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DISSIPATION AND DISPERSION APPROXIMATION TO HYDRODYNAMICAL EQUATIONS AND ASYMPTOTIC LIMIT*

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Dedicated to Professor Li Tatsien on the occasion of his seventieth birthday

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Abstract The compressible Euler equations with dissipation and/or dispersion correction are widely used in the area of applied sciences, for instance, plasma physics, charge transport in semiconductor devices, astrophysics, geophysics, etc. We consider the compressible Euler equation with density-dependent (degenerate) viscosities and capillarity, and investigate the global existence of weak solutions and asymptotic limit.

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1. Introduction

In the area of applied sciences, like plasma physics, transport of charged particles, astrophysics, geophysics, etc, the compressible Euler equations with additional dissipation of the form

$$\partial_t n + \nabla \cdot (nu) = 0, \tag{1.1}$$

$$\partial_t (nu) + \nabla \cdot (nu \otimes u) + \nabla p(n) = \rho F + f_{dis}, \tag{1.2}$$

are often used to simulate the dynamical behaviors of physical observable like the density $n > 0$, velocity u, momentum $J = nu$, and energy $e = e(n, u)$. Here, the Eq. (1.1) and (1.2) respectively express the conservation of mass and the balance of momentum. The force F is taken as the gradient filed of some potential $F = -\nabla\Phi$,

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where Φ represents either electrostatic potential or gravity, and can be determined by the self-consistent Poisson equation

$$\lambda\Delta\Phi = n \tag{1.3}$$

with $\lambda = \mp 1$. The term f_{dis} in (1.2) is chosen based on the different effects caused by the specific physical (dissipative or dispersive) mechanism, like drag friction (lubrication) $-n|u|u$ in the motion of shallow water [1], dispersion effects $\frac{\varepsilon^2}{2}\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})$ with $\varepsilon > 0$ the scaled Planck constant in quantum mechanics [2], Korteweg term $\varepsilon n\nabla\Delta n$ with $\varepsilon > 0$ small parameter in phase transition [1], viscosity $\mu\Delta u$ or $\mu\nabla(n\nabla u)$ with μ viscosity coefficient in fluid-dynamics [3, 4], and so on.

The aim of this paper is to study the dissipative and dispersive approximation to the hydrodynamical system (1.1)–(1.2) as follows

$$\partial_t n + \mathrm{div}(nu) = 0, \tag{1.4}$$

$$\begin{aligned}&\partial_t(nu) + \mathrm{div}(nu\otimes u) + \nabla p(n) + n\nabla\Phi\\&\quad = \varepsilon^2 n\nabla(\varphi'(n)\Delta\varphi(n)) + 2\eta\mathrm{div}\,(\mu(n)D(u)) + \eta\nabla(\lambda(n)\mathrm{div}u) - rn|u|u,\end{aligned} \tag{1.5}$$

where the right hand side terms in (1.5) consist of viscosity, dispersion and nonlinear friction, corresponding to the term f_{dis} in (1.2), and $D(u) = (\nabla u + {}^t\nabla u)/2$ is the stress tensor with degenerate viscosities $\mu(n) \geq 0$, $\lambda(n)$, and $\eta > 0$ a small parameter, which is zero in the appearance of vacuum $n = 0$. The nonlinear dispersion term is also taken into accounted with $\varphi(n) \geq 0$ and $\varepsilon > 0$ a small parameter, and the nonlinear term $-rn|u|u$ represents a drag friction with $r > 0$ a constant. The internal electrostatic potential Φ is chosen through the self-consistent Poisson equation

$$-\Delta\Phi = n - 1. \tag{1.6}$$

We consider the initial value problem of the approximate system (1.4)–(1.5) in $\mathbb{T}^N$ with initial data

$$n(x,0) = n_0(x), \quad nu(x,0) = m_0(x), \qquad x \in \mathbb{T}^N, \tag{1.7}$$

which satisfies

$$n_0 \geq 0 \text{ a.e. on } \mathbb{T}^N, \int n_0(x)\mathrm{d}x = 1, \text{ and } \frac{|m_0|^2}{n_0} = 0 \text{ a.e. on } \{n_0(x) = 0\}. \tag{1.8}$$

The motivation to consider the approximate system (1.4)–(1.6) is the follows. Recently, the quantum hydrodynamic (QHD) model

$$\partial_t n + \nabla\cdot(nu) = 0, \tag{1.9}$$

$$\partial_t(nu) + \nabla\cdot(nu\otimes u) + \nabla p(n) = n\nabla\Phi + \frac{\varepsilon^2}{2}n\nabla(\frac{\triangle\sqrt{n}}{\sqrt{n}}), \tag{1.10}$$

$$-\Delta\Phi = n - 1, \tag{1.11}$$

is derived and studied in the modelings and simulations of semiconductor devices, where the effects of quantum mechanics arises. The basic observation therein is that the energy density consists of one additional new quantum correction term of the order $O(\varepsilon)$ introduced first by Wigner [5] and that the stress tensor contains also an additional quantum correction caused by the quantum Bohm potential $Q(\rho) = -\frac{\varepsilon^2}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$, which is responsible for the quantum effect [2]. Thus, a natural question is whether the quantum hydrodynamic (QHD) model (1.9)–(1.11) will converge to the classical hydrodynamical model as Planck constant ε tends to zero, the so-called semiclassical (dispersion) limit. This problem is only solved recently for global smooth solution [6–8], and is not clear yet for weak solutions. To this end, we first consider the dissipative and dispersive approximation system (1.1)–(1.3) and consider the global existence of weak solutions and the dispersion limit.

There are several prototype models in the literatures of the approximate system (1.4)–(1.6), for instance, the viscous Saint-Venant model for the motion of shallow water when we set in (1.4)–(1.6) that

$$p(n) = n^2, \quad \mu(n) = n, \quad \lambda(n) = 0, \quad \varphi(n) = 0, \quad \Phi \equiv 0,$$

or the model of shallow water in general form if

$$p(n) = n^2, \quad \mu(n) = n, \quad \lambda(n) = 0, \quad \varphi(n) = n, \quad \Phi \equiv 0.$$

The rest part of the paper is arranged as follows. We state the main results on existence and dispersion limit in Section 2; and we establish the a-priori estimates for the system in multi-dimension in Section 3.1, and prove the main results in Section 3.2.

2. Main Results

For simplicity, we consider the approximate system (1.4)–(1.6) for γ-law pressure-density function

$$p(n) = n^\gamma, \quad \gamma \geq 1. \tag{2.1}$$

And the viscosity coefficients and the dispersion term satisfies

$$\mu(n) = \mu_0 n^\alpha,\ \alpha > 0, \quad \varphi(n) = n^\alpha, \quad \lambda(n) = 2(n\mu'(n) - \mu(n)), \tag{2.2}$$

with μ_0 positive constant set to be one below, and define the functions $\pi(n)$ and $\xi(n)$ by

$$n\pi'(n) - \pi(n) = p(n), \quad n\xi'(n) = \mu'(n). \tag{2.3}$$

Let us give the definition of weak solutions of the IVP problem (1.4)–(1.6) and (1.7)–(1.8) in multi-dimensional periodic domain $\mathbb{T}^N$, $N \geq 1$ as follows.

Definition 2.1 *(n,u) with $n\ge 0$ a.e. is said to be a global entropy solution to the IVP problem (1.4)–(1.6) and (1.7)–(1.8), provided that it holds*

$$\left.\begin{aligned} &n\in L^\infty(0,T;L^1(\mathbb{T}^N)\cap L^\gamma(\mathbb{T}^N)), \quad \frac{\nabla\mu(n)}{\sqrt{n}}\in L^\infty(0,T;L^2(\mathbb{T}^N)),\\ &\nabla\mu(n)\in L^\infty(0,T;L^2(\mathbb{T}^N)), \qquad \sqrt{n}u\in L^\infty(0,T;L^2(\mathbb{T}^N)),\\ &\sqrt{\varphi'(n)}\,\Delta\varphi(n)\in L^2(0,T;L^2(\mathbb{T}^N)),\ \sqrt{p'(n)\mu'(n)}\,\nabla\sqrt{n}\in L^2(0,T;L^2(\mathbb{T}^N)),\\ &\sqrt{\mu(n)}\,\nabla u\in L^2(0,T;L^2(\mathbb{T}^N)), \qquad \nabla\Phi\in L^\infty(0,T;L^2(\mathbb{T}^N)). \end{aligned}\right\}\tag{2.4}$$

Moreover, (n,u) satisfies the a-priori entropy estimate

$$\begin{aligned}&\int_{\mathbb{T}^N}(nu^2+\pi(n)+|\tfrac{\nabla\mu(n)}{\sqrt{n}}|^2+|\nabla\Phi|^2+|\varepsilon\nabla\mu(n)|^2)\mathrm{d}x\\ &\quad+\int_0^T\int_{\mathbb{T}^N}((n-1)(\mu(n)-\mu(1))+|\sqrt{p'(n)\mu'(n)}\,\nabla\sqrt{n}\,|^2)\,\mathrm{d}x\mathrm{d}t\\ &\quad+\int_0^T\int_{\mathbb{T}^N}(\varepsilon^2|\sqrt{\mu'(n)}\,\Delta\mu(n)|^2+\eta|\sqrt{\mu(n)}\,\nabla u|^2+n|u|^3)\,\mathrm{d}x\mathrm{d}t\le C_0,\end{aligned}\tag{2.5}$$

for some constant $C_0>0$, and the IVP problem (1.4)–(1.6) and (1.7)–(1.8) in the sense of distribution

$$\int_0^T\int_{\mathbb{T}^N} n\psi_t\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}^N}\sqrt{n}\sqrt{n}u\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t-\int_{\mathbb{T}^N}n\psi\mathrm{d}x\Big|_{t=0}^T=0,\tag{2.6}$$

$$\left.\begin{aligned}&\int_0^T\int_{\mathbb{T}^N}\sqrt{n}\sqrt{n}u\cdot\psi_t\,\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}^N}\sqrt{n}u\otimes\sqrt{n}u:\nabla\psi\,\mathrm{d}x\mathrm{d}t\\ &\quad+\int_0^T\int_{\mathbb{T}^N}p(n)\mathrm{div}\,\psi\,\mathrm{d}x\mathrm{d}t-\int_{\mathbb{T}^N}\sqrt{n}\sqrt{n}u\cdot\psi\mathrm{d}x\Big|_{t=0}^T\\ &\quad-\int_0^T\int_{\mathbb{T}^N}r\sqrt{n}|u|\sqrt{n}u\cdot\psi\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{T}^N}n\nabla\Phi\cdot\psi\,\mathrm{d}x\mathrm{d}t\\ &\quad-\varepsilon^2\int_0^T\int_{\mathbb{T}^N}(\varphi'(n)\Delta\varphi(n)\nabla n\cdot\psi+n\varphi'(n)\Delta\varphi(n)\mathrm{div}\,\psi)\,\mathrm{d}x\mathrm{d}t\\ &\quad-2\eta<\mu(n)D(u),\nabla\psi>-\eta<\lambda(n)\mathrm{div}u,\mathrm{div}\,\psi>=0\end{aligned}\right\},\tag{2.7}$$

$$\int_0^T\int_{\mathbb{T}^N}\nabla\Phi\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{T}^N}n\psi\,\mathrm{d}x\mathrm{d}t=0,\tag{2.8}$$

for any test function $\psi\in C^\infty_{per}([0,T]\times\mathbb{T}^N)$. The nonlinear diffusion terms are well defined in terms of (n,u) as

$$<\mu(n)\nabla u,\Psi>=-\int_0^T\int_{\mathbb{T}^N}\mu(n)u\cdot\mathrm{div}\,\Psi\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{T}^N}(\sqrt{n}u\Psi)\cdot\tfrac{\nabla\mu(n)}{\sqrt{n}}\,\mathrm{d}x\mathrm{d}t,\tag{2.9}$$

$$< \lambda(n)\mathrm{div}u, \psi >= -\int_0^T\int_{\mathbb{T}^N} \lambda(n)u\cdot\nabla\psi\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbb{T}^N} \psi\sqrt{n}u\cdot\frac{\nabla\lambda(n)}{\sqrt{n}}\,\mathrm{d}x\mathrm{d}t, \quad (2.10)$$

for any test function $\Psi \in C^\infty_{per}([0,T]\times\mathbb{T}^N)^{N\times N}$ *and* $\psi \in C^\infty_{per}([0,T]\times\mathbb{T}^N)$.

Remark 2.1 Note here that the dispersion (capillarity) terms and the electrical force appeared in (2.7) are meaningful in multi-dimension in the sense of distribution at least for the case $\alpha \in (\frac{N-1}{N}, 1]$ and $\gamma \geq 2$ (or $\gamma \geq 1$ for one-dimension). In fact, we can conclude from (2.4) that

$$n^{(\alpha+1)/2} \in L^2(0,T;H^1(\mathbb{T}^N)), \quad n \in L^2(0,T;L^2(\mathbb{T}^N)),$$

which together with the entropy estimates (2.5) implies

$$\int_0^T\int_{\mathbb{T}^N} \varphi'(n)\Delta\varphi(n)\nabla n\cdot\psi\,\mathrm{d}x\mathrm{d}t \leq C\|\sqrt{\varphi'(n)}\Delta\varphi(n)\|_{L^2_{x,t}}\|\nabla n^{(\alpha+1)/2}\|_{L^2_{x,t}}\|\psi\|_{L^\infty_{x,t}},$$

$$\int_0^T\int_{\mathbb{T}^N} n\varphi'(n)\Delta\varphi(n)\mathrm{div}\,\psi\,\mathrm{d}x\mathrm{d}t \leq C\|\sqrt{\varphi'(n)}\Delta\varphi(n)\|_{L^2_{x,t}}\|n^\gamma\|^{1/2}_{L^\infty_t L^1_x}\|\psi\|_{L^\infty_{x,t}},$$

and

$$\int_0^T\int_{\mathbb{T}^N} n\nabla\Phi\cdot\psi\,\mathrm{d}x\mathrm{d}t \leq C\|n\|_{L^2([0,T]\times\mathbb{T}^N)}\|\nabla\Phi\|_{L^\infty_t(L^2_x)}\|\psi\|_{L^\infty([0,T]\times\mathbb{T}^N)}.$$

For the approximate system (1.4)–(1.6), we can prove the global in time existence of entropy weak solutions in the sense of Definition 2.1 subject to the initial data (1.7)–(1.8). To this end, additional regularity (2.11) of the initial data (n_0, m_0) is required. In fact, we have

Theorem 2.1 (Global existence) *Let* $N = 1$ *and* $T > 0$. *Let* (2.1)–(2.3) *hold and* $\alpha \in (2/3, \gamma]$, *and* η *and* ε *be fixed positive constants in Eq.* (1.4)–(1.5). *Assume that the initial data* (n_0, m_0) *given by* (1.7)–(1.8) *satisfies*

$$n_0 \in L^\gamma(\mathbb{T}^N), \quad \nabla\mu(n_0) \in L^2(\mathbb{T}^N), \quad \frac{\nabla\mu(n_0)}{\sqrt{n_0}} \in L^2(\mathbb{T}^N), \quad \frac{|m_0|^2}{n_0} \in L^1(\mathbb{T}^N). \quad (2.11)$$

Then, there is a global entropy weak solution (n, u, Φ) *of the IVP problem* (1.4)–(1.8) *in the sense of Definition* 2.1.

Remark 2.2 Although we present only the global existence of entropy weak solution in spatial one-dimension for $\alpha \in (2/3, \gamma]$, (since we aim at the dispersion limit on approximate system (1.4)–(1.6)), the global existence of entropy weak solution in multi-dimension can be established within a similar framework, which is left for further investigation.

For any given global entropy weak solution $(n_\varepsilon, u_\varepsilon, \Phi_\varepsilon)$ of the approximate system (1.4)–(1.6) in the sense of Definition 2.1, we are able to investigate the dispersion limit. Let $\varepsilon \to 0_+$ in the approximate system (1.4)–(1.6) and assume formally

$$n_\varepsilon \to \rho, \quad \sqrt{n_\varepsilon}\, u_\varepsilon \to \sqrt{\rho}\, v, \quad \Phi_\varepsilon \to V,$$

we obtain the Navier-Stokes-Poisson system for (ρ, v, V) as

$$\left.\begin{aligned} &\partial_t \rho + \mathrm{div}(\rho v) = 0, \\ &\partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) + \nabla p(\rho) + \rho \nabla V \\ &\qquad = 2\eta \mathrm{div}(\mu(\rho) D(v)) + \eta \nabla(\lambda(\rho)\mathrm{div} v) - r\rho|v|v, \\ &-\Delta V = \rho - 1. \end{aligned}\right\} \tag{2.12}$$

Without the loss of generality, we set $\eta = 1$, $r = 1$ in Eq. (1.4)–(1.6) and (2.12) throughout this paper. We have the result on the dispersion limit for weak solutions as follows.

Theorem 2.2 (Dispersion limit) *Let $1 \le N \le 3$ and $T > 0$. Assume* (2.1)–(2.3) *hold with $\alpha \in (2/3, 1]$ and $\gamma \ge 2$. Let $(n_\varepsilon, u_\varepsilon, \Phi_\varepsilon)$ be any global entropy weak solution of the IVP problem* (1.4)–(1.8) *in the sense of Definition* 2.1. Then, there exists (ρ, v, V) so that it holds

$$n_\varepsilon \to \rho \quad \text{in} \quad C(0,T; L^1(\mathbb{T}^N)), \qquad \sqrt{n_\varepsilon}\, u_\varepsilon \to \sqrt{\rho}\, v \quad \text{in} \quad L^2(0,T; L^2(\mathbb{T}^N)), \tag{2.13}$$

$$\Phi_\varepsilon \to V \quad \text{in} \quad C(0,T; \dot{H}^2(\mathbb{T}^N)), \tag{2.14}$$

as $\varepsilon \to 0_+$. Here (ρ, v, V), $\rho \ge 0$ a.e., is a global weak solution of the IVP problem (2.12) and (1.7)–(1.8) in the sense of distribution.

Remark 2.3 Theorem 2.2 implies the dispersion limit from the approximate system (1.4)–(1.6) to the Navier-Stokes-Poisson system (2.12). What left for further study is to understand the vanishing viscosity limit $\eta \to 0_+$ for the system (2.12) to the hydrodynamical model (1.1)–(1.3). Note that the global existence of weak solutions to (2.12) without Poisson and drag friction terms is obtained recently in [9] in multi-dimension with symmetry.

3. Proof of Main Results

3.1 The a-priori entropy estimates

Let us establish the energy estimates of physical entropy and the BD entropy developed recently by Bresch and Desjardins [10] that can be satisfied by solutions to the model in general multi-dimension. We only deal with the spatial periodic case $\mathbb{T}^N$, $1 \le N \le 3$. As mentioned before, we set all parameters η, r to be one except ε, since we shall also consider the dispersion limit of the solutions below. We have

Lemma 3.1 *Let $T>0$. Under the assumptions of Theorem 2.1, it holds for any classical solution (n,u,Φ) of the IVP problem (1.4)–(1.8) that*

$$\left.\begin{array}{ll} n\in L^\infty(0,T;L^1(\mathbb{T}^N)\cap L^\gamma(\mathbb{T}^N)), & \dfrac{\nabla\mu(n)}{\sqrt{n}}\in L^\infty(0,T;L^2(\mathbb{T}^N)),\\ \varepsilon\nabla\mu(n)\in L^\infty(0,T;L^2(\mathbb{T}^N)), & \sqrt{n}u\in L^\infty(0,T;L^2(\mathbb{T}^N)),\\ \varepsilon\sqrt{\varphi'(n)}\,\Delta\varphi(n)\in L^2(0,T;L^2(\mathbb{T}^N)), & \nabla n^{(\gamma+\alpha-1)/2}\in L^2(0,T;L^2(\mathbb{T}^N)),\\ \sqrt{\mu(n)}\,\nabla u\in L^2(0,T;L^2(\mathbb{T}^N)), & \nabla\Phi\in L^\infty(0,T;L^2(\mathbb{T}^N)). \end{array}\right\} \tag{3.1}$$

Proof The a-priori estimates (3.1) follow directly from the equalities (3.2) and (3.3) for smooth (approximate) solution sequence of the IVP problem (1.4)–(1.8). (Note here that the estimates (3.1) also hold for weak solution of the IVP problem (1.4)–(1.8), which we can verify later when taking the (sequence) limit of the (approximate) smooth solution). The first one is the classical (physical) entropy for compressible fluids and can be established by the standard arguments. That is, we just have to take inner product in Eulerian space between the transport equation (1.4) and the test function $|u|^2/2$, and between the momentum equation (1.5) and u respectively. Then make summation of the resultant equations and employ the Poisson equation, we have

$$\begin{aligned} &\frac{1}{2}\int(n|u|^2+\varepsilon^2|\nabla\mu(n)|^2+|\nabla\Phi|^2+2\pi(n))(x,t)\mathrm{d}x\\ &\qquad+\int_0^T\int(\mu(n)|D(u)|^2+\lambda(n)|\mathrm{div}u|^2)\mathrm{d}x\mathrm{d}t+\int_0^T\int n|u|^3\mathrm{d}x\mathrm{d}t\\ &\qquad=\frac{1}{2}\int(2\pi(n_0)+\frac{|m_0|^2}{n_0}+\varepsilon^2|\nabla\mu(n_0)|^2+|\nabla\Phi_0|^2)\mathrm{d}x. \end{aligned}\tag{3.2}$$

The second one is a mathematical entropy developed recently by Bresch and Desjardins [10] as

$$\begin{aligned} &\frac{1}{2}\int(n|u+\nabla\xi(n)|^2+\varepsilon^2|\nabla\mu(n)|^2+|\nabla\Phi|^2+2\pi(n))\,\mathrm{d}x\\ &\quad+\int_0^T\int((n-1)(\mu(n)-\mu(1))+\mu(n)|A(u)|^2+n|u|^3+|u|u\cdot\nabla\mu(n))\,\mathrm{d}x\mathrm{d}t\\ &\quad+\varepsilon^2\int_0^T\int\mu'(n)|\Delta\mu(n)|^2\,\mathrm{d}x\mathrm{d}t+\int_0^T\int p'(n)\mu'(n)\frac{|\nabla n|^2}{n}\,\mathrm{d}x\mathrm{d}t\\ &=\frac{1}{2}\int(n_0|u_0+\nabla\xi(n_0)|^2+\varepsilon^2|\nabla\mu(n_0)|^2+|\nabla\Phi|^2+2\pi(n_0))\,\mathrm{d}x. \end{aligned}\tag{3.3}$$

For reader's convenience, let us show the derivation line by line as in [10]. Let us rewrite its derivation to let the paper self-contained. Multiplying the mass equation (1.4) with $\xi'(n)$ and taking gradient, we get for smooth solutions that

$$\partial_t\nabla\xi(\rho)+u\cdot\nabla\nabla\xi(\rho)+\nabla u\cdot\nabla\xi(\rho)+\nabla(\rho\xi'(\rho)\mathrm{div}u)=0$$

with the help of (2.1)–(2.3). Thus, denote $v = \nabla\xi(n)$, we get by using the mass equation (1.4) again

$$\partial_t(nv) + \mathrm{div}(nu \otimes v) + \rho\nabla u \cdot \nabla\xi(n) + n\nabla(n\xi'(n)\mathrm{div}u) = 0,$$

which together with the momentum equation (1.5) and the relation (2.2) gives rise to the equation on $u + v$ as

$$\begin{aligned}\partial_t(n(u+v)) + \mathrm{div}(nu \otimes (u+v)) + \nabla p(n) + n\nabla\Phi \\ = 2\mathrm{div}(\mu(n)A(u)) + \varepsilon^2 n\nabla(\varphi'(n)\Delta\varphi(n)) - n|u|u\end{aligned} \tag{3.4}$$

with $A(u) = \frac{1}{2}(\nabla u - {}^t\nabla u)$ the anti-symmetric stress tensor.

Taking inner product over $\mathbb{T}^N$ between the equation (3.4) and $(u+v)$ and between the mass equation (1.4) and $|u+v|^2/2$ respectively, and adding the resultant equations together, we get easily the new mathematical entropy equality (3.3), where we have make use of the following facts. First of all, we have

$$\int_{\mathbb{T}^N} \mathrm{div}(\mu(n)A(u)) \cdot v\mathrm{d}x = 0$$

since v is a gradient, and from the Poisson equation (1.6) that

$$-\int_{\mathbb{T}^N} n\nabla\Phi \cdot v\mathrm{d}x = -\int_{\mathbb{T}^N} (n-1)(\mu(n) - \mu(1))\mathrm{d}x.$$

The terms coming from the dispersion give for $\mu(n) = \varphi(n)$ that

$$\varepsilon^2 \int_{\mathbb{T}^N} n\nabla(\varphi'(n)\Delta\varphi(n))\cdot(u+v)\mathrm{d}x = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^N} |\varepsilon\nabla\mu(n)|^2\mathrm{d}x - \varepsilon^2\int_{\mathbb{T}^N} \varphi'(n)|\Delta\varphi(n)|^2\mathrm{d}x,$$

and from the pressure lead to

$$\int_{\mathbb{T}^N} \nabla p(n) \cdot (u+v)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^N} \pi(n)\mathrm{d}x + \int_{\mathbb{T}^N} p'(n)\mu'(n)\frac{|\nabla n|^2}{n}\mathrm{d}x.$$

The term coming from the drag friction gives

$$\begin{aligned}-\int_{\mathbb{T}^N} n|u|u \cdot (u+v)\mathrm{d}x &= -\int_{\mathbb{T}^N} n|u|^3\,\mathrm{d}x - \int_{\mathbb{T}^N} |u|u \cdot \nabla\mu(n)\,\mathrm{d}x \\ &= -\int_{\mathbb{T}^N} n|u|^3\mathrm{d}x + \int_{\mathbb{T}^N} \mu(n)\,\mathrm{div}(|u|u)\,\mathrm{d}x \\ &\le -\frac{1}{2}\int_{\mathbb{T}^N} n|u|^3\mathrm{d}x + \int_{\mathbb{T}^N} |\nabla\mu(n)|^2\mathrm{d}x + C\int_{\mathbb{T}^N} \mu(n)|\nabla u|^2\,\mathrm{d}x\end{aligned} \tag{3.5}$$

for $\alpha \in (2/3, 1]$, due to the fact

$$\begin{aligned}
\int_{\mathbb{T}^N} |\mu(n)\,\mathrm{div}(|u|u)|\mathrm{d}x \le & \int_{\mathbb{T}^N} \frac{\sqrt{\mu(n)}}{n^{1/3}} n^{1/3}|u|\sqrt{\mu(n)}|\nabla u|\mathrm{d}x \\
\le & \Big(\int_{0\le n<1} + \int_{n\ge 1}\Big)\frac{\sqrt{\mu(n)}}{n^{1/3}} n^{1/3}|u|\sqrt{\mu(n)}|\nabla u|\,\mathrm{d}x \\
\le & \int_{0\le n<1} n^{2/3}|u|^2 n^{\alpha-2/3}\,\mathrm{d}x + \int_{n\ge 1} n|u|^2 n^{\alpha-1}\,\mathrm{d}x \\
\le & C\int_{\mathbb{T}^N} (\mu(n)|\nabla u|^2 + |\nabla \mu(n)|^2)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^N} n|u|^3\,\mathrm{d}x \qquad (3.6)
\end{aligned}$$

The proof of the Lemma 3.1 is complete.

Next, we can derive the a-priori estimates for smooth solutions based on the entropy estimates given by Lemma 3.1, which allows us to pass into the limit of approximate (smooth) solution sequence, and in particular to take the dispersion limit later for the weak solutions so long as the related estimate derived there is uniform with respect to $\varepsilon > 0$.

Lemma 3.2 *Let $T > 0$. Under the assumptions of Theorem 2.2, it holds for any classical solution (n, u, Φ) with $n > 0$ of the IVP problem (1.4)–(1.8) that*

$$n^{(\alpha+1)/2} \in L^2(0, T; H^1(\mathbb{T}^N)), \qquad (3.7)$$

$$(n^{(\alpha+1)/2})_t \in L^2(0, T; W^{-1,s_0}(\mathbb{T}^N)) \cap L^\infty(0, T; W^{-1,1}(\mathbb{T}^N)), \qquad (3.8)$$

for $\alpha \in (2/3, 1]$ and $\gamma \ge 2$, and $s_0' = \max\{\frac{2\gamma}{\gamma-1}, \frac{2(\alpha+1)N}{2\alpha+N}\} > 2$ if $N = 2, 3$; and that

$$n^\alpha \in L^\infty(0, T; H^1(\mathbb{T}^N)), \quad n \in C([0, T] \times \mathbb{T}^N) \qquad (3.9)$$

uniformly with respect to $\varepsilon > 0$ for $\alpha \in (2/3, 1]$ and $\gamma \ge \max\{1, \alpha\}$ if $N = 1$.

In addition, the electrostatic potential Φ satisfies

$$\Phi \in L^\infty(0, T; \dot{W}^{2,\gamma}(\mathbb{T}^N) \cap \dot{W}^{3,p}(\mathbb{T}^N)), \quad \Phi_t \in L^\infty(0, T; \dot{W}^{1,\frac{2\gamma}{\gamma+1}}(\mathbb{T}^N)) \qquad (3.10)$$

with $p = \frac{2\gamma}{\gamma+2(1-\alpha)} \in (1, 2)$ for $N = 2, 3$, and

$$\Phi \in L^\infty(0, T; \dot{H}^3(\mathbb{T}^N)), \quad \Phi_t \in L^\infty(0, T; \dot{H}^1(\mathbb{T}^N)), \quad N = 1. \qquad (3.11)$$

Proof The estimate (3.7) is established in terms of the estimates (3.1) in Lemma 3.1 as follows. We have the gradient estimate in terms of $n^{(\alpha+1)/2}$ directly

$$\begin{aligned}
\|\nabla n^{(\alpha+1)/2}\|_{L^2}^2 =& \tfrac{(\alpha+1)^2}{4}\int_{\mathbb{T}^N} n^{\alpha-1}|\nabla n|^2\mathrm{d}x = \tfrac{(\alpha+1)^2}{4}\Big(\int_{0\le n<1} + \int_{n\ge 1}\Big) n^{\alpha-1}|\nabla n|^2\mathrm{d}x \\
=& \tfrac{(\alpha+1)^2}{4}\Big(\int_{0\le n<1} n^{2\alpha-3}|\nabla n|^2 n^{2-\alpha}\mathrm{d}x + \int_{n\ge 1} n^{\gamma+\alpha-3}|\nabla n|^2 n^{2-\gamma}\mathrm{d}x\Big)
\end{aligned}$$

$$\leq C \int_{\mathbb{T}^N} (|\frac{\nabla\mu(n)}{\sqrt{n}}|^2 + p'(n)\xi'(n)|\nabla n|^2)\mathrm{d}x \leq C_0, \tag{3.12}$$

in the case $\alpha \in (0, 1]$ and $\gamma \geq 2$ if $N = 2, 3$. Here and below $C_0 > 0$ denotes a constant dependent of initial data. Since it holds

$$\overline{n^{(\alpha+1)/2}} =: \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} n^{(\alpha+1)/2}\mathrm{d}x \leq \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} n^\gamma \mathrm{d}x \leq C_0 \tag{3.13}$$

due to the fact $\max\{\alpha, 1\} \leq \gamma$, we have

$$\begin{aligned} \|n^{(\alpha+1)/2}\|_{L^2}^2 \leq& C(\|n^{(\alpha+1)/2} - \overline{n^{(\alpha+1)/2}}\|_{L^2}^2 + \sqrt{|\mathbb{T}^N|}\ \overline{n^{(\alpha+1)/2}}) \\ \leq& C(\|\nabla n^{(\alpha+1)/2}\|_{L^2}^2 + \sqrt{|\mathbb{T}^N|}\ \overline{n^{(\alpha+1)/2}}) \leq C_0 \end{aligned} \tag{3.14}$$

with the help of Poincáre's inequality.

To prove the regularity (3.8) for $n^{(\alpha+1)/2}$, we make use of the re-normalized equation $\beta(n)_t + \mathrm{div}\,(n\beta'(n)u) - nu \cdot \nabla\beta'(n) = 0$ to have

$$(n^{(\alpha+1)/2})_t = -\tfrac{1+\alpha}{2}\mathrm{div}\,(n^{(\alpha+1)/2}u) + \tfrac{1-\alpha^2}{2\alpha}\sqrt{n}u \cdot \nabla n^{\alpha/2} \in L^\infty(0, T; W^{-1,1}(\mathbb{T}^N)), \tag{3.15}$$

where we have used

$$\|\nabla n^{\alpha/2}\|_{L^2(\mathbb{T}^N)} \leq C(\|\nabla n^{\alpha-1/2}\|_{L^2(\mathbb{T}^N)} + \|\nabla n^\alpha\|_{L^2(\mathbb{T}^N)}).$$

On the other hand, we can re-write the re-normalized equation as

$$\beta(n)_t + \mathrm{div}\,(\beta(n)u) + (n\beta'(n) - \beta(n))\mathrm{div}\,u = 0,$$

in order to have for $\beta(n) = n^{(\alpha+1)/2}$ that

$$(n^{(\alpha+1)/2})_t = -\mathrm{div}\,(n^{\alpha/2}\sqrt{n}u) + \tfrac{1-\alpha}{2}\sqrt{n}\sqrt{\mu(n)}\,\mathrm{div}\,u.$$

It is easy to verify that it holds for $1 < p_0 = \frac{2\gamma}{\gamma+1} < 2$ (due to $\gamma \geq 2$) that

$$\begin{aligned} \|\sqrt{n}\sqrt{\mu(n)}\,\mathrm{div}\,u\|_{L^2(0,T;L^{p_0}(\mathbb{T}^N))} \leq& C\|\sqrt{\mu(n)}\,\mathrm{div}\,u\|_{L_t^2(L_x^2)}\|\sqrt{n}\|_{L_t^\infty(L_x^{2\gamma})} \\ \leq& C\|n\|_{L_t^\infty L_x^\gamma}^{1/2}\|\sqrt{\mu(n)}\,\nabla u(t)\|_{L_{x,t}^2}^2, \end{aligned} \tag{3.16}$$

and for $p = \frac{2(\alpha+1)N}{2\alpha(N-1)+N} \in (1, 2)$ and $q = 2(1 + \frac{1}{\alpha}) > 2$ that

$$\begin{aligned} \|n^{\alpha/2}\sqrt{n}u\|_{L^q(0,T;L^p(\mathbb{T}^N))} \leq& C\|\sqrt{n}u\|_{L_t^\infty L_x^2}\|n^{\alpha/2}\|_{L_t^{\frac{2(\alpha+1)}{\alpha}} L_x^{2N(\alpha+1)/\alpha(N-2)}} \\ \leq& C\|\sqrt{n}u\|_{L_t^\infty L_x^2}\|n^{(\alpha+1)/2}\|_{L_t^2 H_x^1}^{\alpha/2} \end{aligned} \tag{3.17}$$

with the help of the Sobolev embedding

$$n^{(\alpha+1)/2} \in L^2(0, T; H^1(\mathbb{T}^N)) \hookrightarrow L^2(0, T; L^p(\mathbb{T}^N)), \quad p \in [2, 6],\ N = 2, 3. \tag{3.18}$$

Thus, it follows from (3.16) and (3.17) that

$$(n^{(\alpha+1)/2})_t \in L^{\frac{2(\alpha+1)}{\alpha}}(0,T;W^{-1,p}(\mathbb{T}^N)) + L^2(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^N)) \hookrightarrow L^2_t(W_x^{-1,s_0}) \quad (3.19)$$

with $p = \frac{2(\alpha+1)N}{2\alpha(N-1)+N} \in (1,2)$ and $s_0' = \max\{\frac{2\gamma}{\gamma-1}, \frac{2(\alpha+1)N}{2\alpha+N}\} > 2$. This together with (3.15) leads to (3.8).

In the case of spatial one-dimension $N = 1$, we can obtain the upper bound of density under the assumption of $\gamma \geq \max\{1, \alpha\}$ and $\alpha > 1/2$

$$\begin{aligned} n^{\alpha-1/2} \leq & \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} n^{\alpha-1/2}\,\mathrm{d}x + \int_{\mathbb{T}^N} |\nabla n^{\alpha-1/2}|\,\mathrm{d}x \\ \leq & C(\|n^\gamma\|_{L_t^\infty L_x^1}^{(\alpha-1/2)/\gamma} + \|\tfrac{\nabla\mu(n)}{\sqrt{n}}\|_{L_t^\infty L_x^2}) \leq C_0. \end{aligned} \quad (3.20)$$

Thus, we have for $\alpha \in (1/2, 1]$ that

$$n \in L^\infty(0,T;H^1(\mathbb{T}^N)), \quad N = 1, \quad (3.21)$$

and

$$\|\nabla\sqrt{n}\|^2_{L^2(\mathbb{T}^N)} \leq \tfrac{1}{(2\alpha-1)^2} \int_{\mathbb{T}^N} |\nabla n^{\alpha-1/2}|^2\,\mathrm{d}x \leq C\|\tfrac{\nabla\mu(n)}{\sqrt{n}}\|_{L^2(\mathbb{T}^N)} \leq C_0. \quad (3.22)$$

The regularity (3.21) on the density n together with the re-normalized equation for $\mu(n)$ in one-dimension gives

$$\mu(n)_t = -\mathrm{div}\,(\mu(n)u) - (\mu'(n)n - \mu(n))\,\mathrm{div}\,u \in L^2(0,T;W^{-1,2}(\mathbb{T}^N)). \quad (3.23)$$

We conclude from (3.21) and (3.23) the continuity of the function $\mu(\rho) = n^\alpha$ for any $\alpha \in (1/2, 1]$ and the density n as

$$\mu(n) \in C([0,T]\times\mathbb{T}^N), \quad n \in C([0,T]\times\mathbb{T}^N). \quad (3.24)$$

The regularity (3.9) follows from (3.21), (3.22), and (3.24).

The regularities (3.10)–(3.11) on the electrostatic potential Φ follows from the Poisson equation (1.6), (3.1), (3.21), Eq. (1.4), and the Sobolev embedding inequality. The proof of the Lemma 3.2 is complete.

Remark 3.1 Note here that the dispersion (capillarity) terms appeared in (2.7) is meaningful in multi-dimension in the sense of distribution at least for the case $\alpha \in (2/3, 1]$ and $\gamma \geq 2$ (*or* $\gamma \geq 1$ for one-dimension). In fact, we conclude from Lemma 3.2 that

$$n^{(\alpha+1)/2} \in L^2(0,T;H^1(\mathbb{T}^N)), \quad (3.25)$$

which together with the regularity (2.4) of (n, u) implies

$$\int_0^T \int_{\mathbb{T}^N} \varphi'(n)\Delta\varphi(n)\nabla n\cdot\psi\,\mathrm{d}x\mathrm{d}t \leq C\|\sqrt{\varphi'(n)}\Delta\varphi(n)\|_{L^2_{x,t}}\|\nabla n^{(\alpha+1)/2}\|_{L^2_{x,t}}, \quad (3.26)$$

$$\int_0^T\int_{\mathbb{T}^N} n\varphi'(n)\Delta\varphi(n)\mathrm{div}\,\psi\,\mathrm{d}x\mathrm{d}t \le C\|\sqrt{\varphi'(n)}\Delta\varphi(n)\|_{L^2_{x,t}}\|n^\gamma\|^{1/2}_{L^\infty_t L^1_x}, \tag{3.27}$$

with $q\in(2,\infty]$. Where we have used the regularity (3.7) on the density.

To justify the meaningfulness of the electrical force appeared in (2.7), we note that it follows from (2.4), (3.7) and Sobolev embedding theorem that

$$\begin{cases} n\in L^\infty([0,T]\times\mathbb{T}^N), \quad \text{for } N=1,\\ n\in L^{\alpha+1}(0,T;L^p(\mathbb{T}^N))\cap L^\infty(0,T;L^\gamma(\mathbb{T}^N)),\ p\in[\alpha+1,\infty) \quad \text{for } N=2,\\ n\in L^{\alpha+1}(0,T;L^p(\mathbb{T}^N))\cap L^\infty(0,T;L^\gamma(\mathbb{T}^N)),\ p\in[\alpha+1,\frac{(\alpha+1)N}{N-2}] \quad \text{for } N=3. \end{cases}$$

Therefore, we are able to estimate the electrical force for $N=1$ as

$$\int_0^T\int_{\mathbb{T}^N} n\nabla\Phi\cdot\psi\,\mathrm{d}x\mathrm{d}t \le C\|n\|_{L^\infty([0,T]\times\mathbb{T}^N)}\|\nabla\Phi\|_{L^\infty_t(L^2_x)}\|\psi\|_{L^\infty(L^2_x)} \tag{3.28}$$

and for $N=2,3$ that

$$\int_0^T\int_{\mathbb{T}^N} n\nabla\Phi\cdot\psi\,\mathrm{d}x\mathrm{d}t \le C\|n\|_{L^2([0,T]\times\mathbb{T}^N)}\|\nabla\Phi\|_{L^\infty_t(L^2_x)}\|\psi\|_{L^\infty([0,T]\times\mathbb{T}^N)}. \tag{3.29}$$

3.2 Compactness and convergence

Proof of Theorem 2.1 With the help of the a-priori estimates, we are able to prove the existence of global entropy weak solutions of the IVP problem (1.4)–(1.6) and (1.7)–(1.8) in spatial one-dimension. Assume that (n_k,u_k,Φ_k) with $n_k>0$, $k\ge1$, is a classical solution sequence for the IVP problem (1.4)–(1.6) and (1.7)–(1.8) and satisfies the a-priori estimates given by Lemma 3.1. Indeed, with the standard argument as used in [11], we can prove the existence of a sequence of approximate solutions (n_k,u_k,Φ_k) with $n_k>0$, $k\ge1$, which satisfies the a-priori estimates given by Lemma 3.1. Let us make use of the standard compactness framework developed in (refer to [11] and references therein for instance) with modification to show the existence of (n,u,Φ) which is the limit of (n_k,u_k,Φ_k) as $k\to\infty$. In fact, we can verify by a straightforward argument as proving Lemma 2.5 that it holds

$$n_k^\alpha\in L^\infty(0,T;H^1(\mathbb{T}^N)),\quad \partial_t n_k^\alpha\in L^2(0,T;W^{-1,2}(\mathbb{T}^N)),\quad N=1, \tag{3.30}$$

for $\alpha\in(2/3,1]$ and $\gamma\ge\max\{1,\,\alpha\}$, and

$$0<C_k\le n_k\le C_0 \tag{3.31}$$

with $C_k\to0$ as $k\to\infty$. On the other hand, we also have from the estimate (3.1) that

$$\|\sqrt{\mu'(n_k)}\,\Delta\mu(n_k)\|^2_{L^2_{x,t}} = \left(\int_{0<n_k\le1}+\int_{n_k>1}\right)\alpha n_k^{\alpha-1}|\Delta\mu(n_k)|^2\mathrm{d}x\mathrm{d}t\le C, \tag{3.32}$$

which together with (3.31) implies

$$\|\Delta\mu(n_k)\|^2_{L^2((0,T)\times\mathbb{T}^N)} \le C, \quad N=1. \tag{3.33}$$

This together with the Lions-Aubin lemma implies that there is $n\in C([0,T]\times\mathbb{T}^N)\cap L^\infty(0,T;H^1(\mathbb{T}^N))$ so that it holds after extracting a subsequence that

$$n_k \longrightarrow n \quad \text{in } C([0,T]\times\mathbb{T}^N), \tag{3.34}$$

$$(n_k^\alpha)_x \to (n^\alpha)_x \quad \text{and} \quad (n_k^{(\alpha+\gamma-1)/2})_x \to (n^{(\alpha+\gamma-1)/2})_x \quad \text{in } L^2((0,T)\times\mathbb{T}^N), \tag{3.35}$$

$$\tfrac{\mu(n_k)_x}{\sqrt{n}} \rightharpoonup \tfrac{\mu(n)_x}{\sqrt{n}} \quad \text{weakly in } L^2((0,T)\times\mathbb{T}^N), \quad \alpha>1/2. \tag{3.36}$$

The (3.34) together with (1.4) implies the conservation of mass for n

$$\int n(x,t)\mathrm{d}x = \int n_0(x)\mathrm{d}x = 1 \tag{3.37}$$

and strong convergence of electric field

$$\nabla\Phi_k \longrightarrow \nabla\Phi \quad \text{in } C([0,T]\times H^2(\mathbb{T}^N)). \tag{3.38}$$

Now, we can prove, by applying a similar argument as used in [12], that

$$\sqrt{n_k}u_k \longrightarrow \sqrt{n}u \quad \text{in } L^2((0,T)\times\mathbb{T}^N), \quad N=1. \tag{3.39}$$

Indeed, we can first prove that there is a m so that $m_k=n_ku_k$ converges to m strongly in $L^2(0,T;L^2(\mathbb{T}^1))$, due to the facts $n_ku_k\in L^\infty(0,T;L^2(\mathbb{T}^N))\cap L^3(0,T;L^3(\mathbb{T}^N))$,

$$\nabla(n_ku_k) = n_k^{1-\alpha/2}n_k^{\alpha/2}\,\nabla u + n_k^{7/6-\alpha}\nabla n_k^{\alpha-1/2}\otimes n_k^{1/3}u \in L^2(0,T;L^{\frac{6}{5}}(\mathbb{T}^N)),\ N=1,$$

where we have used (3.31), and, in terms of the momentum equation (1.5),

$$\begin{aligned}(n_ku_k)_t =& -\operatorname{div}(n_ku_k^2) - (n_k^\gamma)_x - n_k\Phi_{kx} - n_k|u_k|u_k \\ &+ \alpha(n_k^\alpha u_{kx})_x + \varepsilon^2 n_k(\mu'(n_k)\mu(n_k)_{xx})_x \ \in\ L^1(0,T;W^{-1,1}(\mathbb{T}^N)).\end{aligned}$$

In particular, we have $m(x,t)=0$ a.e. in $\{n(x,t)=0\}$. Moreover, since $\frac{m_k}{\sqrt{n_k}}$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{T}^N)$, if we define $\frac{m^2}{n}$ to be zero when $m=0$. Then by Fatou's lemma, we have

$$\int_{\mathbb{T}^N}\frac{m^2}{n}\mathrm{d}x \le C.$$

Because $\sqrt{n_k}u_k$ is uniformly bounded in $L^3(0,T;L^3(\mathbb{T}^N))$, it is thus enough to prove the strong convergence

$$\sqrt{n_k}u_k \longrightarrow \sqrt{n}u \quad \text{in } L^1((0,T)\times\mathbb{T}^N), \tag{3.40}$$

as $k\to\infty$. To this end, we denote the set of vacuum by $\mathcal{V}=:\{\rho(x,t)=0\}$. Notes that $\sqrt{n_k}u_k$ converges almost everywhere to $\frac{m}{\sqrt{n}}$ in the region $\mathcal{V}^c$. To control the convergence of $\sqrt{n_k}u_k$ on the vacuum set $\mathcal{V}$, one sets $\mathcal{V}_{k,M} = \{n_k^{\frac{1}{3}} u_k \geq \mathrm{M}\}$ for any given $\mathrm{M} > 0$. It holds obviously

$$\int |\sqrt{n_k}u_k - \frac{m}{\sqrt{n}}|\mathrm{d}x\mathrm{d}t = \left(\int_{(\mathcal{V}_{k,M})^c\setminus\mathcal{V}} + \int_{(\mathcal{V}_{k,M})^c\cap\mathcal{V}} + \int_{\mathcal{V}_{k,M}}\right) |\sqrt{n_k}u_k - \frac{m}{\sqrt{n}}|\mathrm{d}x\mathrm{d}t. \quad (3.41)$$

The right hand side terms of (3.41) can be estimated as follows. Since $\sqrt{n_k}u_k \in L^\infty(0,T;L^2(\mathbb{T}^N))$ uniformly with respect to k, and $\sqrt{n_k}u_k$ converges almost everywhere to $\frac{m}{\sqrt{\rho}}$ in the region $\mathcal{V}^c$, it holds

$$\int_{(\mathcal{V}_{k,M})^c\setminus\mathcal{V}} |\sqrt{n_k}u_k - \frac{m}{\sqrt{n}}|\mathrm{d}x\mathrm{d}t \to 0, \text{ as } k \to \infty.$$

On the other hand, the fact that $\sqrt{n_k}u_k$ is uniformly bounded in $L^\infty(0,T;L^3(\mathbb{T}^N))$ together with Tchebychev's inequality leads to

$$|\mathcal{V}_{k,M}| \leq \frac{C}{\mathrm{M}^2},$$

from which it follows as $\mathrm{M} \to \infty$ that

$$\int_{\mathcal{V}_{k,M}} |\sqrt{n_k}u_k - \frac{m}{\sqrt{n}}|\mathrm{d}x\mathrm{d}t \leq \sqrt{|\mathcal{V}_{k,M}|}(\|\sqrt{n_k}u_k\|_{L^2} + \|\tfrac{m}{\sqrt{n}}\|_{L^2}) \leq \frac{C}{\mathrm{M}} \to 0.$$

What left is to control the second term on the right hand side of (3.41). Since it holds on the region $(\mathcal{V}_{k,M})^c \cap \mathcal{V}$, where $n_k \to 0$ as $k \to \infty$, that

$$|\sqrt{n_k}u_k| \leq \mathrm{M}(n_k)^{\frac{1}{2}-\frac{1}{3}} \to 0,$$

and $\sqrt{n_k}u_k$ is uniformly bounded in $L^\infty(0,T;L^2(\mathbb{T}^N)$ with respect to n, we conclude as $k \to \infty$ that

$$\int_{(\mathcal{V}_{k,M})^c\cap\mathcal{V}} |\sqrt{n_k}u_k|\mathrm{d}x\mathrm{d}t \to 0.$$

This together with the fact that $\frac{m}{\sqrt{\rho}} = 0$ on $\mathcal{V}$ yields $1_{(\mathcal{V}_{k,M})^c\cap\mathcal{V}}\frac{m}{\sqrt{\rho}}(x,t) = 0$, a.e. for any integer k. Thus, we can conclude that the second term of (3.41) goes to zero as $k \to \infty$. Combining all the arguments above, using the diagonal principle, we have that $\sqrt{n_k}\, u_k$ converges to $\sqrt{n}u =: \frac{m}{\sqrt{n}}$ in $L^1((0,T)\times\mathbb{T}^N)$ strongly as $k \to \infty$. Therefore, we also have

$$n_k|u_k|u_k \longrightarrow n|u|u \quad \text{ in } L^{3/2}((0,T)\times\mathbb{T}^N). \quad (3.42)$$

The limits of the nonlinear diffusion, nonlinear dispersion, and electric potential force can be justified as follows. There exist functions $\overline{n^\alpha u_x} \in L^2(\Omega\times(0,T))$ and $\overline{\sqrt{\varphi'(n)}\Delta\varphi(n)} \in L^2(\Omega\times(0,T))$ so that

$$n_k^\alpha u_{kx} \rightharpoonup \overline{n^\alpha u_x} \quad \text{ weakly in } L^2(\Omega\times(0,T)), \quad (3.43)$$

$$\sqrt{\varphi'(n_k)}\Delta\varphi(n_k) \rightharpoonup \overline{\sqrt{\varphi'(n)}\Delta\varphi(n)} \quad \text{weakly in } L^2(\Omega\times(0,T)). \tag{3.44}$$

In one-dimension, the nonlinear diffusion term becomes

$$2\eta\mathrm{div}\,(\mu(n)D(u)) + \eta\nabla(\lambda(n)\mathrm{div}u) = \eta\alpha(n^\alpha u_x)_x.$$

It is easy to derive for any test function ψ defined on $[0,T]\times\mathbb{T}^N$ that

$$\begin{aligned}
\int \mu_\varepsilon(n_k)u_{kx}\psi\mathrm{d}x\mathrm{d}t &= \int n_k^\alpha u_{kx}\psi\mathrm{d}x\mathrm{d}t + \varepsilon\int n_k^\theta u_{kx}\psi\mathrm{d}x\mathrm{d}t \\
&= -\int n_k^\alpha u_k\psi_x\mathrm{d}x\mathrm{d}t - \int u_k(n_k^\alpha)_x\psi\mathrm{d}x\mathrm{d}t + \varepsilon\int n_k^\theta u_{kx}\psi\mathrm{d}x\mathrm{d}t \\
&= -\int n_k^\alpha u_k\psi_x\mathrm{d}x\mathrm{d}t - \int n_k^{1/2}u_k\frac{(n_k^\alpha)_x}{\sqrt{n_k}}\psi\mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon\int n_k^\theta u_{kx}\psi\mathrm{d}x\mathrm{d}t \\
&\longrightarrow -\int n^\alpha u\psi_x\mathrm{d}x\mathrm{d}t - \int\sqrt{n}\,u\overline{\tfrac{\mu(n)_x}{\sqrt{n}}}\psi\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.45}$$

The limiting nonlinear dispersion term and the limiting electric force term are obtained, in the sense of distribution, as

$$\begin{aligned}
-\int n_k(\varphi'(n_k)\varphi(n_k)_{xx})_x\psi\mathrm{d}x\mathrm{d}t &= \int (n_k\varphi'(n_k)\varphi(n_k)_{xx}\psi_x + n_{kx}\varphi'(n_k)\varphi(n_k)_{xx}\psi)\,\mathrm{d}x\mathrm{d}t \\
&= \int n_k\varphi'(n_k)\varphi(n_k)_{xx}\psi_x\mathrm{d}x\mathrm{d}t + \int \varphi(n_k)_x\varphi(n_k)_{xx}\psi\mathrm{d}x\mathrm{d}t \\
&= -\int((n_k\varphi'(n_k))_x\varphi(n_k)_x\psi_x + n_k\varphi'(n_k)\varphi(n_k)_x\psi_{xx})\,\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int\varphi(n_k)_x^2\psi_x\mathrm{d}x\mathrm{d}t \\
&\longrightarrow -\int((n\varphi'(n))_x\varphi(n)_x\psi_x + n\varphi'(n)\varphi(n)_x\psi_{xx})\,\mathrm{d}x\mathrm{d}t - \frac{1}{2}\int\varphi(n)_x^2\psi_x\mathrm{d}x\mathrm{d}t \\
&= -\int n(\sqrt{\varphi'(n)}\,\overline{\sqrt{\varphi'(n)}\varphi(n)_{xx}}\,)_x\psi\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{3.46}$$

and

$$\int n_k\Phi_{kx}\psi\mathrm{d}x\mathrm{d}t \longrightarrow \int n\Phi_x\psi\mathrm{d}x\mathrm{d}t. \tag{3.47}$$

In addition, one can verify easily that the limiting function (n,u,Φ) satisfies the equations (1.4)–(1.6) in the sense of distribution and the following energy estimates

$$\begin{aligned}
\sup_{0\le t\le T}\int_{\mathbb{T}^N}(|\overline{\tfrac{\mu(n)_x}{\sqrt{n}}}(t)|^2 + \varepsilon^2|\mu(n)_x(t)|^2 + |\Phi_x(t)|^2 + |\sqrt{n}u(t)|^2 + n^\gamma)\mathrm{d}x \\
+ \int_0^T\int_{\mathbb{T}^N}(|\left(n^{(\alpha+\gamma-1)/2}\right)_x|^2 + |n^{\alpha/2}u_x|^2 + |n^{1/3}u|^3 \\
+ \varepsilon^2|\overline{\sqrt{\varphi'(n)}\Delta\varphi(n)}\,|^2)\mathrm{d}x\mathrm{d}t \le C_0,
\end{aligned} \tag{3.48}$$

with C_0 independent of T. This completes the proof of Theorem 2.1.

Proof of Theorem 2.2 We consider the zero dispersion limit and prove the convergence of weak solutions $(n_\varepsilon, u_\varepsilon, \Phi_\varepsilon)$ of the IVP problem (1.4)–(1.8) to the weak solution (ρ, v, V) of the Navier-Stokes-Poisson system (2.12), (1.7) and (1.8). Indeed, the dispersion limit and the convergence from the equations (1.4)–(1.6) to the Navier-Stokes-Poisson system (2.12) can be justified by a similar compactness argument to those used in the proof of existence theory under modification, with the help of the a-priori estimates (2.5) and the Lions-Aubin lemma. Without the loss of generality, we only consider the cases $N = 2, 3$ here, since the dispersion limit for $N = 1$ can be established by a similar arguments in proving Theorem 2.1.

For any global weak solution $(n_\varepsilon, u_\varepsilon, \Phi_\varepsilon)$ of the IVP problem (1.4)–(1.8) in the sense of Definition 2.1, it follows from (2.5) with $\alpha \in (2/3, 1]$ and $\gamma \geq 2$ that

$$\left.\begin{array}{l} n_\varepsilon^\alpha \in L^\infty(0,T; L^{\gamma/\alpha}(\mathbb{T}^N)) \\ \nabla n_\varepsilon^\alpha \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^N)) \end{array}\right\} \Rightarrow n_\varepsilon^\alpha \in L^\infty(0,T; W^{1,\frac{2\gamma}{\gamma+1}}(\mathbb{T}^N)),$$

$$\left.\begin{array}{l} n_\varepsilon^{\alpha/2} \in L^\infty(0,T; L^{2\gamma/\alpha}(\mathbb{T}^N)) \\ \nabla n_\varepsilon^{\alpha/2} \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1-\alpha}}(\mathbb{T}^N)) \end{array}\right\} \Rightarrow n_\varepsilon^{\alpha/2} \in L^\infty(0,T; W^{1,\frac{2\gamma}{\gamma+1-\alpha}}(\mathbb{T}^N)),$$

$$(n_\varepsilon^\alpha)_t \in L^2(0,T; W^{-1,1}(\mathbb{T}^N)), \quad (n_\varepsilon^{\alpha/2})_t \in L^2(0,T; W^{-1,1}(\mathbb{T}^N)).$$

This implies the existence of $0 \leq \rho \in C(0,T; L^p(\mathbb{T}^N))$ with $p \in [1, \frac{2\gamma\alpha N}{\gamma(N-2)+N})$, so that it holds after extracting a subsequence that

$$\left.\begin{array}{lll} n_\varepsilon^{\alpha/2} \longrightarrow \rho^{\alpha/2} & \text{in } C(0,T; L^p(\mathbb{T}^N)), & p \in [1, \frac{2\gamma N}{\gamma(N-2)+N(1-\alpha)}), \\ n_\varepsilon \longrightarrow \rho & \text{in } C(0,T; L^p(\mathbb{T}^N)), & p \in [1, \frac{2\gamma\alpha N}{\gamma(N-2)+N}), \\ n_\varepsilon^\gamma \longrightarrow \rho^\gamma & \text{in } L^1(0,T; L^1(\mathbb{T}^N)), & \end{array}\right\} \tag{3.49}$$

and then

$$\nabla\Phi_\varepsilon \longrightarrow \nabla V \quad \text{in } C([0,T] \times H^1(\mathbb{T}^N)).$$

Next, we show, with the help of (2.5), that there are v and J so that for some fixed small $\delta_1 > 0$ it holds

$$J_\varepsilon = n_\varepsilon u_\varepsilon \longrightarrow J \quad \text{in } L^2(0,T; L^{1+\delta_1}(\mathbb{T}^N)), \tag{3.50}$$

$$\sqrt{n_\varepsilon} u_\varepsilon \longrightarrow \sqrt{\rho} v \quad \text{in } L^1((0,T) \times \mathbb{T}^N), \tag{3.51}$$

$$\int_{\mathbb{T}^N} \frac{|J|^2}{\rho} \mathrm{d}x \leq C. \tag{3.52}$$

Indeed, we can prove that $J_\varepsilon = n_\varepsilon u_\varepsilon$ converges strongly to some J in $L^2(0,T; L^{1+\delta_1}(\mathbb{T}^N))$ with $\delta_1 > 0$ small, due to the facts $J_\varepsilon \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^N))$, and

$$\nabla(n_\varepsilon u_\varepsilon) = n_\varepsilon^{1-\alpha/2} n_\varepsilon^{\alpha/2} \nabla u + n_\varepsilon^{7/6-\alpha} \nabla n_\varepsilon^{\alpha-1/2} \otimes n_\varepsilon^{1/3} u \in L^1(0,T; L^{p_1}(\mathbb{T}^N))$$

for some constant $p_1 > 1$ so that $p_1^{-1} = 5/6 + r_1^{-1}$ with any $r_1 \in (6, \infty)$ for $N = 2$, or $r_1 = \frac{6N(\alpha+1)}{(7-6\alpha)(N-2)} > 6$ for $N = 3$, where we recall $n_\varepsilon \in L^{\alpha+1}(0, T; L^p(\mathbb{T}^N))$ with $p \in [1, \frac{N(\alpha+1)}{N-2})$ for $N = 3$, and

$$\begin{aligned}(n_\varepsilon u_\varepsilon)_t =& - \operatorname{div}(n_\varepsilon u_\varepsilon \otimes u_\varepsilon) - \nabla n_\varepsilon^\gamma - n_\varepsilon \nabla \Phi_\varepsilon \\ &+ \operatorname{div}(n_\varepsilon^\alpha \nabla u_\varepsilon) + (\alpha - 1)\nabla(n_\varepsilon^\alpha \operatorname{div} u_\varepsilon) \\ &- n_\varepsilon |u_\varepsilon| u_\varepsilon + \varepsilon^2 n_\varepsilon \nabla(\mu'(n_\varepsilon)\Delta\mu(n_\varepsilon)) \quad \in L^1(0, T; W^{-1,1}(\mathbb{T}^N)).\end{aligned}$$

Moreover, since $\frac{J_k}{\sqrt{n_k}}$ is uniformly bounded in $L^\infty(0, T; L^2(\mathbb{T}^N))$, and $J(x, t) = 0$ a.e. on $\{\rho(x, t) = 0\}$, thus, if we define $\frac{|J|^2}{\rho}$ to be zero as $\rho = 0$, then by Fatou's lemma, we have (3.52). Again, with a similar argument as above, we can prove the strong convergence (3.51). The strong convergence

$$n_\varepsilon |u_\varepsilon| u_\varepsilon \longrightarrow \sqrt{\rho}\, |v| \sqrt{\rho}\, v \quad \text{ in } L^1((0, T) \times \mathbb{T}^N) \tag{3.53}$$

follows from (3.51) and the fact

$$\sqrt{n_\varepsilon}\, u_\varepsilon = n_\varepsilon^{1/6} n_\varepsilon^{1/3} u_\varepsilon \in L^3(0, T; L^{\frac{6\gamma}{2\gamma+1}}(\mathbb{T}^N)) \hookrightarrow L^{\frac{6\gamma}{2\gamma+1}}((0, T) \times \mathbb{T}^N).$$

Similarly, the weak convergence of nonlinear term $\rho_\varepsilon^\alpha \nabla u_\varepsilon$ can be obtained too.

We can now perform dispersion limit on the global solutions $(n_\varepsilon, u_\varepsilon, \Phi_\varepsilon)$. Indeed, we can write the system with test function $\psi \in C_{per}^\infty([0, T] \times \mathbb{T}^N)$ as

$$0 = \int\int (n_\varepsilon \partial_t \psi + \sqrt{n_\varepsilon}\sqrt{n_\varepsilon} u_\varepsilon \cdot \nabla\psi)\, \mathrm{d}x\mathrm{d}t \to \int\int (\rho\partial_t\psi + \sqrt{\rho}\sqrt{\rho} v \cdot \nabla\psi)\, \mathrm{d}x\mathrm{d}t,$$

$$\begin{aligned}0 =& \int\int (n_\varepsilon u_\varepsilon \cdot \partial_t \psi + n_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla\psi + n_\varepsilon^\gamma \operatorname{div}\psi)\, \mathrm{d}x\mathrm{d}t \\ &- \int\int \mu(n_\varepsilon) D(u_\varepsilon) : D(\psi)\, \mathrm{d}x\mathrm{d}t - \int\int \lambda(n_\varepsilon) \operatorname{div} u_\varepsilon \operatorname{div}\psi\, \mathrm{d}x\mathrm{d}t \\ &+ \int\int n_\varepsilon |u_\varepsilon| u_\varepsilon \cdot \psi\, \mathrm{d}x\mathrm{d}t \\ &- \int\int n\psi \cdot \nabla\Phi_\varepsilon\, \mathrm{d}x\mathrm{d}t - \varepsilon^2 \int\int n_\varepsilon \psi \cdot \nabla(\mu'(n_\varepsilon)\Delta(\mu(n_\varepsilon)))\, \mathrm{d}x\mathrm{d}t \qquad (3.54) \\ &\longrightarrow \int\int (\sqrt{\rho}\sqrt{\rho} v \cdot \partial_t\psi + \sqrt{\rho} v \otimes \sqrt{\rho} v : \nabla\psi + \rho^\gamma \operatorname{div}\psi)\, \mathrm{d}x\mathrm{d}t \\ &\qquad - \int\int \sqrt{\mu(\rho)}\sqrt{\mu(\rho)} D(v) : D(\psi)\, \mathrm{d}x\mathrm{d}t - \int\int \sqrt{\lambda(\rho)}\sqrt{\lambda(\rho)} \operatorname{div} v \operatorname{div}\psi\, \mathrm{d}x\mathrm{d}t \\ &\qquad + \int\int \sqrt{\rho}\, |v| \sqrt{\rho}\, v \cdot \psi\, \mathrm{d}x\mathrm{d}t - \int\int \rho\nabla V \cdot \psi\, \mathrm{d}x\mathrm{d}t \qquad (3.55)\end{aligned}$$

and

$$0 = \int\int \nabla\Phi_\varepsilon \cdot \nabla\psi\, \mathrm{d}x\mathrm{d}t + \int\int n_\varepsilon\psi\, \mathrm{d}x\mathrm{d}t \to \int\int \nabla V \cdot \nabla\psi\, \mathrm{d}x\mathrm{d}t + \int\int \rho\psi\, \mathrm{d}x\mathrm{d}t$$

with the help of above convergence (3.49)–(3.51), and the fact

$$\varepsilon^2 \int\int n_\varepsilon \psi \cdot \nabla(\mu'(n_\varepsilon)\Delta\mu(n_\varepsilon)))\,\mathrm{d}x\mathrm{d}t \to 0 \tag{3.56}$$

for ψ any test function, which is derived due to

$$\begin{aligned}\varepsilon^2 \int\int n_\varepsilon \psi \cdot \nabla(\mu'(n_\varepsilon)\Delta\mu(n_\varepsilon))\mathrm{d}x\mathrm{d}t &= -\varepsilon^2 \int\int \mathrm{div}(n_\varepsilon\psi)\mu'(n_\varepsilon)\Delta\mu(n_\varepsilon)\mathrm{d}x\mathrm{d}t \\ \le & C\varepsilon\|\sqrt{\mu'(n_\varepsilon)}\,\Delta\mu(n_\varepsilon)\|_{L^2}\cdot\varepsilon\|n_\varepsilon^{(\alpha+1)/2}\|_{L^2} + C\varepsilon\|\sqrt{\mu'(n_\varepsilon)}\,\Delta\mu(n_\varepsilon)\|_{L^2}\cdot\varepsilon\|\frac{\nabla\mu(n_\varepsilon)}{\sqrt{\mu'(n_\varepsilon)}}\|_{L^2} \\ \le & C\varepsilon(\|n_\varepsilon^{(\alpha+1)/2}\|_{L^2} + \|\nabla n_\varepsilon^{(\alpha+1)/2}\|_{L^2}) \to 0\end{aligned} \tag{3.57}$$

as $\varepsilon \to 0_+$. The proof of Theorem 2.2 is completed.

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