and Theorem 1.2 is due to that the coefficient depends nonlinearly on the first order derivative, and may decay to zero. This leads to the difficulty for the proof of compactness of a sequence of first order derivative of approximate solutions, this is only in $L^1(0,T_e;H^{-1}(\Omega))$, we shall use, as in Alber and Zhu [3], the generalized form of the Aubin-Lions lemma, given by Roubíček [4] or Simon [5], which is valid in $L^1$. To prove the existence of classical solution we make use of an existence theorem from the book by Ladyzenskaya etc. [2].

We now recall some references which are related to this article. Some authors have investigated the existence of special solutions of the Cauchy problems to the equation (1.1), see e.g. Broadbridge [6], Kitada and Umehara [7]. As for the conserved Mullins equation which is a fourth order nonlinear parabolic equation, we refer to Broadbridge and Tritscher [8], Tritscher and Broadbridge [9], etc. As for the problems for the equations where the coefficient of the principle part depends on the unknown and decay to zero as the unknown tends to zero, for instance, the porous media equation is a famous model, we refer to R. Dal Passo and Luckhaus [10], Aronson, Crandall and Peletier [11], Brezis and Crandall [12], etc.

The remaining part of this article is organized as follows: In Section 2 we are going to establish some *a priori* estimates for both weak and classical solutions. Then in Section 3 we discuss the limits of the approximate solutions by using those estimates, thus prove the existence of solutions. Also we investigate the large time behavior of the classical solutions.

## 2. A priori estimates

In this section we are going to prove the existence of classical solution and to derive some *a priori* estimates for the following approximate problem

$$u_t = \left(\left(1+u_x^2\right)^{-1} + \kappa\right) u_{xx}, \tag{2.1}$$

$$u|_{\partial\Omega} = 0, \quad \text{or} \quad u_x|_{\partial\Omega} = 0, \tag{2.2}$$

$$u|_{t=0} = u_0^\kappa. \tag{2.3}$$

Here, $\kappa$ is a positive number, we assume that

$$0 < \kappa < 1.$$

The initial data $u_0^\kappa \in C_0^\infty(\Omega)$ is a smooth approximation of $u_0$ such that

$$\|u_0^\kappa - u_0\|_{H^1(\Omega)} \to 0$$

as $\kappa \to 0$.

Employing a theorem in the book by Ladyzenskaya etc. [2], we can prove easily the existence of classical solution $u^\kappa$ to the problem (2.1) – (2.3). Moreover we can establish the following *a priori* estimates which are uniformly bounded in $\kappa$, and the bounds $C$ are independent of $t$. To denote a constant depending probably on $t$ we use $C_T$.

**Lemma 2.1** (The basic energy estimate) *There hold for any* $t \in [0,T]$ *that*

$$\|u_x^\kappa\|^2 + \int_0^t \int_\Omega \left(\frac{1}{1+(u_x^\kappa)^2} + \kappa\right) |u_{xx}^\kappa|^2 dx d\tau \le C, \tag{2.4}$$

$$\int_0^t \int_\Omega \left(\left|\frac{u_{xx}^\kappa}{1+(u_x^\kappa)^2}\right|^2 + (u_t^\kappa)^2\right) dx d\tau \le C, \tag{2.5}$$

$$\int_0^t \int_\Omega |\mathrm{arctg}(u_x^\kappa)|^2 = \int_0^t \int_\Omega \left|\int_0^{u_x^\kappa} \frac{dy}{1+y^2} dy\right|^2 dx d\tau \le C_T. \tag{2.6}$$

*When the initial data* $u_0 \in W^{1,\infty}(\Omega)$, *the solution satisfies for any* $t \in [0,T]$

$$\|u_x^\kappa\|_{L^\infty(Q_T)} \le C. \tag{2.7}$$

**Proof** Suppose that (2.4) holds. Then (2.6) and (2.5) follow easily from the simple inequalities

$$\left|\int_0^{u_x^\kappa}\frac{dy}{1+y^2}\right| \le \int_0^{|u_x^\kappa|}\frac{dy}{1+y^2} \le \int_0^{\infty}\frac{dy}{1+y^2}dy \le C < \infty.$$

and that

$$\frac{1}{1+(u_x^\kappa)^2} \le 1.$$

So we only need to prove (2.4).

Multiplying (2.1) by $-u_{xx}^\kappa$ and integrating the resulting equation over $Q_T$ yield

$$\frac{1}{2}\|u_x^\kappa\|^2 + \int_0^t\int_\Omega\left(\frac{1}{1+(u_x^\kappa)^2}+\kappa\right)(u_{xx}^\kappa)^2dxd\tau = \frac{1}{2}\|u_{0x}^\kappa\|^2. \tag{2.8}$$

which implies (2.4). As for (2.7), we follow an idea in [13], however that technique is used here to get the $L^p-$bounds of the derivative of the unknown while it was used for getting $L^p-$bounds of the unknown. Multiplying (2.1) by $-\left(\int^{u_x^\kappa} y^{2n+1}dy\right)_x$ for any integer $n \ge 0$ and integrating it over $Q_T$ yield

$$\begin{aligned}
0 &= \left(u_t^\kappa, -\left(\int^{u_x^\kappa} y^{2n+1}dy\right)_x\right) + \int_\Omega \frac{(u_x^\kappa)^{2n+1}}{1+(u_x^\kappa)^2}(u_{xx}^\kappa)^2dx \\
&\ge \left(u_{xt}^\kappa, \int^{u_x^\kappa} y^{2n+1}dy\right) \\
&= \frac{1}{2n+2}\frac{d}{dt}\int_\Omega |u_x^\kappa|^{2n+2}dx.
\end{aligned} \tag{2.9}$$

Thus, we obtain

$$\left(\int_\Omega |u_x^\kappa(t)|^{2n+2}dx\right)^{\frac{1}{2n+2}} \le \left(\int_\Omega |u_x^\kappa(0)|^{2n+2}dx\right)^{\frac{1}{2n+2}} \le \|u_{0x}\|_{L^\infty(\Omega)}.$$

Letting $n \to \infty$ yields (2.7), and we complete the proof of the lemma.

**Lemma 2.2** *There holds for any $t \in [0,T]$ that*

$$\left\|\partial_t\int^{u_x^\kappa}\frac{dy}{1+y^2}dy\right\|_{L^1(0,T;H^{-1}(\Omega))} \le C. \tag{2.10}$$

**Proof** Differentiating the equation (2.1) with respect to $x$, multiplying it by $\varphi/(1+(u_x^\kappa)^2)$, where $\varphi$ is a test function in $L^\infty(0,T;H_0^1(\Omega))$, then integrating the resulting equation over $Q_T$ we obtain

$$\left(\frac{u_{xt}^\kappa}{1+(u_x^\kappa)^2},\varphi\right)_T = \left(\partial_x\left(\frac{u_{xx}^\kappa}{1+(u_x^\kappa)^2}+\kappa u_{xx}^\kappa\right),\frac{\varphi}{1+(u_x^\kappa)^2}\right)_T. \tag{2.11}$$

Here, we used the property that $u^\kappa_{xt} \in L^2(Q_T)$ for a classical solution $u^\kappa$ to problem (2.1) – (2.3).

The above equation can be rewritten as

$$\left(\partial_t \int^{u^\kappa_x} \frac{dy}{1+y^2} dy, \varphi\right)_T = -\left(\frac{u^\kappa_{xx}}{1+(u^\kappa_x)^2} + \kappa u^\kappa_{xx}, \partial_x\left(\frac{\varphi}{1+(u^\kappa_x)^2}\right)\right)_T. \tag{2.12}$$

Thus we have

$$\begin{aligned}\left|\left(\partial_t \int^{u^\kappa_x} \frac{dy}{1+y^2} dy, \varphi\right)_T\right| =& \left|\left(\frac{u^\kappa_{xx}}{1+(u^\kappa_x)^2} + \kappa u^\kappa_{xx}, \partial_x\left(\frac{\varphi}{1+(u^\kappa_x)^2}\right)\right)_T\right| \\ \le& \left|\left(\frac{u^\kappa_{xx}}{1+(u^\kappa_x)^2} + \kappa u^\kappa_{xx}, \frac{\varphi_x}{1+(u^\kappa_x)^2}\right)_T\right| \\ &+ \left|\left(\frac{u^\kappa_{xx}}{1+(u^\kappa_x)^2} + \kappa u^\kappa_{xx}, \frac{-2\varphi u^\kappa_x u^\kappa_{xx}}{(1+(u^\kappa_x)^2)^2}\right)_T\right| \\ \le& \left(\left\|\frac{u^\kappa_{xx}}{1+(u^\kappa_x)^2}\right\|_T + \kappa\|u^\kappa_{xx}\|\right)\|\varphi_x\|_T \\ &+ \left(\left\|\frac{u^\kappa_{xx}}{1+(u^\kappa_x)^2}\right\|_T^2 + \kappa^2\|u^\kappa_{xx}\|^2\right)\|\varphi\|_{L^\infty(Q_T)} \\ \le& C\|\varphi\|_{L^\infty(0,T;H^1_0(\Omega))}.\end{aligned} \tag{2.13}$$

Here we used the Sobolev imbedding theorem, the basic inequality $2ab \le a^2 + b^2$ for any $a, b \ge 0$, and estimate (2.5). We obtain (2.10). And the proof of this lemma is completed.

To conclude the existence of classical solution, we need more estimate and assume that

$$u_0 \in C^{2,\alpha}(\bar\Omega).$$

We shall apply the maximum principle for parabolic equations to the difference quotient

$$u^\kappa_h(t) = \frac{u^\kappa(t+h) - u^\kappa(t)}{h}$$

for any $T > h > 0$. Hereafter, we use $u^\kappa(t)$ to denote $u^\kappa(t, x)$ for the sake of simplicity.

**Lemma 2.3** *There holds*

$$\|u^\kappa_t\|_{L^\infty(Q_T)} \le \|u_0\|_{W^{2,\infty}(\Omega)}. \tag{2.14}$$

**Proof** Suppose that there exists a classical solution $u^\kappa$ to the problem (2.1) – (2.3). We omit the upper-script $\kappa$ and the argument $t$ (however we leave the argument $t + h$) of unknowns, and write $u^\kappa$ as $u$ for simplicity, in the following proof of this lemma. After straightforward computations we then see that $u_h$ satisfies

$$\begin{aligned} u_{ht} &= \frac{1}{h}\left(\frac{u_{xx}}{1+u_x^2}(t+h) - \frac{u_{xx}}{1+u_x^2}\right) + \frac{\kappa}{h}(u_{xx}(t+h) - u_{xx}) \\ &= \left(\frac{1+u_x^2}{(1+u_x^2(t+h))(1+u_x^2)} + \kappa\right) u_{hxx} - \frac{(u_x(t+h)+u_x)\, u_{xx}}{(1+u_x^2(t+h))(1+u_x^2)} u_{hx}, \end{aligned} \tag{2.15}$$

and the boundary and initial data are

$$u_h|_{\partial\Omega} = 0,$$
$$u_h|_{t=0} = \hat{u}_0,$$

where

$$\hat{u}_0 = \frac{1}{h}\int_0^h u_t(\tau)d\tau, \text{ and } \hat{u}_0 \to \left(1+u_{0x}^2\right)^{-1} u_{0xx} \quad \text{as } h \to 0. \tag{2.16}$$

Therefore applying the maximum principle to the function $u_h$ which is a classical solution to the above problem, we then obtain

$$\|u_h\|_{L^\infty(Q_{T-h})} \le C\|\hat{u}_0\|_{L^\infty(Q_T)}.$$

Letting $h \to 0$, using (2.16), we get (2.14). And the proof of this lemma is completed.

**Remark** For the Neumann condition, we can only obtain the estimates for $u_t^\kappa, u_{xx}^\kappa$ in $L^\infty(0,T;L^2(\Omega))$, however they are enough to get the Hölder estimate for $u_x^\kappa$, with a smaller exponent in (3.7) and (3.8). To prove these estimates, we apply the energy estimate and use (2.7), and omit the detail.

**Corollary 2.4** *There hold for any $t \in [0,T]$ that*

$$\|u^\kappa\|_{L^\infty(0,T;W^{2,\infty}(\Omega))} \le C, \tag{2.17}$$

$$\|u_t^\kappa\|_T + \|u_{xx}^\kappa\|_T \le C. \tag{2.18}$$

**Proof** Suppose that (2.17) is true, we infer (2.18) easily from the equation (2.1). Thus we only need to prove (2.17).

From Eq. (2.1) and (2.14) we get

$$\|u_{xx}^\kappa\|_{L^\infty(Q_T)} \le \|u_t^\kappa\|_{L^\infty(Q_T)}\|(1+|u_x^\kappa|^2)\|_{L^\infty(Q_T)} \le C\left(1+\|u_x^\kappa\|^2_{L^\infty(Q_T)}\right). \tag{2.19}$$

By the Nirenberg inequality one has

$$\|u_x^\kappa\|_{L^\infty(\Omega)} \le C\|u_{xx}^\kappa\|^{\frac{1}{3}}_{L^\infty(\Omega)}\|u_x^\kappa\|^{\frac{2}{3}} + C'\|u_x^\kappa\|,$$

so (2.19) becomes

$$\begin{aligned} \|u_{xx}^\kappa\|_{L^\infty(Q_T)} &\le C\left(1+\sup_{0\le t\le T}\|u_{xx}^\kappa(t)\|^{\frac{2}{3}}_{L^\infty(\Omega)}\right) \\ &\le \sup_{0\le t\le T}\frac{1}{2}\|u_{xx}^\kappa(t)\| + C = \frac{1}{2}\|u_{xx}^\kappa\|_{L^\infty(Q_T)} + C, \end{aligned} \tag{2.20}$$

from which we arrive at (2.17). Here we used the estimate (2.4) and the Young inequality. Thus the proof of this lemma is completed.

## 3. Existence and large time behavior

This section is concerned with the limits of approximate solutions and with the large time behavior of classical solution. The proof of the existence of weak solution is based on the following two results:

**Theorem 3.1** *Let $B_0$ be a normed linear space imbedded compactly into another normed linear space $B$ which is continuously imbedded into a Hausdorff locally convex space $B_1$, and $1 \leq p < +\infty$. If $v, v_i \in L^p(0, T_e; B_0), i \in \mathbb{N}$, the sequence $\{v_i\}_{i\in\mathbb{N}}$ converges weakly to $v$ in $L^p(0, T_e; B_0)$, and $\{\frac{\partial v_i}{\partial t}\}_{i\in\mathbb{N}}$ is bounded in $L^1(0, T_e; B_1)$, then $v_i$ converges to $v$ strongly in $L^p(0, T_e; B)$.*

**Lemma 3.2** *Let $(0, T_e) \times \Omega$ be an open set in $\mathbb{R}^+ \times \mathbb{R}^n$. Suppose functions $g_n, g$ are in $L^q((0, T_e) \times \Omega)$ for any given $1 < q < \infty$, which satisfy*

$$\|g_n\|_{L^q((0,T_e)\times\Omega)} \leq C, \quad g_n \to g \text{ almost everywhere in } (0, T_e) \times \Omega.$$

*Then $g_n$ converges to $g$ weakly in $L^q((0, T_e) \times \Omega)$.*

Theorem 3.1 is a general version of Aubin-Lions lemma valid under the weak assumption $\partial_t v_i \in L^1(0, T_e; B_1)$. This version, which we need here, is proved in [4] and [5], separately. A proof of Lemma 3.2 can be found in [14, p.12].

**Proof of existence of weak solution** Let us first prove the existence of weak solution. From the estimate $\|u_x^\kappa\| \leq C$ we assert that there exists a subsequence and a function $u$ such that

$$u^\kappa \rightharpoonup^* u,$$

in $L^\infty(0, T; H^1(\Omega))$. We next prove that $u$ is a weak solution to the problem.

Choosing a test function $\varphi \in C_0^\infty((-\infty, T) \times \Omega)$ (for the Neumann condition, $\varphi \in C_0^\infty((-\infty, T) \times \mathbb{R})$, we don't point out this again later on), multiplying (2.1) by $\varphi$, integrating the resulting equation with respect to $t, x$, and using integration by parts we get

$$\begin{aligned} 0 &= (u_t^\kappa, \varphi)_T + ((\mathrm{arctg}(u_x^\kappa))_x, \varphi)_T \\ &= -(u^\kappa, \varphi_t)_T - (u^\kappa(0), \varphi(0)) - (\mathrm{arctg}(u_x^\kappa), \varphi_x)_T . \end{aligned} \tag{3.1}$$

We shall see (1.6) is satisfied provided that we prove the following results are true: For $\kappa \to 0$, there hold

$$(u^\kappa, \varphi_t)_T \to (u, \varphi_t)_T, \tag{3.2}$$

$$(\mathrm{arctg}(u_x^\kappa), \varphi_x)_T \to (\mathrm{arctg}(u_x), \varphi_x)_T , \tag{3.3}$$

$$(u^\kappa(0), \varphi(0)) \to (u_0, \varphi(0)). \tag{3.4}$$

The relations (3.2) and (3.4) are easy to check. We are going to prove (3.3). To this end, we apply Theorem 3.1, and choose

$$v_\kappa := \mathrm{arctg}(u_x^\kappa),\ p = 2$$

and

$$B_0 = H_0^1(\Omega), \quad B = L^2(\Omega), \quad B_1 = H^{-1}(\Omega).$$

(for the Neumann condition, $H_0^1(\Omega)$ is changed to $H^1(\Omega)$.) The spaces $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ are reflexive and the Sobolev imbedding theorem implies that $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. From Theorem 3.1 we thus conclude that there is a subsequence, still denote it by $v_n$ such that

$$\|v_\kappa - v\|_T \to 0,$$

as $\kappa \to 0$. Thus we can select a subsequence $v_\kappa$ which converges almost everywhere to $v$. Recalling that $v_\kappa = \arctan(u_x^\kappa)$, we assert that

$$u_x^\kappa \text{ converges to } u_x \text{ almost everywhere.} \tag{3.5}$$

thus

$$\mathrm{arctg}(u_x^\kappa) \to \mathrm{arctg}(u_x) \text{ almosteverywhere.}$$

It is easy to show that

$$\|\mathrm{arctg}(u_x^\kappa)\|_T \le C,$$

applying Lemma 3.2 we conclude that

$$\mathrm{arctg}(u_x^\kappa) \rightharpoonup \mathrm{arctg}(u_x)$$

in $L^2(Q_T)$, whence

$$(\mathrm{arctg}(u_x^\kappa), \varphi_x)_T \to (\mathrm{arctg}(u_x), \varphi_x)_T$$

as $\kappa \to 0$, for any test function $\varphi \in C_0^\infty((-\infty, T) \times \Omega)$.

Therefore (3.1) becomes

$$0 = -(u, \varphi_t)_T - (u_0, \varphi(0)) - (\mathrm{arctg}(u_x), \varphi_x)_T\,. \tag{3.6}$$

For the uniqueness of weak solution, we can prove it easily by the monotonicity of the function $\mathrm{arctg}(y)$, hence the proof of the existence and uniqueness of weak solution is completed.

**Proof of existence of strong solution** By definition, to prove the existence of strong solution we only need to examine the semi-regularity. For the weak solution $u$ we have obtained that

$$u \in L^\infty(0, T; H^1(\Omega)), \quad u_t \in L^2(Q_T).$$

From the estimates (2.5) and (2.7) one can obtain easily that

$$\|u\|_{L^2(0,T;H^2(\Omega))} \le C, \quad \text{so} \ \ u \in L^2(0,T;H^2(\Omega)).$$

And the proof of existence is completed.

**Proof of existence of classical solution** To prove global existence of classical solution we need the lemma see, e.g. [2]

**Lemma 3.3** *Let $f(t,x)$ be a function on $Q_T$ such that*

i) *$f$ is uniformly (with respect to $x$) Hölder continuous in $t$, with exponent $0 < \alpha \le 1$, that is $|f(t,x) - f(s,x)| \le C|t-s|^\alpha$, and*

ii) *$f_x$ is uniformly (with respect to $t$) Hölder continuous in $x$, with exponent $0 < \beta \le 1$, that is $|f_x(t,x) - f_x(t,y)| \le C'|y-x|^\beta$.*

*Then $f_x$ is uniformly Hölder continuous in $t$ with exponent $\gamma = \alpha\beta/(1+\beta)$, such that*

$$|f_x(t,x) - f_x(s,x)| \le C''|t-s|^\gamma, \ \forall x \in \bar{\Omega}, 0 \le s \le t \le T.$$

*where $C''$ is a constant which may depend on $C, C'$ and $\alpha, \beta$.*

Now from the estimates (2.14) and (2.17) we obtain easily that

$$\begin{aligned} |u^\kappa(t,x) - u^\kappa(s,x)| &= \left| \int_s^t u_t^\kappa(\tau,x)d\tau \right| \\ &\le \|u_t^\kappa\|_{L^\infty(Q_T)}|t-s| \le C|t-s|, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} |u_x^\kappa(t,x) - u_x^\kappa(t,y)| &= \left| \int_x^y u_{xx}^\kappa(t,\xi)d\xi \right| \\ &\le \|u_{xx}^\kappa\|_{L^\infty(Q_T)}|y-x| \le C|y-x|. \end{aligned} \tag{3.8}$$

This shows that to apply Lemma 3.3 to the function $u_x^\kappa$, we can choose the parameters as follows

$$\alpha = \beta = 1, \ \text{whence} \ \gamma = 1.$$

Therefore, for any $\alpha < 1$ we also have

$$u_x^\kappa \in C^{\frac{\alpha}{2},\alpha}(Q_T), \ \text{and} \ \|u_x\|_{C^{\frac{\alpha}{2},\alpha}(Q_T)} \le C.$$

Hence there exist two constants $\lambda, \Lambda$ which are independent of $\kappa \in (0,1]$ and depend only on the norm of $u_x^\kappa$ in $L^\infty(Q_T)$, such that

$$\lambda \le \frac{1}{1+(u_x^\kappa)^2} + \kappa \le \Lambda.$$

Thus, by the estimate of Schauder type see e.g. Friedman[15] for the uniformly parabolic equations we obtain

$$\|u^\kappa\|_{C^{1+\frac{\alpha}{2},2+\alpha}(Q_T)} \le C, \tag{3.9}$$

from which we can conclude easily the existence of classical solution to the problem (1.1) – (1.3).

It remains to show that $u_{tx}, u_{xxx} \in L^2(Q_T)$. To this end, we use (2.15). For any fixed $h > 0$ we can easily find the limit version of (2.15) as $\kappa \to 0$. Multiplying it by $u_h$, integrating the resulting equation and using integration by parts we obtain

$$\begin{aligned}
0 =& \frac{1}{2}\frac{d}{dt}\|u_h\|^2 + \int_\Omega \frac{1+u_x^2(t)}{(1+u_x^2(t+h))(1+u_x^2(t))}|u_{hx}|^2 dx \\
&+ \int_\Omega \left(\frac{1+u_x^2(t)}{(1+u_x^2(t+h))(1+u_x^2(t))}\right)_x u_{hx}u_h dx \\
&+ \int_\Omega \frac{(u_x(t+h)+u_x(t))\,u_{xx}(t)}{(1+u_x^2(t+h))(1+u_x^2(t))} u_{hx}(t)u_h dx \\
=& \frac{1}{2}\frac{d}{dt}\|u_h\|^2 + I_1 + I_2 + I_3.
\end{aligned} \tag{3.10}$$

By estimate (2.17) we have that there exists a constant $C$ such that

$$I_1 \geq C\|u_{hx}\|^2, \tag{3.11}$$

$$|I_2|, \quad |I_3| \leq \varepsilon\|u_{hx}\|^2 + C_\varepsilon\|u_h\|^2, \tag{3.12}$$

Thus, combination of (3.10) – (3.12) yields

$$\frac{1}{2}\frac{d}{dt}\|u_h\|^2 + C\|u_{hx}\|^2 \leq \varepsilon\|u_{hx}\|^2 + C_\varepsilon\|u_h\|^2. \tag{3.13}$$

Letting $\varepsilon \leq \frac{1}{2}C$ and integrating (3.13) with respect to $t$ we arrive at

$$\|u_h(t)\|^2 + C\int_0^t \|u_{hx}\|^2 d\tau \leq C\int_0^t \|u_h\|^2 d\tau + \|u_h(0)\|^2. \tag{3.14}$$

Since we investigate the classical solution $u$, sending $h \to 0$ and using the lower semi-continuity of the $L^2-$norm $\|\cdot\|_T$, we infer from (3.14) that

$$\begin{aligned}
\|u_t(t)\|^2 + C\int_0^t \|u_{tx}\|^2 d\tau \leq& \lim_{h\to 0}\|u_h(t)\|^2 + C\liminf_{h\to 0}\int_0^t \|u_{hx}\|^2 d\tau \\
\leq& C\liminf_{h\to 0}\left(\int_0^t \|u_h\|^2 d\tau + \|u_h(0)\|^2\right) \\
=& C\int_0^t \|u_t\|^2 d\tau + \|u_t(0)\|^2 \leq C.
\end{aligned} \tag{3.15}$$

Here, we used the estimate (2.18) and eq. (2.1). Therefore we obtain

$$\int_0^t \|u_{tx}\|^2 d\tau \leq C, \tag{3.16}$$

from this and eq. (2.1), using the estimates (2.17) and (2.18) we get further that

$$\int_0^t \|u_{xxx}\|^2 d\tau \le C. \tag{3.17}$$

We turn to study the asymptotic behavior as time goes to infinity. Firstly we prove that

$$\|u_x(t)\| \to 0, \tag{3.18}$$

as $t \to \infty$. For simplicity of notations, we set

$$y(t) = \|u_x(t)\|^2.$$

Once we have

$$\int_0^\infty y(t)dt \le C, \tag{3.19}$$

$$\int_0^\infty \left|\frac{d}{dt}y(t)\right| dt \le C. \tag{3.20}$$

then we conclude easily that $y(t) \to 0$ as $t \to \infty$. Thus it remains to prove (3.19) and (3.20).

We are now going to prove (3.19). Invoking the boundary condition $u|_{x=a,b} = 0$, by the mean value theorem, we assert that for any given $t \ge 0$ there exists a point $x_0 = x_0(t)$ such that

$$u_x(t, x_0(t)) = 0,$$

thus integrating (2.1) with respect to $x$ over $(x_0, x)$ one has

$$u_x(t,x) = u_x(t, x_0(t)) + \int_{x_0(t)}^x (1+u_x^2)u_t dy = \int_{x_0(t)}^x (1+u_x^2)u_t dy,$$

and

$$\begin{aligned}\int_0^t \left|\int_\Omega u_x(t,x)dx\right|^2 d\tau =& \int_0^t \left|\int_\Omega \int_{x_0(t)}^x (1+u_x^2)u_t dydx\right|^2 d\tau \\ \le& C\int_0^t \int_\Omega \left(u_t^2 + u_x^2\right) dxd\tau \le C. \end{aligned} \tag{3.21}$$

Therefore, by the Poincaré inequality and the estimates (2.4)and (2.17) we obtain that

$$\begin{aligned}\int_0^\infty y(t)dt \le& C\int_0^\infty \left(\|u_x - \bar{u}_x\|^2 + \|\bar{u}_x\|^2\right) dt \\ \le& C\int_0^\infty \|u_{xx}\|^2 + \left(\int_\Omega u_x dx\right)^2 dt \le C. \end{aligned} \tag{3.22}$$

For the Neumann boundary value problem, it is easy to handle. We then prove (3.19).

We now want to prove (3.20). Multiplying (1.1) by $-u_{xx}$ and integrating the resulting equation, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_x(s)\|^2 = -\int_\Omega \frac{|u_{xx}|^2}{1+(u_x)^2}(s)dx. \tag{3.23}$$

Integrating (3.23) with respect to $t$ yields

$$\|u_x(t)\|^2 + 2\int_0^t\int_\Omega \frac{u_{xx}^2}{1+u_x^2}dxd\tau = \|u_x(0)\|^2 \le C.$$

This is the limit version of (2.8). Thus, from (3.23) we arrive at

$$\int_0^t \left|\frac{d}{dt}\|u_x\|^2\right| d\tau = 2\int_0^t\int_\Omega \frac{u_{xx}^2}{1+u_x^2}dxd\tau \le C, \tag{3.24}$$

that is

$$\int_0^\infty \left|\frac{d}{dt}y(t)\right| dt \le C.$$

By combining the above with (3.19) we show that

$$y(t)\to 0, \text{ as } t\to\infty.$$

Using the Nirenberg inequality

$$\|u\|_{L^\infty(\Omega)} \le C\|u_x\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}},$$

and estimate $\|u\|\le C$, one has

$$\|u(t)\|_{L^\infty(\Omega)} \le C\|u_x(t)\|^{\frac{1}{2}} \to 0$$

as $t\to\infty$. Thus the proof of Theorem 1.1 is completed.

For the Neumann boundary, we prove this in a different way. Integrating the equation (1.1) with respect to $x$ yields

$$\frac{d}{dt}\int_\Omega u(t,x)dx = \int_\Omega \frac{u_{xx}}{1+u_x^2}(t,x)dx = \int^{u_x}\frac{dy}{1+y^2}\bigg|_a^b = 0.$$

Thus $\int_\Omega u(t,x)dx = \int_\Omega u_0(x)dx$. Applying the Poincaré inequality we have

$$\|u(t)-\bar{u}_0\| = \|u(t)-\bar{u}(t)\| \le C\|u_x(t)\| \to 0.$$

So

$$\|u(t)-\bar{u}_0\|_{H^1(\Omega)} \to 0.$$

By combination of the above with the Sobolev imbedding theorem we get

$$\|u(t)-\bar{u}_0\|_{L^\infty(\Omega)} \to 0, \text{ as } t\to\infty.$$

Hence the proof of Theorem 1.2 is completed.

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