

A CAHN-HILLIARD TYPE EQUATION WITH GRADIENT DEPENDENT POTENTIAL*

Yin Jingxue and Huang Rui

(Department of Mathematics, Jilin University, Changchun 130012, China)

(E-mail: yjx@mail.jlu.edu.cn)

Dedicated to the 70th birthday of Professor Li Tatsien

(Received Oct. 30, 2007)

Abstract We investigate a Cahn-Hilliard type equation with gradient dependent potential. After establishing the existence and uniqueness, we pay our attention mainly to the regularity of weak solutions by means of the energy estimates and the theory of Campanato Spaces.

Key Words Cahn-Hilliard equation; existence; uniqueness; regularity.

2000 MR Subject Classification 35K55, 35G30, 35D10.

Chinese Library Classification O175.4, O175.29.

1. Introduction

In this paper, we consider the initial boundary value problem for the following Cahn-Hilliard type equation with gradient dependent potential

$$\frac{\partial u}{\partial t} + \operatorname{div} \left(K \nabla \Delta u - \vec{\Phi}(\nabla u) \right) = 0, \quad (x, t) \in Q_T, \quad (1.1)$$

$$\nabla u \cdot \nu \Big|_{\partial\Omega} = \mu \cdot \nu \Big|_{\partial\Omega} = 0, \quad t \in [0, T], \quad (1.2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega, \quad (1.3)$$

where Ω is a bounded domain in \mathbb{R}^N with smooth boundary, $Q_T = \Omega \times (0, T)$, ν denotes the unit exterior normal to the boundary $\partial\Omega$, $\mu = K \nabla \Delta u - \vec{\Phi}(\nabla u)$ is the flux, K is the positive diffusion coefficient, and $\vec{\Phi} = (\Phi_1, \Phi_2, \dots, \Phi_N)$ is a smooth vector function from \mathbb{R}^N to \mathbb{R}^N .

The problem (1.1)–(1.3) models many interesting phenomena in mathematical biology, fluid mechanics, phase transition, etc. Recently, such type of equations, especially in the case of one spatial dimension have arisen interests to many mathematicians.

*This work is supported by NNSF of China (No. 10531040).

For example, Myers [1] considered the following one dimensional fourth-order diffusion equation,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(C \frac{u^3}{3} \frac{\partial^3 u}{\partial x^3} + f(u, u_x, u_{xx}) \right) = 0,$$

which has been proposed to describe the surface tension phenomena in some particular case of thin films. We refer the readers to [2–6] for more examples of one-dimensional models. However, many models in multi-dimensional case may occur from practical problems, see for example [7,8]. Here, we briefly introduce the derivation of the equation (1.1) based on the continuum model for epitaxial thin film growth from King, Stein and Winkler [9]. Let $u(x, t)$ be the height of the film at point x and time t . Then u satisfies the following basic equation

$$\frac{\partial u}{\partial t} = g - \nabla \cdot \vec{j} + \eta, \quad (1.4)$$

where $g = g(x, t)$ denotes the deposition flux, $\vec{j} = \vec{j}(x, t)$ comprises all processes which move atoms along the surface, $\eta = \eta(x, t)$ is some Gaussian noise. The pivotal step in the phenomenological approach is to expand \vec{j} in ∇u and powers thereof keeping only “sensible” terms (see [10]). Then, we have

$$\vec{j} = A_1 \nabla u + A_2 \nabla \Delta u + A_3 |\nabla u|^2 \nabla u + A_4 \nabla |\nabla u|^2, \quad (1.5)$$

where A_1, A_2, A_3 and A_4 are constants. It can be informed from the work of Ortiz [11] that $A_4 = 0$ if Onsager’s reciprocity relations hold. Then, after neglecting the effect of the noise term η and the deposition flux g , the equation (1.4) reads

$$\frac{\partial u}{\partial t} + A_2 \Delta^2 u + A_1 \nabla \cdot \left(\frac{A_3}{A_1} |\nabla u|^2 + 1 \right) \nabla u = 0. \quad (1.6)$$

It can be informed from [11] that the case of $A_1 > 0, A_3 < 0$ is significative and interesting. After relabeling the constants, we have the following fourth-order diffusion equation

$$\frac{\partial u}{\partial t} + \alpha \Delta^2 u - \beta \nabla \cdot (|\nabla u|^2 \nabla u) + \gamma \Delta u = 0,$$

where α, β and γ are positive constants. Moreover, from a mathematical point of view it is more satisfactory to generalize the term involving second-order diffusion, and then we have

$$\frac{\partial u}{\partial t} + \alpha \Delta^2 u - \beta \nabla \cdot \vec{\Phi}(\nabla u) = 0, \quad (1.7)$$

where $\vec{\Phi}$ is a smooth vector with $\vec{\Phi}(0) = 0$. In fact, a lot of references (for example [12–15] etc) show that the diffusion coefficient is usually dependent on the concentration in the models governed by the Cahn-Hilliard equations or Thin-Film equations. Then, the equality (1.5) should be rewritten as

$$\vec{j} = m(u) (A_1 \nabla u + A_2 \nabla \Delta u + A_3 |\nabla u|^2 \nabla u + A_4 \nabla |\nabla u|^2),$$

where $m(s)$ is a positive smooth real value function, denotes the diffusion coefficient or mobility. Substituting the above equality into the basic equation (1.4) and redoing the same deducing procedure as above, we can see that the equation (1.7) can be generalized to the following equation

$$\frac{\partial u}{\partial t} + \operatorname{div} \left[m(u)(K \nabla \Delta u - \vec{\Phi}(\nabla u)) \right] = 0, \quad (1.8)$$

where K is a positive constant.

It was King, Stein and Winkler [9] who first paid their attention to the theoretical investigation of equations like (1.8) with constant mobility, namely the equation (1.1) subject to the boundary value conditions (1.2) and initial value condition (1.3). The typical case considered in [9] is related to the following gradient dependent potential

$$\vec{\Phi}(\nabla u) = |\nabla u|^{p-2} \nabla u - \nabla u,$$

where $p > 2$. In fact, part of their structure conditions are

$$\begin{aligned} \vec{\Phi}(\xi) \cdot \xi &\geq C_1(|\xi|^p - |\xi|^2), \\ \left| \vec{\Phi}(\xi) \right| &\leq C_2(|\xi|^{p-1} + 1), \quad \left| \vec{\Phi}'(\xi) \right| \leq C_3(|\xi|^{p-2} + 1), \end{aligned}$$

under which $\vec{\Phi}(\xi) \cdot \xi$ is restricted to be bounded below and goes to infinity as $|\xi| \rightarrow \infty$, namely

$$\vec{\Phi}(\xi) \cdot \xi \geq C_4, \quad \lim_{|\xi| \rightarrow \infty} \vec{\Phi}(\xi) \cdot \xi = +\infty.$$

Our interest in the current paper is to find a more suitable structure condition for $\vec{\Phi}(\xi)$, under which we are able to discuss the global solvability of weak solutions, while if such a condition is not valid, then the local solution might blow-up in finite time. It will be shown that such a condition is

$$\vec{\Phi}(\xi) \cdot \xi \geq -C|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad (1.9)$$

where C is a positive constant.

We adopt the following definition of weak solutions with more regularity than that formulated in [9].

Definition 1.1 *A function $u(x, t)$ is said to be a weak solution of the problem (1.1)–(1.3), if $u \in L^\infty(0, T; H^2(\Omega)) \cap C([0, T]; H^1(\Omega))$, $\vec{\Phi}(\nabla u) \in L^2(Q_T)$, $\frac{\partial u}{\partial t} \in L^2(Q_T)$, $u(x, 0) = u_0(x)$ in $L^2(\Omega)$, namely,*

$$\lim_{t \rightarrow 0^+} \int_{\Omega} |u(x, t) - u_0(x)|^2 dx = 0,$$

and u satisfies

$$\iint_{Q_T} \frac{\partial u}{\partial t} \varphi dx dt - K \iint_{Q_T} \Delta u \Delta \varphi dx dt + \iint_{Q_T} \vec{\Phi}(\nabla u) \nabla \varphi dx dt = 0 \quad (1.10)$$

for all $\varphi \in E \equiv \{\varphi \in L^2(0, T; H^2(\Omega)); \nabla \varphi \cdot \nu|_{\partial\Omega} = 0\}$.

The existence and uniqueness we obtain is in the following theorem.

Theorem 1.1 *Assume that $\vec{\Phi}$ is Lipschitz continuous and satisfies the structure condition (1.9). If $u_0(x) \in H^2(\Omega)$, $\Psi(\nabla u_0(x)) \in L^1(\Omega)$, where $\nabla_\xi \Psi = \vec{\Phi}(\xi)$, $\forall \xi \in \mathbb{R}^N$, then the problem (1.1)–(1.3) admits a unique weak solution.*

It is worthy noticing that if the condition (1.9) is not valid, then the solutions might blow-up in finite time. In fact, we have

Theorem 1.2 *If $\nabla u_0(x) \not\equiv 0$ and*

$$\vec{\Phi}(\xi) = -\gamma_1 |\xi|^2 \xi + \gamma_2 \xi, \quad \forall \xi \in \mathbb{R}^N,$$

where γ_1 and γ_2 are positive constants. Then there exists a positive constant Γ such that the problem (1.1)–(1.3) admits no global weak solution if $\gamma_1 > \Gamma$.

Finally, we supplement a regularity discussion for weak solutions. Owing to the weakness of the structure condition (1.9), up to now, we can only obtain a fine regularity result for two dimensional case, stated in the following theorem.

Theorem 1.3 *If $N = 2$, $u_0(x) \in C^{4+\alpha}(\overline{\Omega})$, $\vec{\Phi} = (\Phi_1, \Phi_2)$ satisfies the condition (1.9), and $\Phi_1(\xi), \Phi_2(\xi) \in C^{1+\alpha}(\mathbb{R}^2)$, then the solution of the problem (1.1)–(1.3) is classical, namely, $u \in C^{4+\alpha, 1+\alpha/4}(\overline{Q_T})$.*

As far as the regularity of the solutions is considered, a key step is to establish the a priori Schauder type estimates. It is obvious that the Schauder's estimates are certain kind of pointwise estimates and in many cases it is quite difficult to derive pointwise estimates directly from the differential equation considered. However, to derive integral estimates is relatively easy. In fact, the Campanato spaces can be used to describe the integral characteristic of the Hölder continuous functions. Our method in this section is based on the theory of Campanato spaces. To shorten the length of this paper, we omit the definition and properties of the Campanato spaces which can be found in [15–18]. Comparing to the standard Cahn-Hilliard equation, the equation (1.1) owns a gradient dependent potential. Due to the structure condition satisfied by the potential, we can not able to use the interpolation inequality in the proof directly. So, we have to give the L^∞ norm estimate of ∇u which restrict us to get the regularity of the solutions in two spatial dimension.

This paper is organized as follows. In the second section, we investigate the existence and uniqueness of weak solutions of the problem (1.1)–(1.3) by means of the Galerkin approach for any spatial dimension. After establishing some necessary compactness estimates on the approximate solutions and using a compactness result of Sobolev spaces provided by [19], we obtain the existence of weak solutions by passing to the limits on the approximate solutions. In the third section, we prove that if $\vec{\Phi}(\xi)$ does not satisfies the condition (1.9), then the nontrivial solution to the problem (1.1)–(1.3) might blow up in a finite time. In the fourth section of this paper, we investigate the regularity of the solutions in two spatial dimension.

2. Existence and Uniqueness of Weak Solutions

In this section we employ the Galerkin approximation to prove the existence of weak solutions of the problem (1.1)–(1.3). For this purpose, let $\{\phi_i\}_{i \in \mathbb{N}}$ be the eigenfunctions of the Laplace operator with Neumann boundary value conditions, i.e.

$$-\Delta \phi_i = \lambda_i \phi_i, \quad x \in \Omega,$$

$$\nabla \phi_i \cdot \nu = 0, \quad x \in \partial\Omega.$$

The eigenfunctions ϕ_i are orthogonal in the $H^1(\Omega)$ and the $L^2(\Omega)$ scalar product. We normalize ϕ_i such that $(\phi_i, \phi_j)_{L^2(\Omega)} = \delta_{ij}$. Consider the following problem

$$u^k(x, t) = \sum_{i=1}^k c_i^k(t) \phi_i(x), \quad (2.1)$$

$$\int_{\Omega} \partial_t u^k \phi_j dx + K \int_{\Omega} \Delta u^k \Delta \phi_j dx + \int_{\Omega} \vec{\Phi}(\nabla u^k) \nabla \phi_j dx = 0, \quad j = 1, \dots, k, \quad (2.2)$$

$$u^k(x, 0) = \sum_{i=1}^k (u_0, \phi_i)_{L^2(\Omega)} \phi_i. \quad (2.3)$$

This gives an initial value problem for a system of ordinary differential equations for (c_1, \dots, c_k)

$$\frac{d}{dt} c_j^k(t) = -\lambda_j^2 K c_j^k(t) - \int_{\Omega} \vec{\Phi} \left(\sum_{i=1}^k c_i^k(t) \nabla \phi_i(x) \right) \nabla \phi_j dx, \quad (2.4)$$

$$c_j^k(0) = (u_0, \phi_j)_{L^2(\Omega)}. \quad (2.5)$$

Since the right hand side in (2.4) depends continuously on c_1, \dots, c_k , the initial value problem (2.4)–(2.5) admits a local solution.

Lemma 2.1 *The initial value problem (2.1)–(2.3) admits a global solution. Furthermore, there exists a positive constant C which is independent of k , such that*

$$\|u^k\|_{L^\infty(0, T; H^2(\Omega))} \leq C, \quad \left\| \frac{\partial u^k}{\partial t} \right\|_{L^2(Q_T)} \leq C. \quad (2.6)$$

Proof We first discuss the solvability of the problem (2.1)–(2.3). For this purpose, we need some a priori estimates on the possible solutions u^k . We use u^k as a test function to conclude

$$\int_{\Omega} \frac{\partial u^k}{\partial t} u^k dx + K \int_{\Omega} |\Delta u^k|^2 dx + \int_{\Omega} \vec{\Phi}(\nabla u^k) \nabla u^k dx = 0.$$

By the assumption (1.9), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} |u^k|^2 dx + K \int_{\Omega} |\Delta u^k|^2 dx \\ & \leq C \int_{\Omega} |\nabla u^k|^2 dx = -C \int_{\Omega} \Delta u^k \cdot u^k dx \\ & \leq \frac{K}{2} \int_{\Omega} |\Delta u^k|^2 dx + \frac{C^2}{2K} \int_{\Omega} |u^k|^2 dx. \end{aligned}$$

Thus

$$\frac{d}{dt} \int_{\Omega} |u^k|^2 dx + K \int_{\Omega} |\Delta u^k|^2 dx \leq \frac{C_1^2}{K} \int_{\Omega} |u^k|^2 dx.$$

The Gronwall inequality yields

$$\sup_{t \in [0, T]} \int_{\Omega} |u^k(x, t)|^2 dx \leq C, \quad (2.7)$$

$$\iint_{Q_T} |\Delta u^k(x, t)|^2 dx dt \leq C.$$

The inequality (2.7) implies that $c_i^k(t)$ are bounded and therefore a global solution to the initial value problem (2.4)–(2.5) exists. Thus, the initial value problem (2.1)–(2.3) admits a global solution.

Now, we prove the two estimates on the approximate solutions u^k in (2.6). To this end, we define

$$E^k(t) = \frac{K}{2} \int_{\Omega} |\Delta u^k(x, t)|^2 dx + \int_{\Omega} \Psi(\nabla u^k(x, t)) dx,$$

where $\nabla_{\xi} \Psi = \vec{\Phi}(\xi), \forall \xi \in \mathbb{R}^N$. Without loss of generality, we can assume $\Psi(0) = 0$. Then,

$$\begin{aligned} \frac{dE^k(t)}{dt} &= K \int_{\Omega} \Delta u^k \Delta \frac{\partial u^k}{\partial t} dx + \int_{\Omega} \vec{\Phi}(\nabla u^k) \cdot \nabla \frac{\partial u^k}{\partial t} dx \\ &= \int_{\Omega} \left(K \Delta^2 u^k - \operatorname{div} \vec{\Phi}(\nabla u^k) \right) \frac{\partial u^k}{\partial t} dx \\ &= - \int_{\Omega} \left| \frac{\partial u^k}{\partial t} \right|^2 dx \leq 0. \end{aligned}$$

Thus

$$\frac{K}{2} \int_{\Omega} |\Delta u^k(x, t)|^2 dx + \int_{\Omega} \Psi(\nabla u^k(x, t)) dx + \int_0^t \int_{\Omega} \left| \frac{\partial u^k}{\partial t} \right|^2 dx dt = E^k(0), \quad \forall t \in [0, T].$$

Let $\Psi(t\xi) = \psi(t)$. It follows from the assumption (1.9) that

$$\begin{aligned} \Psi(\xi) &= \psi(1) = \psi(1) - \psi(0) \\ &= \int_0^1 \psi'(t) dt = \int_0^1 \vec{\Phi}(t\xi) \cdot \xi dt \\ &\geq -C \int_0^1 t |\xi|^2 dt = -\frac{C}{2} |\xi|^2 \end{aligned}$$

holds for any $\xi \in \mathbb{R}^N$. Then

$$\begin{aligned} &\frac{K}{2} \int_{\Omega} |\Delta u^k(x, t)|^2 dx + \int_0^t \int_{\Omega} \left| \frac{\partial u^k}{\partial t} \right|^2 dx dt \\ &\leq E^k(0) + \frac{C}{2} \int_{\Omega} |\nabla u^k(x, t)|^2 dx \\ &\leq E^k(0) - \frac{C}{2} \int_{\Omega} u^k \Delta u^k dx \\ &\leq E^k(0) + \frac{K}{4} \int_{\Omega} |\Delta u^k(x, t)|^2 dx + \frac{C^2}{4K} \int_{\Omega} |u^k|^2 dx \end{aligned}$$

holds for all $t \in [0, T]$. Noticing the inequality (2.7) and the condition satisfied by the initial value u_0 , we have

$$\frac{K}{4} \int_{\Omega} |\Delta u^k(x, t)|^2 dx + \int_0^t \int_{\Omega} \left| \frac{\partial u^k}{\partial t} \right|^2 dx dt \leq C, \quad \forall t \in [0, T].$$

Thus we have

$$\left\| \frac{\partial u^k}{\partial t} \right\|_{L^2(Q_T)} \leq C,$$

and

$$\sup_{0 < t < T} \int_{\Omega} |\Delta u^k|^2 dx \leq C.$$

Combining the above inequality with (2.7), we have

$$\|u^k\|_{L^\infty(0, T; H^2(\Omega))} \leq C.$$

The proof of this lemma is complete.

The following lemma is used to prove the existence of weak solutions of the problem (1.1)–(1.3). One can find its proof in Simon [19].

Lemma 2.2 *Let X, Y and Z be Banach spaces with a compact embedding $X \hookrightarrow Y$ and a continuous embedding $Y \hookrightarrow Z$. Then the embeddings*

$$\left\{ u \in L^2(0, T; X), \frac{\partial u}{\partial t} \in L^2(0, T; Z) \right\} \hookrightarrow L^2(0, T; Y)$$

and

$$\left\{ u \in L^\infty(0, T; X), \frac{\partial u}{\partial t} \in L^2(0, T; Z) \right\} \hookrightarrow C([0, T]; Y)$$

are compact.

Now we are in a position to prove the existence and uniqueness of weak solutions of the problem (1.1)–(1.3).

Proof of the Theorem 1.1 By Lemma 2.1, we can choose $X = H^2(\Omega)$, $Y = H^1(\Omega)$ and $Z = L^2(\Omega)$ in Lemma 2.2 to obtain a subsequence of ∇u^k which is still denoted by itself and

$$\nabla u^k \rightarrow \nabla u, \quad \text{strongly in } L^2(Q_T) \quad k \rightarrow +\infty.$$

By the assumptions on $\vec{\Phi}$, we have

$$\vec{\Phi}(\nabla u^k) \rightarrow \vec{\Phi}(\nabla u), \quad \text{strongly in } L^2(Q_T), \quad k \rightarrow +\infty.$$

Letting $k \rightarrow +\infty$, by (2.6) and Lemma 2.2 we have

$$\begin{aligned} u^k &\overset{*}{\rightharpoonup} u, & \text{in } L^\infty(0, T; H^2(\Omega)), \\ u^k &\rightarrow u, & \text{in } C([0, T]; H^1(\Omega)) \quad \text{strongly,} \\ \frac{\partial u^k}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t}, & \text{in } L^2(Q_T), \\ \Delta u^k &\rightharpoonup \Delta u, & \text{in } L^2(Q_T), \end{aligned}$$

except for a subsequence. Now we can pass to the limit in (2.2) and (2.3) to see that (1.10) holds. The strong convergence of u^k in $C([0, T]; H^1(\Omega))$ and the fact that $u^k(x, 0) \rightarrow u_0(x)$ in $L^2(\Omega)$ gives $u(x, 0) = u_0(x)$ in $L^2(\Omega)$. Then the problem (1.1)–(1.3) admits a weak solution.

Next, we prove the uniqueness of the weak solution. Suppose the problem (1.1)–(1.3) admits two weak solutions u_1 and u_2 . Set $w = u_1 - u_2$. Then the function w satisfies the following problem

$$\frac{\partial w}{\partial t} + K\Delta^2 w - \operatorname{div} \left(\vec{\Phi}(\nabla u_1) - \vec{\Phi}(\nabla u_2) \right) = 0, \quad (x, t) \in Q_T, \quad (2.8)$$

$$\nabla w \cdot \nu \Big|_{\partial\Omega} = \hat{\mu} \cdot \nu \Big|_{\partial\Omega} = 0, \quad t \in [0, T], \quad (2.9)$$

$$w(x, 0) = 0, \quad x \in \Omega, \quad (2.10)$$

where $\hat{\mu} = K\nabla\Delta w - \left(\vec{\Phi}(\nabla u_1) - \vec{\Phi}(\nabla u_2)\right)$. Multiplying both sides of the equation (2.8) by w and integrating the result over $\Omega \times (0, t)$, we have

$$\int_0^t \int_{\Omega} \frac{\partial w}{\partial t} w dx dt + K \int_0^t \int_{\Omega} |\Delta w|^2 dx dt + \int_0^t \int_{\Omega} \left(\vec{\Phi}(\nabla u_1) - \vec{\Phi}(\nabla u_2)\right) \nabla w dx dt = 0.$$

Since $\vec{\Phi}$ is Lipchitz continuous, we have

$$\begin{aligned} \int_0^t \int_{\Omega} \frac{\partial}{\partial t} w^2 dx dt + 2K \int_0^t \int_{\Omega} |\Delta w|^2 dx dt &\leq C \int_0^t \int_{\Omega} |\nabla w|^2 dx dt \\ &\leq K \int_0^t \int_{\Omega} |\Delta w|^2 dx dt + C \int_0^t \int_{\Omega} w^2 dx dt. \end{aligned}$$

Noticing that $w(x, 0) = 0$, we have

$$\int_{\Omega} w^2(x, t) dx \leq C \int_0^t \int_{\Omega} w^2(x, t) dx dt.$$

It follows from Gronwall's inequality that

$$\int_{\Omega} w^2(x, t) dx = 0, \quad \text{a.e. } [0, T].$$

Thus $w = 0$ a.e. in Q_T . The proof of this Theorem is complete.

3. Non-Existence of Global Solutions

In the previous section, we have shown that the problem (1.1)–(1.3) admits a unique weak solution under the condition (1.9) satisfied by $\vec{\Phi}(\xi)$. In this section, we will prove that if $\vec{\Phi}(\xi)$ does not satisfy the condition (1.9), then the problem (1.1)–(1.3) might not have global weak solutions.

Proof of Theorem 1.2 Suppose to the contrary, namely, the problem (1.1)–(1.3) admits a global weak solution u . Integrating the equation (1.1) with respect to x over Ω and using the boundary value conditions (1.2), we have

$$\int_{\Omega} u(x, t) dx = \int_{\Omega} u_0(x) dx \equiv M, \quad \forall t \geq 0.$$

Without loss of generality, we assume $M = 0$. Otherwise, we may consider the initial boundary value problem satisfied by $v(x, t) = u(x, t) - M$. Then, it follows from the Poincaré inequality that

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}. \quad (3.1)$$

Define

$$E(t) = \frac{K}{2} \int_{\Omega} |\Delta u(x, t)|^2 dx + \int_{\Omega} \Psi(\nabla u(x, t)) dx,$$

where

$$\Psi(\xi) = -\frac{\gamma_1}{4}|\xi|^4 + \frac{\gamma_2}{2}|\xi|^2, \quad \forall \xi \in \mathbb{R}^N.$$

A simple calculation gives that $\frac{dE(t)}{dt} \leq 0$. Thus

$$E(t) = \frac{K}{2} \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} \Psi(\nabla u) dx \leq E(0),$$

and hence

$$K \int_{\Omega} |\Delta u|^2 dx \leq 2E(0) + \frac{\gamma_1}{2} \int_{\Omega} |\nabla u|^4 dx - \gamma_2 \int_{\Omega} |\nabla u|^2 dx. \quad (3.2)$$

Multiplying both sides of the equation (1.1) by u and integrating the result with respect to x over Ω , we obtain from the equality (3.1), (3.2) and the Hölder inequality that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx &= -K \int_{\Omega} |\Delta u|^2 dx - \int_{\Omega} \vec{\Phi}(\nabla u) \nabla u dx \\ &= -K \int_{\Omega} |\Delta u|^2 dx + \gamma_1 \int_{\Omega} |\nabla u|^4 dx - \gamma_2 \int_{\Omega} |\nabla u|^2 dx \\ &\geq \frac{\gamma_1}{2} \int_{\Omega} |\nabla u|^4 dx - 2E(0) \geq \frac{\gamma_1}{2|\Omega|} \left(\int_{\Omega} |\nabla u|^2 dx \right)^2 - 2E(0) \\ &\geq \frac{\gamma_1}{2|\Omega|C^4} \left(\int_{\Omega} u^2 dx \right)^2 - 2E(0), \end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of the domain Ω , and C is the constant in the Poincaré inequality (3.1). Noticing that

$$E(0) = \frac{K}{2} \int_{\Omega} |\Delta u_0(x)|^2 dx - \frac{\gamma_1}{4} \int_{\Omega} |\nabla u_0(x)|^4 dx + \frac{\gamma_2}{2} \int_{\Omega} |\nabla u_0(x)|^2 dx,$$

and $\nabla u_0(x) \not\equiv 0$, we know that there exists a positive constant Γ , such that $E(0) \leq 0$ if $\gamma_1 > \Gamma$. So

$$\frac{d}{dt} \int_{\Omega} u^2 dx \geq \frac{\gamma_1}{|\Omega|C^4} \left(\int_{\Omega} u^2 dx \right)^2.$$

Then

$$\int_{\Omega} |u(x, t)|^2 dx \geq \frac{\int_{\Omega} |u_0(x)|^2 dx}{1 - \frac{\gamma_1 t}{|\Omega|C^4} \int_{\Omega} |u_0(x)|^2 dx},$$

which contradicts the global existence of the solution u . The proof is complete.

4. Regularity of Weak Solutions

In this section we discuss the regularity of weak solutions of the problem (1.1)–(1.3). The key point of the proof is the Schauder type estimates on the solutions, and we will adopt the theory of Campanato spaces combined with the energy estimates to obtain the desired a priori estimates.

Firstly, we prove the boundedness of the L^∞ norm of solutions.

Lemma 4.1 *The solution u of the problem (1.1)–(1.3) satisfies*

$$\|u\|_{L^\infty(Q_T)} \leq C, \quad \|\nabla u\|_{L^\infty(Q_T)} \leq C, \quad (4.1)$$

where C is a positive constant.

Proof Multiplying both sides of the equation (1.1) by u and integrating the resulting relation with respect to x over Ω , we have

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u^2 dx + 2K \int_{\Omega} |\Delta u|^2 dx &\leq C \int_{\Omega} |\nabla u|^2 dx \\ &\leq K \int_{\Omega} |\Delta u|^2 dx + C \int_{\Omega} u^2 dx, \end{aligned}$$

namely,

$$\frac{d}{dt} \int_{\Omega} u^2 dx + K \int_{\Omega} |\Delta u|^2 dx \leq C \int_{\Omega} u^2 dx.$$

It follows from the Gronwall inequality that

$$\sup_{0 < t < T} \int_{\Omega} u^2 dx \leq C. \quad (4.2)$$

Define

$$E(t) = \frac{K}{2} \int_{\Omega} |\Delta u(x, t)|^2 dx + \int_{\Omega} \Psi(\nabla u(x, t)) dx,$$

where

$$\nabla_{\xi} \Psi = \vec{\Phi}(\xi), \quad \forall \xi \in \mathbb{R}^2.$$

Without loss of generality, we assume that $\Psi(0) = 0$. Then, a simple calculation similar to that in the proof of Lemma 2.1 gives that

$$E(t) = \frac{K}{2} \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} \Psi(\nabla u) dx \leq E(0).$$

Furthermore, we have

$$\sup_{0 < t < T} \int_{\Omega} |\Delta u(x, t)|^2 dx \leq C,$$

which, together with the inequality (4.2), gives that

$$\|u(x, t)\|_{L^\infty(0, T; H^2(\Omega))} \leq C.$$

By the Sobolev embedding theorem, we have

$$\|u\|_{L^\infty(Q_T)} \leq C.$$

It follows from the Nirenberg inequality that

$$\|\nabla u\|_{L^\infty(Q_T)} \leq C_1 \|u\|_{L^\infty(0, T; H^2(\Omega))} + C_2 \|u\|_{L^\infty(Q_T)} \leq C.$$

The proof of this lemma is complete.

Now, we are in a position to establish the estimate on the Hölder norm of ∇u . Let $(x^0, t_0) \in \Omega \times (0, T)$ be fixed and define

$$\varphi(u, \rho) = \iint_{S_\rho} (|\nabla u - (\nabla u)_\rho|^2 + \rho^4 |\nabla \Delta u|^2) dx dt,$$

where

$$S_\rho = (t_0 - \rho^4, t_0 + \rho^4) \times B_\rho(x^0), \quad (\nabla u)_\rho = \frac{1}{|S_\rho|} \iint_{S_\rho} \nabla u dx dt$$

and $B_\rho(x^0)$ is the ball centers at x^0 with radius ρ .

We split the solution u of the problem (1.1)–(1.3) on S_R as $u = u_1 + u_2$, where u_1 is the solution of the problem

$$\frac{\partial u_1}{\partial t} + \Delta^2 u_1 = 0, \quad (x, t) \in S_R, \quad (4.3)$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u}{\partial \nu}, \quad \frac{\partial \Delta u_1}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu}, \quad (x, t) \in (t_0 - R^4, t_0 + R^4) \times \partial B_R(x^0), \quad (4.4)$$

$$u_1 = u, \quad t = t_0 - R^4, \quad x \in B_R(x^0), \quad (4.5)$$

and u_2 solves the problem

$$\frac{\partial u_2}{\partial t} + \Delta^2 u_2 = \operatorname{div} \vec{\Phi}(\nabla u), \quad (x, t) \in S_R, \quad (4.6)$$

$$\frac{\partial u_2}{\partial \nu} = 0, \quad \frac{\partial \Delta u_2}{\partial \nu} = 0, \quad (x, t) \in (t_0 - R^4, t_0 + R^4) \times \partial B_R(x^0), \quad (4.7)$$

$$u_2 = 0, \quad t = t_0 - R^4, \quad x \in B_R(x^0). \quad (4.8)$$

By the classical linear theory, the above decomposition is uniquely determined by u .

Lemma 4.2

$$\sup_{(t_0 - R^4, t_0 + R^4)} \int_{B_R(x^0)} |\nabla u_2(x, t)|^2 dx + \iint_{S_R} |\nabla \Delta u_2|^2 dx dt \leq C \sup_{S_R} \left| \vec{\Phi}(\nabla u) \right|^2 R^6.$$

Proof Multiplying both sides of the equation (4.6) by Δu_2 and integrating the result relation over $(t_0 - R^4, t) \times B_R(x^0)$, we have

$$\begin{aligned} & \frac{1}{2} \int_{B_R(x^0)} |\nabla u_2(x, t)|^2 dx + \int_{t_0 - R^4}^t \int_{B_R(x^0)} |\nabla \Delta u_2|^2 dx dt \\ &= \int_{t_0 - R^4}^t \int_{B_R(x^0)} \vec{\Phi}(\nabla u) \nabla \Delta u_2 dx dt \\ &\leq \frac{1}{2} \int_{t_0 - R^4}^t \int_{B_R(x^0)} \left| \vec{\Phi}(\nabla u) \right|^2 dx dt + \frac{1}{2} \int_{t_0 - R^4}^t \int_{B_R(x^0)} |\nabla \Delta u_2|^2 dx dt \end{aligned}$$

$$\leq C \sup_{S_R} \left| \vec{\Phi}(\nabla u) \right|^2 R^6 + \frac{1}{2} \int_{t_0-R^4}^t \int_{B_R(x^0)} |\nabla \Delta u_2|^2 dx dt,$$

which implies that

$$\sup_{(t_0-R^4, t_0+R^4)} \int_{B_R(x^0)} |\nabla u_2(x, t)|^2 dx + \iint_{S_R} |\nabla \Delta u_2|^2 dx dt \leq C \sup_{S_R} \left| \vec{\Phi}(\nabla u) \right|^2 R^6.$$

The proof of this lemma is complete.

Lemma 4.3

$$\begin{aligned} & \frac{|\nabla u_1(x^1, t_1) - \nabla u_1(x^2, t_2)|^2}{|x^1 - x^2| + |t_1 - t_2|^{1/4}} \\ & \leq C \sup_{(t_0-\rho^4, t_0+\rho^4)} \int_{B_\rho(x^0)} \left(\rho^{-3} |\nabla u_1 - (\nabla u_1)_\rho|^2 + \rho |\nabla \Delta u_1|^2 \right) dx \\ & \quad + C \iint_{S_\rho} \left(\rho^{-3} |\nabla \Delta u_1|^2 + \rho |\nabla \Delta^2 u_1| \right) dx dt. \end{aligned}$$

Proof From the Sobolev embedding theorem, we get, for any $(x^1, t_1), (x^2, t_2) \in S_\rho$, that

$$\begin{aligned} & \frac{|\nabla u_1(x^1, t_1) - \nabla u_1(x^2, t_2)|^2}{|x^1 - x^2| + |t_1 - t_2|^{1/4}} \\ & \leq C \sup_{(t_0-\rho^4, t_0+\rho^4)} \int_{B_\rho(x^0)} \left(\rho^{-3} |\nabla u_1 - (\nabla u_1)_\rho|^2 + \rho |\nabla \Delta u_1|^2 \right) dx. \end{aligned}$$

Then, by using the equation (4.3) itself we can obtain the desired estimate at once. The proof of this lemma is complete.

Lemma 4.4 (Caccioppelli-type inequality)

$$\begin{aligned} & \sup_{(t_0-(R/2)^4, t_0+(R/2)^4)} \int_{B_{R/2}(x^0)} |\nabla u_1 - (\nabla u_1)_R|^2 dx + \iint_{S_{R/2}} |\nabla \Delta u_1|^2 dx dt \\ & \leq \frac{C}{R^4} \iint_{S_R} |\nabla u_1 - (\nabla u_1)_R|^2 dx dt; \\ & \sup_{(t_0-(R/2)^4, t_0+(R/2)^4)} \int_{B_{R/2}(x^0)} |\Delta u_1|^2 dx + \iint_{S_{R/2}} |\Delta^2 u_1|^2 dx dt \\ & \leq \frac{C}{R^4} \iint_{S_R} |\Delta u_1|^2 dx dt \leq \frac{C}{R^6} \iint_{S_{2R}} |\nabla u_1 - (\nabla u_1)_R|^2 dx dt; \\ & \sup_{(t_0-(R/2)^4, t_0+(R/2)^4)} \int_{B_{R/2}(x^0)} |\nabla \Delta u_1|^2 dx + \iint_{S_{R/2}} |\nabla \Delta^2 u_1|^2 dx dt \\ & \leq \frac{C}{R^4} \iint_{S_R} |\nabla \Delta u_1|^2 dx dt. \end{aligned}$$

Proof As an example, we only prove the first inequality, since the other two can be shown similarly. Choose a cut-off function $\chi(x)$ defined on $B_R(x^0)$ such that $\chi(x) = 1$ in $B_{R/2}(x^0)$ and

$$|\nabla\chi| \leq \frac{C}{R}, \quad |\Delta\chi| \leq \frac{C}{R^2}, \quad |\nabla\Delta\chi| \leq \frac{C}{R^3}, \quad |\Delta^2\chi| \leq \frac{C}{R^4}.$$

Let $g(t) \in C_0^\infty(\mathbb{R})$ with $0 \leq g(t) \leq 1, 0 \leq g'(t) \leq \frac{C}{R^4}$ for all $t \in \mathbb{R}$, $g(t) = 1$ for $t \geq t_0 - (R/2)^4$ and $g(t) = 0$ for $t \leq t_0 - R^4$. Multiplying both sides of the equation (4.3) by $g(t)\nabla \cdot [\chi^4(\nabla u_1 - (\nabla u_1)_R)]$ and integrating the resulting relation over $(t_0 - R^4, t) \times B_R(x^0)$, we have

$$\begin{aligned} & \int_{t_0-R^4}^t g(t)dt \int_{B_R(x^0)} \frac{\partial u_1}{\partial t} \nabla \cdot [\chi^4(\nabla u_1 - (\nabla u_1)_R)] dxdt \\ & + \int_{t_0-R^4}^t g(t)dt \int_{B_R(x^0)} \Delta^2 u_1 \nabla \cdot [\chi^4(\nabla u_1 - (\nabla u_1)_R)] dxdt = 0. \end{aligned} \quad (4.9)$$

The first term of the left hand side in the above equality can be written to

$$\begin{aligned} & \int_{t_0-R^4}^t g(t)dt \int_{B_R(x^0)} \frac{\partial u_1}{\partial t} \nabla \cdot [\chi^4(\nabla u_1 - (\nabla u_1)_R)] dxdt \\ & = - \int_{t_0-R^4}^t g(t)dt \int_{B_R(x^0)} \frac{\partial \nabla u_1}{\partial t} \chi^4(\nabla u_1 - (\nabla u_1)_R) dxdt \\ & = - \frac{1}{2} \int_{t_0-R^4}^t g(t)dt \int_{B_R(x^0)} \chi^4 \frac{\partial}{\partial t} |\nabla u_1 - (\nabla u_1)_R|^2 dxdt \\ & = - \frac{1}{2} \int_{t_0-R^4}^t \frac{d}{dt} \int_{B_R(x^0)} g(t) \chi^4 |\nabla u_1 - (\nabla u_1)_R|^2 dxdt \\ & \quad + \frac{1}{2} \int_{t_0-R^4}^t \int_{B_R(x^0)} g'(t) \chi^4 |\nabla u_1 - (\nabla u_1)_R|^2 dxdt \\ & = - \frac{1}{2} \int_{B_R(x^0)} g(t) \chi^4 |\nabla u_1 - (\nabla u_1)_R|^2 dx \\ & \quad + \frac{1}{2} \int_{t_0-R^4}^t \int_{B_R(x^0)} g'(t) \chi^4 |\nabla u_1 - (\nabla u_1)_R|^2 dxdt. \end{aligned}$$

For the second term of (4.9), we just notice that

$$\begin{aligned} & \int_{B_R(x^0)} \Delta^2 u_1 \nabla \cdot [\chi^4(\nabla u_1 - (\nabla u_1)_R)] dx \\ & = - \int_{B_R(x^0)} \nabla \Delta u_1 \Delta [\chi^4(\nabla u_1 - (\nabla u_1)_R)] dx \\ & = - \int_{B_R(x^0)} \chi^4 |\nabla \Delta u_1|^4 dx - 2 \int_{B_R(x^0)} \nabla \chi^4 \nabla \Delta u_1 \Delta u_1 dx \end{aligned}$$

$$\begin{aligned}
& - \int_{B_R(x^0)} \nabla \Delta u_1 (\nabla u_1 - (\nabla u_1)_R) \Delta \chi^4 dx \\
& \equiv -I_1 - I_2 - I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_2 &= 2 \int_{B_R(x^0)} \nabla \chi^4 \nabla \Delta u_1 \Delta u_1 dx = 8 \int_{B_R(x^0)} \chi \nabla \chi \nabla \Delta u_1 \Delta u_1 dx \\
&\geq -\frac{1}{8} \int_{B_R(x^0)} \chi^4 |\nabla \Delta u_1|^2 dx - 128 \int_{B_R(x^0)} |\chi \nabla \chi|^2 |\Delta u_1|^2 dx \\
&\equiv -\frac{1}{8} I_1 + I_4,
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= -128 \int_{B_R(x^0)} |\chi \nabla \chi|^2 |\Delta u_1|^2 dx \\
&\geq -\frac{C}{R^2} \int_{B_R(x^0)} \chi^2 |\Delta u_1|^2 dx \\
&= -\frac{C}{R^2} \int_{B_R(x^0)} \chi^2 \Delta u_1 \nabla \cdot (\nabla u_1 - (\nabla u_1)_R) dx \\
&= \frac{C}{R^2} \int_{B_R(x^0)} (\nabla u_1 - (\nabla u_1)_R) \nabla (\chi^2 \Delta u_1) dx \\
&= \frac{C}{R^2} \int_{B_R(x^0)} (\nabla u_1 - (\nabla u_1)_R) \chi^2 \nabla \Delta u_1 dx \\
&\quad + \frac{C}{R^2} \int_{B_R(x^0)} \chi \nabla \chi \Delta u_1 (\nabla u_1 - (\nabla u_1)_R) dx \\
&\geq -\frac{1}{16} \int_{B_R(x^0)} \chi^4 |\nabla \Delta u_1|^2 dx - \frac{C}{R^4} \int_{B_R(x^0)} |\nabla u_1 - (\nabla u_1)_R|^2 dx \\
&\quad - 64 \int_{B_R(x^0)} |\chi \nabla \chi|^2 |\Delta u_1|^2 dx.
\end{aligned}$$

Then we have

$$I_4 \geq -\frac{1}{8} I_1 - \frac{C}{R^4} \int_{B_R(x^0)} |\nabla u_1 - (\nabla u_1)_R|^2 dx.$$

Thus

$$\begin{aligned}
I_2 &\geq -\frac{1}{4} I_1 - \frac{C}{R^4} \int_{B_R(x^0)} |\nabla u_1 - (\nabla u_1)_R|^2 dx. \\
I_3 &= \int_{B_R(x^0)} \nabla \Delta u_1 (\nabla u_1 - (\nabla u_1)_R) \Delta \chi^4 dx \\
&= \int_{B_R(x^0)} \nabla \Delta u_1 (\nabla u_1 - (\nabla u_1)_R) (4\chi^3 \Delta \chi + 12\chi^2 |\nabla \chi|^2) dx
\end{aligned}$$

$$\begin{aligned}
&\geq -\frac{1}{4} \int_{B_R(x^0)} \chi^4 |\nabla \Delta u_1|^2 dx - 32 \int_{B_R(x^0)} |\chi \Delta \chi|^2 |\nabla u_1 - (\nabla u_1)_R|^2 dx \\
&\quad - 288 \int_{B_R(x^0)} |\nabla \chi|^4 |\nabla u_1 - (\nabla u_1)_R|^2 dx \\
&\geq -\frac{1}{4} I_1 - \frac{C}{R^4} \int_{B_R(x^0)} |\nabla u_1 - (\nabla u_1)_R|^2 dx
\end{aligned}$$

and hence

$$I_1 + I_2 + I_3 \geq \frac{1}{2} I_1 - \frac{C}{R^4} \int_{B_R(x^0)} |\nabla u_1 - (\nabla u_1)_R|^2 dx.$$

Then we can obtain the following estimate on the second term of (4.9)

$$\begin{aligned}
&\int_{t_0-R^4}^t g(t) dt \int_{B_R(x^0)} \Delta^2 u_1 \nabla \cdot [\chi^4 (\nabla u_1 - (\nabla u_1)_R)] dx \\
&\quad = - \int_{t_0-R^4}^t g(t) (I_1 + I_2 + I_3) dt \\
&\quad \leq -\frac{1}{2} \int_{t_0-R^4}^t g(t) dt \int_{B_R(x^0)} \chi^4 |\nabla \Delta u_1|^2 dx \\
&\quad \quad + \frac{C}{R^4} \int_{t_0-R^4}^t g(t) dt \int_{B_R(x^0)} |\nabla u_1 - (\nabla u_1)_R|^2 dx,
\end{aligned}$$

which, together with the estimate on the first term of (4.9), implies that

$$\begin{aligned}
&\frac{1}{2} \int_{B_R(x^0)} g(t) \chi^4 |\nabla u_1 - (\nabla u_1)_R|^2 dx + \frac{1}{2} \int_{t_0-R^4}^t g(t) dt \int_{B_R(x^0)} \chi^4 |\nabla \Delta u_1|^2 dx \\
&\quad \leq \frac{1}{2} \int_{t_0-R^4}^t \int_{B_R(x^0)} g'(t) \chi^4 |\nabla u_1 - (\nabla u_1)_R|^2 dx dt \\
&\quad \quad + \frac{C}{R^4} \int_{t_0-R^4}^t g(t) dt \int_{B_R(x^0)} |\nabla u_1 - (\nabla u_1)_R|^2 dx \\
&\quad \leq \frac{C}{R^4} \iint_{S_R} |\nabla u_1 - (\nabla u_1)_R|^2 dx dt.
\end{aligned}$$

By the definition of $g(t)$ and χ we immediately obtain the desired first inequality of this lemma, and then complete the proof.

Lemma 4.5 For any $\rho \in (0, R)$,

$$\varphi(u_1, \rho) \leq C \left(\frac{\rho}{R}\right)^7 \varphi(u_1, R).$$

Proof It is sufficient to show the inequality for $\rho \leq R/2$. By the mean value theorem, there exists a point $(x^*, t_*) \in S_\rho$ such that

$$(\nabla u_1)_\rho = \nabla u_1(x^*, t_*).$$

Then, by Lemma 4.3 and Lemma 4.4, one has

$$\begin{aligned}
& \iint_{S_\rho} |\nabla u_1 - (\nabla u_1)_\rho|^2 dx dt \\
&= \iint_{S_\rho} |\nabla u_1 - \nabla u_1(x^*, t_*)|^2 dx dt \\
&\leq C\rho^6 \sup_{(x,t) \in S_\rho} |\nabla u_1 - \nabla u_1(x^*, t_*)|^2 \\
&\leq C\rho^7 \sup_{t \in (t_0 - (R/2)^4, t_0 + (R/2)^4)} \int_{B_{R/2}(x^0)} (R^{-3} |\nabla u_1 - (\nabla u_1)_R|^2 + R |\nabla \Delta u_1|^2) dx \\
&\quad + C\rho^7 \iint_{S_{R/2}} (R^{-3} |\nabla \Delta u_1|^2 + R |\nabla \Delta^2 u_1|^2) dx dt \\
&\leq C \left(\frac{\rho}{R}\right)^7 \iint_{S_R} (|\nabla u_1 - (\nabla u_1)_R|^2 + R^4 |\nabla \Delta u_1|^2) dx dt,
\end{aligned}$$

and

$$\begin{aligned}
\iint_{S_\rho} \rho^4 |\nabla \Delta u_1|^2 dx dt &\leq C\rho^8 \sup_{t \in (t_0 - \rho^4, t_0 + \rho^4)} \int_{B_\rho(x^0)} |\nabla \Delta u_1|^2 dx \\
&\leq C\rho^7 R \sup_{t \in (t_0 - (R/2)^4, t_0 + (R/2)^4)} \int_{B_{R/2}(x^0)} |\nabla \Delta u_1|^2 dx \\
&\leq C \left(\frac{\rho}{R}\right)^7 \iint_{S_R} R^4 |\nabla \Delta u_1|^2 dx dt.
\end{aligned}$$

The proof of this lemma is complete.

The following technical lemma is required to estimate the Höler norm of ∇u . One can find its proof in Giaquinta [20].

Lemma 4.6 *Let $\varphi(\rho)$ be a nonnegative and nondecreasing function satisfying*

$$\varphi(\rho) \leq A \left(\frac{\rho}{R}\right)^\alpha \varphi(R) + BR^\beta, \quad \forall 0 < \rho \leq R \leq R_0,$$

where A, B, α, β are positive constants with $\beta < \alpha$. Then there exists a positive constant C depending only on α, β and A , such that

$$\varphi(\rho) \leq C \left(\frac{\rho}{R}\right)^\beta \left[\varphi(R) + BR^\beta\right], \quad \forall 0 < \rho \leq R \leq R_0.$$

Lemma 4.7 *For $\lambda \in (6, 7)$,*

$$\varphi(u, \rho) \leq C \left(1 + \sup_{S_R} \left| \vec{\Phi}(\nabla u) \right|^2\right) \rho^\lambda, \quad \forall 0 < \rho \leq R \leq R_0,$$

where $R_0 \triangleq \min \left\{ \text{dist}(x^0, \partial\Omega), t_0^{1/4} \right\}$.

Proof A simple calculation gives that

$$(\nabla u)_\rho = (\nabla u_1)_\rho + (\nabla u_2)_\rho,$$

and

$$\iint_{S_\rho} |\nabla u - (\nabla u)_\rho|^2 dx dt \leq \iint_{S_\rho} |\nabla u|^2 dx dt.$$

Then, by Cauchy's inequality and using Lemmas 4.2 and Lemma 4.5, we have

$$\begin{aligned} \varphi(u, \rho) &\leq 2\varphi(u_1, \rho) + 2\varphi(u_2, \rho) \\ &\leq C \left(\frac{\rho}{R}\right)^7 \varphi(u_1, R) + 2\varphi(u_2, R) \\ &\leq C \left(\frac{\rho}{R}\right)^7 \varphi(u, R) + 2 \iint_{S_R} (|\nabla u_2|^2 + R^4 |\nabla \Delta u_2|^2) dx dt \\ &\leq C \left(\frac{\rho}{R}\right)^7 \varphi(u, R) + C \sup_{S_R} \left| \vec{\Phi}(\nabla u) \right|^2 R^{10} \\ &\leq C \left(\frac{\rho}{R}\right)^7 \varphi(u, R) + C \sup_{S_R} \left| \vec{\Phi}(\nabla u) \right|^2 R^\lambda, \end{aligned}$$

where $6 < \lambda < 7$ is a constant. Then, by Lemma 4.6, one can complete the proof of this lemma immediately.

Now we give the proof of Theorem 1.3 as follows.

Proof of the Theorem 1.3 From the integral characteristic of the Hölder continuous functions (see [15, 17, 20] for details) and the Lemma 4.7, one has

$$\frac{|\nabla u(x^1, t_1) - \nabla u(x^2, t_2)|}{|x^1 - x^2|^{(\lambda-6)/2} + |t_1 - t_2|^{(\lambda-6)/8}} \leq C \left(1 + \sup_{S_R} \left| \vec{\Phi}(\nabla u) \right| \right).$$

By Lemma 4.1, we have

$$|\nabla u(x^1, t_1) - \nabla u(x^2, t_2)| \leq C \left(|x^1 - x^2|^{(\lambda-6)/2} + |t_1 - t_2|^{(\lambda-6)/8} \right).$$

Noticing that one can rewrite the equation (1.1) to the following form

$$\frac{\partial u}{\partial t} + K \Delta^2 u - B_1(x, t) \nabla u_{x_1} - B_2(x, t) \nabla u_{x_2} = 0,$$

where $B_i(x, t) = \nabla_\xi \Phi_i(\nabla u)$, $i = 1, 2$. By the Hölder continuity of ∇u and $\vec{\Phi}$, one can see that $B_i(x, t)$ is Hölder continuous. Using the classical Schauder theory for parabolic equations, one can conclude that the solution of the problem (1.1)–(1.3) is classical in the interior points of Q_T .

For the regularity of the solution near the boundary of Q_T , we can deal with it in the same way. Let $(x^0, t_0) \in \partial\Omega \times (0, T)$ be fixed and assume that $\partial\Omega$ can be explicitly expressed by a function $y = \phi(x)$ in some neighborhood of x^0 . We split u as $u_1 + u_2$ in

$\hat{S}_R = (t_0 - R^4, t_0 + R^4) \times \Omega_R(x_0)$ with $\Omega_R(x_0) = B_R(x^0) \cap \Omega$. u_1 solves the following problem

$$\begin{aligned} \frac{\partial u_1}{\partial t} + \Delta^2 u_1 &= 0, \quad (x, t) \in \hat{S}_R, \\ \frac{\partial u_1}{\partial \nu} &= \frac{\partial u}{\partial \nu}, \quad \frac{\partial \Delta u_1}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu}, \quad (x, t) \in (t_0 - R^4, t_0 + R^4) \times \partial \Omega_R(x^0), \\ u_1 &= u, \quad t = t_0 - R^4, \quad x \in \Omega_R(x^0), \end{aligned}$$

and u_2 solves the problem

$$\begin{aligned} \frac{\partial u_2}{\partial t} + \Delta^2 u_2 &= \operatorname{div} \vec{\Phi}(\nabla u), \quad (x, t) \in \hat{S}_R, \\ \frac{\partial u_2}{\partial \nu} &= 0, \quad \frac{\partial \Delta u_2}{\partial \nu} = 0, \quad (x, t) \in (t_0 - R^4, t_0 + R^4) \times \partial \Omega_R(x^0), \\ u_2 &= 0, \quad t = t_0 - R^4, \quad x \in \Omega_R(x^0). \end{aligned}$$

We can modify the function $\varphi(u, \rho)$ as

$$\varphi(u, \rho) = \iint_{S_\rho} (|\partial_n u|^2 + |\partial_\tau u - (\partial_\tau u)_\rho|^2 + \rho^4 |\nabla \Delta u|^2) \, dx dt,$$

where

$$\partial_n = \phi'(x) \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \quad \partial_\tau = \frac{\partial}{\partial x_1} + \phi'(x) \frac{\partial}{\partial x_2}$$

denote the normal and tangential derivatives respectively. The remaining part of the proof is similar to that in the proof of the previous lemmas, and we omit the details. The proof of the theorem is complete.

References

- [1] Myers T. Thin films with high surface tension[J]. *SIAM Review*, 1998, **40**: 441–462.
- [2] Ansini L, Giacomelli L. Shear-thinning liquid films: macroscopic and asymptotic behaviour by quasi-self-similar solutions[J]. *Nonlinearity*, 2002, **15**: 2147–2164.
- [3] Ansini L, Giacomelli L. Doubly nonlinear Thin-Film equations in one space dimension[J]. *Arch Ration Mech Anal*, 2004, **173**(1): 89–131.
- [4] Bertozzi A. Symmetric singularity formation in lubrication-type equations for interface motion[J]. *SIAM J Appl Math*, 1996, **56**: 681–714.
- [5] Jin C, Yin J, Yin L. Existence and blow-up of solutions of a fourth order nonlinear diffusion equation[J]. *Nonlinear Analysis: Real World Applications*, in press.
- [6] King J. Two generalisations of the thin film equation[J]. *Math Comput Modelling*, 2001, **34**(7-8): 737–756.
- [7] Betelú S, King J. Explicit solutions of a two-dimensional fourth-order nonlinear diffusion equation[J]. *Mathematical and Computer Modelling*, 2003, **37**: 395–403.
- [8] Xu M, Zhou S. Existence and uniqueness of weak solutions for a generalized thin film equation[J]. *Nonlinear Analysis*, 2005, **60**: 755–774.

-
- [9] King B, Stein O, Winkler M. A fourth-order parabolic equation modeling epitaxial thin film growth[J]. *J Math Anal Appl*, 2003, **286**: 459–490.
- [10] Zangwill A. Some causes and a consequence of epitaxial roughening[J]. *J Crystal Growth*, 1996, **163**: 8–21.
- [11] Ortiz M, Repetto E, Si H. A continuum model of kinetic roughening and coarsening in thin films[J]. *J Mech Phys Solids*, 1999, **47**: 697–730.
- [12] Elliott C, Garcke H. On the Cahn-Hilliard equation with degenerate mobility[J]. *SIAM J Math Anal*, 1996, **27**(2): 404–423.
- [13] Gurtin M. Generalized Ginzburg–Landau and Cahn-Hilliard equations based on a micro-force balance[J]. *Physica D*, 1996, **92**: 178–192.
- [14] Yin J. On the existence of nonnegative continuous solutions of the Cahn-Hilliard equation[J]. *J Diff Eqs*, 1992, **97**(2): 310–327.
- [15] Yin J. On the Cahn-Hilliard equation with nonlinear principal part[J]. *Journal of Partial Differential Equations*, 1994, **7**: 77–96.
- [16] Campanato S. Propriets di Holderianita di alcuna classi di funzioni[J]. *Ann Sc Norm Sup Pisa*, 1963, **17**: 175–188.
- [17] Wu Z, Yin J, Wang C. *Elliptic & Parabolic Equations*[M]. Singapore: World Scientific Publishing Co Pte Ltd, 2006.
- [18] Yin J, Liu C. Regularity of solutions of the Cahn-Hilliard equation with concentration dependent mobility[J]. *Nonlinear Analysis*, 2001, **45**: 543–554.
- [19] Simon J. Compact sets in the space $L^p(0, T; B)$ [J]. *Annali di Mathematica pura ed applicata*, 1987, **146**: 65–96.
- [20] Giaquinta M. *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*[M]. Princeton, New Jersey: Princeton Univ Press, 1983.