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HYPERBOLIC-PARABOLIC CHEMOTAXIS SYSTEM WITH NONLINEAR PRODUCT TERMS*

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Abstract We prove the local existence and uniqueness of week solution of the hyperbolic-parabolic Chemotaxis system with some nonlinear product terms. For one dimensional case, we prove also the global existence and uniqueness of the solution for the problem.

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1. Introduction

Let $u(x,t)$ and $v(x,t)$ represent the population of an organism and an external signal at place $x \in \Omega \subset R^N$ and time t respectively, in general speaking, the external signal is produced by the individuals and decays, which is described by a nonlinear function $g(v,u)$. Under the spatial spread of the external signal is driven by diffusion, the full system for u and v reads (see [1-3])

$$u_t = \nabla(d\nabla u - \chi(v)\nabla v \cdot u), \tag{1}$$

$$v_t = d\Delta v + g(v,u). \tag{2}$$

In the case of that the external stimulus were based on the light (or the electro-magnetic wave), H. Chen and S. Wu [4] studied following hyperbolic-parabolic type chemotaxis system:

$$u_t = \nabla(d\nabla u - \chi(v)\nabla v \cdot u), \tag{3}$$

$$v_{tt} = d\Delta v + g(v,u), \tag{4}$$

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where v represents the potential function of the external signal, for example, if the external signal is the electromagnetic field, then v would be voltage (in this case ∇v denotes the electromagnetic field).

The result of [4] gives the existence and uniqueness of the solution for the system (3)-(4) with Neumann boundary value condition on a smoothly bounded open domain Ω and $g(v,u) = -v + f(u)$. In this paper, we shall study the case for more general nonlinear term $g(v,u)$.

Throughout this article, we assume that we can choose a constant σ, satisfying

$$1 < \sigma < 2, \tag{5}$$

$$N < 2\sigma < N+2, \tag{6}$$

$$\sigma - 1 \geq \frac{N}{4}, \tag{7}$$

where $1 \leq N \leq 3$ are space dimensions.

It is easy to check that there exists some constant σ such that the three conditions above can be simultaneously satisfied in the cases of $1 \leq N \leq 3$. In fact, we can choose $\sigma = \frac{5}{4}$ for $N = 1$, $\sigma = \frac{13}{8}$ for $N = 2$ and $\sigma = \frac{15}{8}$ for $N = 3$.

Next, we define

$$\begin{aligned}
&X_{t_0} = C([0,t_0], H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \;\; on \;\; \partial\Omega\}),\\
&X_\infty = C([0,+\infty), H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \;\; on \;\; \partial\Omega\}),\\
&Y_{t_0} = C([0,t_0], H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \;\; on \;\; \partial\Omega\}) \cap C^1([0,t_0], H^1(\Omega)),\\
&Y_\infty = C([0,+\infty), H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \;\; on \;\; \partial\Omega\}) \cap C^1([0,+\infty), H^1(\Omega)),
\end{aligned}$$

and

$$\begin{aligned}
&Z_{t_0} = C^1([0,t_0], L^2(\Omega)), \qquad Z_\infty = C^1([0,\infty), L^2(\Omega))),\\
&W_{t_0} = C^2([0,t_0], L^2(\Omega)), \qquad W_\infty = C^2([0,\infty), L^2(\Omega)).
\end{aligned}$$

2. Local Existence and Uniqueness for $g(u,v) = \alpha uv$

In this section we consider following system in which the nonlinear function $g(u,v)$ is a product term:

$$\begin{cases}
u_t = \nabla(\nabla u - \chi u \nabla v), \quad \text{in} \quad \Omega \times (0,T),\\
v_{tt} = \Delta v + \alpha uv, \quad \text{in} \quad \Omega \times (0,T),\\
\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \quad \text{on} \quad \partial\Omega \times (0,T),\\
u(0,\cdot) = u_0, \quad v(0,\cdot) = \varphi, \quad v_t(0,\cdot) = \psi, \quad \text{in} \quad \Omega,
\end{cases} \tag{8}$$

where $\Omega \subset R^N$, a bounded open domain Ω with smooth boundary, α is a constant and χ is a nonnegative constant.

We have following result

Theorem 2.1 *Under the conditions* (5), (6) *and* (7), *for each initial data* $u_0 \in H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$, $\varphi \in H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\}$ *and* $\psi \in H^1(\Omega)$, *the problem* (8) *has a unique local solution* $(u, v) \in (X_{t_0} \cap Z_{t_0}) \times (Y_{t_0} \cap W_{t_0})$.

Let us divide the system (8) into two parts,

$$\begin{cases} u_t = \nabla(\nabla u - \chi u \nabla v), & \text{in } \Omega \times (0, T), \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0, \cdot) = u_0, & \text{in } \Omega, \end{cases} \tag{9}$$

and

$$\begin{cases} v_{tt} = \Delta v + \alpha u v, & \text{in } \Omega \times (0, T), \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega \times (0, T), \\ v(0, \cdot) = \varphi, \quad v_t(0, \cdot) = \psi, & \text{in } \Omega. \end{cases} \tag{10}$$

For $v \in Y_{t_0}$ fixed, by using the proposition [5, p.273], we can deduce that for each $u_0 \in H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$ and $v \in Y_{t_0}$, the problem (9) has a unique solution $u \in X_{t_0} \cap Z_{t_0}$ (also see the proof of [4, Lemma 3.2]).

Secondly we have

Lemma 2.2 *Assume* σ *satisfies the conditions* (5), (6) *and* (7), *then for the solution* $u \in X_{t_0} \cap Z_{t_0}$ *of* (9), *there exists a constant* C, *which is independent of* t_0, *such that*

$$\|u\|_{X_{t_0}} \le C \|u_0\|_{H^\sigma} + C t_0^{1-\frac{\sigma}{2}} \|v\|_{Y_{t_0}} \cdot \|u\|_{X_{t_0}}. \tag{11}$$

Proof See [4, Lemma 3.3].

Lemma 2.3 *If* $u \in X_{t_0}$, $v \in L^2(0, t_0; W^{1,4}(\Omega))$, *then* $uv \in L^2(0, t_0; H^1(\Omega))$ *and*

$$\|uv\|_{H^1} \le c(\|u\|_{L^4} \cdot \|v\|_{W^{1,4}} + \|v\|_{L^4} \cdot \|u\|_{W^{1,4}}), \quad 0 \le t \le t_0, \tag{12}$$

for some positive constant c.

Proof Since $u \in X_{t_0}$, we have that $u(t, \cdot) \in H^\sigma(\Omega)$ for each t $(0 \le t \le t_0)$. By Sobolev imbedding theorem, we know that $H^\sigma \subset W^{1, \frac{2N}{N-2(\sigma-1)}}$.

The condition (7) implies that $\frac{2N}{N-2(\sigma-1)} \ge 4$, which means $u \in W^{1,4}$. By virtue of Cauchy inequality, we have

$$\|uv\|_{L^2} \le \|u\|_{L^4} \cdot \|v\|_{L^4}, \quad 0 \le t \le t_0. \tag{13}$$

Furthermore $\nabla(uv) = \nabla u \cdot v + u \cdot \nabla v$, which implies that

$$\|\nabla(uv)\|_{L^2} \le \|\nabla u \cdot v\|_{L^2} + \|u \cdot \nabla v\|_{L^2}, \quad 0 \le t \le t_0. \tag{14}$$

Thus we use Cauchy inequality to get

$$\|\nabla u\cdot v\|_{L^2}\leq\|\nabla u\|_{L^4}\cdot\|v\|_{L^4}\quad 0\leq t\leq t_0,\tag{15}$$

$$\|\nabla v\cdot u\|_{L^2}\leq\|\nabla v\|_{L^4}\cdot\|u\|_{L^4}\quad 0\leq t\leq t_0.\tag{16}$$

Lemma 2.3 is proved.

Similar to [4, Lemma 3.1], we can deduce that

Lemma 2.4 *For each $T>0$, assume*

$$\varphi\in H^2(\Omega)\cap\{\frac{\partial u}{\partial n}=0\quad on\quad\partial\Omega\},\qquad\psi\in H^1(\Omega),\qquad u\in\quad X_{t_0},$$

then the problem (10) *has a unique solution $v\in Y_{t_0}\cap W_{t_0}$ and*

$$\|v\|_{Y_{t_0}}\leq ce^{|\alpha|t_0}(\|\varphi\|_{H^2}+\|\psi\|_{H^1}+t_0\|u\|_{X_{t_0}}\cdot\|v\|_{Y_{t_0}}).\tag{17}$$

Proof of Theorem 2.1 Let $g\in X_{t_0}$, $g(0)=u_0$ and $v=v(g)$ denotes the corresponding solution of following equation

$$\begin{cases}v_{tt}=\Delta v+\alpha gv,\\ \dfrac{\partial v}{\partial n}=0,\quad\text{on}\quad\partial\Omega\times(0,T),\\ v(0,\ \cdot)=\varphi,\quad v_t(0,\ \cdot)=\psi.\end{cases}\tag{18}$$

By using Lemma 2.4, we know that the problem (18) has a unique local solution $v\in Y_{t_0}$, which satisfies

$$\|v\|_{Y_{t_0}}\leq ce^{|\alpha|t_0}(\|\varphi\|_{H^2}+\|\psi\|_{H^1})+ct_0e^{|\alpha|t_0}\|g\|_{X_{t_0}}\cdot\|v\|_{Y_{t_0}}.\tag{19}$$

For the solution v above, let $u=u(v(g))$ be the corresponding solution of

$$\begin{cases}u_t=\nabla(\nabla u-\chi u\nabla v),\\ \dfrac{\partial u}{\partial n}=0,\quad\text{on}\quad\partial\Omega\times(0,T),\\ u(0,\cdot)=u_0=g(0).\end{cases}\tag{20}$$

The lemma 2.2 shows that the solution $u\in X_{t_0}$, thus we have obtained a mapping $G:X_{t_0}\to X_{t_0}$, defined by $Gg=u(v(g))$, and the estimate (11) gives

$$\|Gg\|_{X_{t_0}}\leq C\|u_0\|_{H^\sigma}+Ct_0^{1-\frac{\sigma}{2}}\cdot\|v\|_{Y_{t_0}}\cdot\|Gg\|_{X_{t_0}}.\tag{21}$$

Next, we choose a ball $B_M=\{g\in X_{t_0}\,|g(0)=u_0,\|g(t,\cdot)\|_{H^\sigma}\leq M\ ,\ 0\leq t\leq t_0\}$, where $M=2C\|u_0\|_{H^\sigma}$ and the constant $C(\geq 1)$ is given by (11).

For $g\in B_M$, we know that from the estimate (19)

$$\begin{aligned}\|v\|_{Y_{t_0}}&\leq ce^{|\alpha|t_0}(\|\varphi\|_{H^2}+\|\psi\|_{H^1})+ct_0e^{|\alpha|t_0}\|g\|_{X_{t_0}}\cdot\|v\|_{Y_{t_0}}\\&\leq ce^{|\alpha|t_0}(\|\varphi\|_{H^2}+\|\psi\|_{H^1})+ct_0e^{|\alpha|t_0}M\cdot\|v\|_{Y_{t_0}}.\end{aligned}\tag{22}$$

Thus we choose $t_0 > 0$ small enough, then

$$\|v\|_{Y_{t_0}} \le 2ce^{|\alpha|t_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1}). \tag{23}$$

In view of (21) and (22), we obtain

$$\|Gg\|_{X_{t_0}} \le C\|u_0\|_{H^\sigma} + Ct_0^{1-\frac{\sigma}{2}} \cdot 2ce^{|\alpha|t_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1}) \cdot \|Gg\|_{X_{t_0}}.$$

Choosing t_0 small enough, so that $2ct_0 \cdot e^{|\alpha|t_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1}) \le \frac{1}{2}$, then $\|Gg\|_{X_{t_0}} \le 2C\|u_0\|_{H^\sigma} = M$, which indicates that G maps B_M into B_M itself.

Next, we shall prove that G is contract mapping for sufficiently small t_0.

Let us consider that g_1, $g_2 \in X_{t_0}$, and let v_i $(i = 1, 2)$ be the corresponding solutions of the problem (18). Thus the difference $Gg_1 - Gg_2$ satisfies:

$$\begin{aligned}
Gg_1 - Gg_2 &= u_1 - u_2 \\
&= -\chi \int_0^t T(t-s)u_1\Delta v_1 \mathrm{d}s - \chi \int_0^t T(t-s)\nabla u_1 \nabla v_1 \mathrm{d}s \\
&\quad + \chi \int_0^t T(t-s)u_2\nabla v_2 \mathrm{d}s + \chi \int_0^t T(t-s)\nabla u_2 \nabla v_2 \mathrm{d}s \\
&= -\chi \int_0^t T(t-s)(u_1\Delta v_1 - u_2\Delta v_2)\mathrm{d}s - \chi \int_0^t T(t-s)(\nabla u_1\nabla v_1 - \nabla u_2\nabla v_2)\mathrm{d}s,
\end{aligned}$$

where $T(t) = e^{t\Delta}$ is a mapping from $L^2(\Omega)$ to $H^\sigma(\Omega)$ with norm $c_\sigma t^{-\frac{\sigma}{2}}$. Thus we have

$$\begin{aligned}
&\left\|\int_0^t T(t-s)(u_1\Delta v_1 - u_2\Delta v_2)\mathrm{d}s\right\|_{H^\sigma} \\
&\quad \le \left\|\int_0^t T(t-s)u_1(\Delta v_1 - \Delta v_2)\mathrm{d}s\right\|_{H^\sigma} + \left\|\int_0^t T(t-s)(u_1 - u_2)\Delta v_2 \mathrm{d}s\right\|_{H^\sigma},
\end{aligned}$$

and

$$\begin{aligned}
\left\|\int_0^t T(t-s)u_1(\Delta v_1 - \Delta v_2)\mathrm{d}s\right\|_{H^\sigma} &\le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|u_1(\Delta v_1 - \Delta v_2)\|_2 \\
&\le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|u_1\|_{L^\infty} \cdot \|\Delta(v_1 - v_2)\|_2,
\end{aligned}$$

where $\|\cdot\|_p$ as the norm of L^p, and $\sigma > \frac{N}{2}$, so

$$\begin{aligned}
\left\|\int_0^t T(t-s)u_1(\Delta v_1 - \Delta v_2)\mathrm{d}s\right\|_{H^\sigma} &\le Ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|u_1\|_{H^\sigma} \cdot \|\Delta(v_1 - v_2)\|_2 \\
&\le CMt_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|v_1 - v_2\|_{H^2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left\|\int_0^t T(t-s)(u_1-u_2)\Delta v_2 \mathrm{d}s\right\|_{H^\sigma} &\le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|(u_1-u_2)\Delta v_2\|_2 \\ &\le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|v_2\|_{H^2}\cdot\|u_1-u_2\|_{L^\infty} \\ &\le ct_0^{1-\frac{\sigma}{2}} \|v_2\|_{Y_{t_0}}\cdot\|u_1-u_2\|_{X_{t_0}},\end{aligned}$$

which means

$$\begin{aligned}&\left\|\int_0^t T(t-s)(u_1\Delta v_1-u_2\Delta v_2)\mathrm{d}s\right\|_{H^\sigma} \\ &\quad\le Ct_0^{1-\frac{\sigma}{2}}\|v_1-v_2\|_{Y_{t_0}}+Ct_0^{1-\frac{\sigma}{2}}\|v_2\|_{Y_{t_0}}\cdot\|u_1-u_2\|_{X_{t_0}}, \quad 0\le t\le t_0.\end{aligned}$$

Secondly, we have

$$\begin{aligned}\left\|\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s\right\|_{H^\sigma} &\le \left\|\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_1)\mathrm{d}s\right\|_{H^\sigma} \\ &\quad+\left\|\int_0^t T(t-s)(\nabla u_2\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s\right\|_{H^\sigma}.\end{aligned}$$

Here

$$\begin{aligned}&\left\|\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_1)\mathrm{d}s\right\|_{H^\sigma} \\ &\quad\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le t\le t_0}\|\nabla v_1\cdot\nabla(u_1-u_2)\|_2, \quad 0\le t\le t_0.\end{aligned}$$

By Sobolev imbedding theorem, $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$ for $N=1$, we have

$$\begin{aligned}\|\nabla f_1\nabla f_2\|_2 &\le \|\nabla f_1\|_\infty\cdot\|\nabla f_2\|_2 \\ &\le c\|f_1\|_{H^2}\cdot\|f_2\|_{H^1} \\ &\le c\|f_1\|_{H^2}\cdot\|f_2\|_{H^\sigma}.\end{aligned}$$

For $N=2,3$, we have $H^1(\Omega)\hookrightarrow L^{\frac{N}{\sigma-1}}(\Omega)$, $H^{\sigma-1}(\Omega)\hookrightarrow L^{\frac{2N}{N-2(\sigma-1)}}(\Omega)$, thus $f_1^2\in L^{\frac{N}{2(\sigma-1)}}$, $f_2^2\in L^{\frac{N}{N-2(\sigma-1)}}$ if $f_1\in H^1$ and $f_2\in H^{\sigma-1}$. Hence Cauchy inequality yields

$$\left\|f_1^2f_2^2\right\|_1\le\left\|f_1^2\right\|_{\frac{N}{2(\sigma-1)}}\cdot\left\|f_2^2\right\|_{\frac{N}{N-2(\sigma-1)}},$$

which implies $\|f_1f_2\|_2\le\|f_1\|_{\frac{N}{\sigma-1}}\cdot\|f_2\|_{\frac{2N}{N-2(\sigma-1)}}$. Thus

$$\begin{aligned}\|\nabla f_1\nabla f_2\|_2 &\le \|\nabla f_1\|_{\frac{N}{\sigma-1}}\cdot\|\nabla f_2\|_{\frac{2N}{N-2(\sigma-1)}} \\ &\le c\|\nabla f_1\|_{H^1}\cdot\|\nabla f_2\|_{\frac{2N}{N-2(\sigma-1)}} \\ &\le c\|f_1\|_{H^2}\cdot\|\nabla f_2\|_{H^{\sigma-1}}\le c\|f_1\|_{H^2}\cdot\|f_2\|_{H^\sigma}.\end{aligned}$$

Therefore, we obtain

$$\left\|\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_1)\mathrm{d}s\right\|_{H^\sigma}\le Ct_0^{1-\frac{\sigma}{2}}\left\|v_1\right\|_{Y_{t_0}}\cdot\left\|u_1-u_2\right\|_{X_{t_0}},0\le t\le t_0.$$

Similarly

$$\begin{aligned}\left\|\int_0^t T(t-s)(\nabla u_2\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s\right\|_{H^\sigma}&\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le t\le t_0}\left\|\nabla u_2\cdot\nabla(v_1-v_2)\right\|_2\\&\le ct_0^{1-\frac{\sigma}{2}}\left\|u_2\right\|_{X_{t_0}}\cdot\left\|v_1-v_2\right\|_{Y_{t_0}}\\&\le cMt_0^{1-\frac{\sigma}{2}}\left\|v_1-v_2\right\|_{Y_{t_0}},\quad 0\le t\le t_0.\end{aligned}$$

Hence we have deduced that

$$\begin{aligned}\|Gg_1-Gg_2\|_{X_{t_0}}&=\|u_1-u_2\|_{X_{t_0}}\\&\le 2ct_0^{1-\frac{\sigma}{2}}\|v_1-v_2\|_{Y_{t_0}}+ct_0^{1-\frac{\sigma}{2}}(\|v_2\|_{Y_{t_0}}+\|v_1\|_{Y_{t_0}})\cdot\|Gg_1-Gg_2\|_{X_{t_0}}.\end{aligned}\tag{24}$$

Next, the difference v_1-v_2 satisfies

$$\begin{cases}(v_1-v_2)_{tt}=\Delta(v_1-v_2)+\alpha(g_1v_1-g_2v_2)\\ \qquad\qquad\;=\Delta(v_1-v_2)+\alpha v_1(g_1-g_2)+\alpha g_2(v_1-v_2),\\ \dfrac{\partial v_1}{\partial n}=\dfrac{\partial v_2}{\partial n}=0,\quad \text{on}\quad \partial\Omega\times(0,T),\\ (v_1-v_2)(0,\cdot)=0,\quad (v_1-v_2)_t(0,\cdot)=0.\end{cases}$$

Applying Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}\left\|v_1-v_2\right\|_{Y_{t_0}}\le&ce^{|\alpha|t_0}\Big(\int_0^{t_0}\left\|\alpha v_1(g_1-g_2)+\alpha g_2(v_1-v_2)\right\|_{H^1}\mathrm{d}\tau\Big)\\\le&c\left|\alpha\right|e^{|\alpha|t_0}\cdot t_0\cdot\sup_{0\le\tau\le t_0}\left\|v_1(g_1-g_2)+g_2(v_1-v_2)\right\|_{H^1}\\\le&c\left|\alpha\right|e^{|\alpha|t_0}\cdot t_0\cdot\sup_{0\le\tau\le t_0}\left\|v_1(g_1-g_2)\right\|_{H^1}\\&+c\left|\alpha\right|e^{|\alpha|t_0}\cdot t_0\cdot\sup_{0\le\tau\le t_0}\left\|g_2(v_1-v_2)\right\|_{H^1}\\\le&c\left|\alpha\right|e^{|\alpha|t_0}\cdot t_0\cdot\sup_{0\le\tau\le t_0}(\left\|v_1\right\|_{H^2}\cdot\left\|g_1-g_2\right\|_{H^\sigma})\\&+c\left|\alpha\right|e^{|\alpha|t_0}\cdot t_0\cdot\sup_{0\le\tau\le t_0}(\left\|g_2\right\|_{H^\sigma}\cdot\left\|v_1-v_2\right\|_{H^2})\\\le&c\left|\alpha\right|e^{|\alpha|t_0}\cdot t_0\cdot\left\|v_1\right\|_{Y_{t_0}}\cdot\left\|g_1-g_2\right\|_{X_{t_0}}\\&+c\left|\alpha\right|e^{|\alpha|t_0}\cdot t_0\cdot\left\|g_2\right\|_{X_{t_0}}\cdot\left\|v_1-v_2\right\|_{Y_{t_0}}.\end{aligned}$$

Thus we choose t_0 small enough to get

$$\left\|v_1-v_2\right\|_{Y_{t_0}}\le 2c\left|\alpha\right|t_0e^{|\alpha|t_0}\cdot\left\|v_1\right\|_{Y_{t_0}}\cdot\left\|g_1-g_2\right\|_{X_{t_0}}.\tag{25}$$

For t_0 small enough, we use the estimate (23) directly to get

$$\|v_i\|_{Y_{t_0}} \le c_1 e^{|\alpha|t_0} \cdot (\|\varphi\|_{H^2} + \|\psi\|_{H^1}), \quad i = 1, 2. \tag{26}$$

Since $1 - \frac{\sigma}{2} > 0$, thus combining the estimates (24), (25) and (26) and choosing t_0 small enough, we can deduce that the mapping G is contract. So it follows that the problem (8) has a unique local solution $(u, v) \in (X_{t_0} \cap Z_{t_0}) \times (Y_{t_0} \cap W_{t_0})$, Theorem 2.1 is proved.

3. Local and Global Existence and Uniqueness for $g(u, v) = h(v^2)v + f(u)$

Consider the following system

$$\begin{cases} u_t = \nabla(\nabla u - \chi u \nabla v), \quad \text{in} \quad \Omega \times (0, T), \\ v_{tt} = \Delta v + h(v^2)v + f(u), \quad \text{in} \quad \Omega \times (0, T), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \quad \text{on} \quad \partial\Omega \times (0, T), \\ u(0, \cdot) = u_0, \quad v(0, \cdot) = \varphi, \quad v_t(0, \cdot) = \psi, \quad \text{in} \quad \Omega, \end{cases} \tag{27}$$

where $\Omega \subset R^N$ $(1 \le N \le 3)$ is a bounded open domain with smooth boundary, and the functions $h(x)$, $f(x) \in C^2(R)$ satisfying following conditions (cf. [6]):

(a) For $N = 1$, h is bounded on bounded sets and $h'(w)w$ is bounded on bounded sets of R^+.

(b) For $N = 2$, $|h(w)| \le c(w^k + 1)$ and $|h'(w)w| \le c(w^k + 1)$, for $k \ge 0$.

(c) For $N = 3$, $|h(w)| \le c(w^k + 1)$, and $|h'(w)w| \le c(w^k + 1)$, for $0 \le k \le 1$.

Theorem 3.1 *Under the conditions* (a), (b) *and* (c), *if σ satisfies the conditions* (5), (6) *and* (7), *then for each initial data $u_0 \in H^\sigma(\Omega) \cap \{\frac{\partial u}{\partial n} = 0$ on $\partial\Omega\}$, $\varphi \in H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0$ on $\partial\Omega\}$ and $\psi \in H^1(\Omega)$, the problem* (27) *has a unique local solution $(u, v) \in (X_{t_0} \cap Z_{t_0}) \times (Y_{t_0} \cap W_{t_0})$. In the case of $N = 1$, $\sigma = \frac{5}{4}$ and $u_0 \ge 0$, then the one-dimensional problem* (27) *has a unique global solution $(u, v) \in (X_\infty \cap Z_\infty) \times (Y_\infty \cap W_\infty)$.*

The result of Theorem 3.1 has following obviously extension:

Corollary 3.2 *For a positive integer m, if $u_0 \in H^{\sigma+m}(\Omega) \cap \{\frac{\partial u}{\partial n} = 0$ on $\partial\Omega\}$, $\varphi \in H^{2+m}(\Omega) \cap \{\frac{\partial v}{\partial n} = 0$ on $\partial\Omega\}$, $\psi \in H^{1+m}(\Omega)$ and $f \in C^{2+m}(\mathbf{R})$ (or $f \in C_0^{2+m}(\mathbf{R})$ in the case of $N = 1$ and $\sigma = \frac{5}{4}$), then the solution (u, v) of* (27) *also belongs to $X_{t_0}^m \times Y_{t_0}^m$ (or $X_\infty^m \times Y_\infty^m$ in the case of $N = 1$ and $\sigma = \frac{5}{4}$). Where*

$$X_{t_0}^m = C([0, t_0], H^{\sigma+m}(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \ \ on \ \ \partial\Omega\}),$$

$$X_\infty^m = C([0, +\infty), H^{\sigma+m}(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \ \ on \ \ \partial\Omega\}),$$

$$Y_{t_0}^m = C([0, t_0], H^{2+m}(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \quad on \ \ \partial\Omega\}) \cap C^1([0, t_0], H^{1+m}(\Omega)),$$

$$Y_\infty^m = C([0, +\infty), H^{2+m}(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \quad on \ \ \partial\Omega\}) \cap C^1([0, +\infty), H^{1+m}(\Omega)).$$

In order to prove Theorem 3.1, we divide the system (27) into two parts,

$$\begin{cases} u_t = \nabla(\nabla u - \chi u \nabla v), \quad \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial u}{\partial n} = 0, \quad \text{on} \quad \partial\Omega \times (0,T), \\ u(0,\cdot) = u_0, \quad \text{in} \quad \Omega, \end{cases} \tag{28}$$

and

$$\begin{cases} v_{tt} = \Delta v + h(v^2)v + f(u), \quad \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial v}{\partial n} = 0, \quad \text{on} \quad \partial\Omega \times (0,T), \\ v(0,\ \cdot) = \varphi, \quad v_t(0,\ \cdot) = \psi, \quad \text{in} \quad \Omega. \end{cases} \tag{29}$$

We have

Lemma 3.3 *For each $T > 0$, suppose*

$$\varphi \in H^2(\Omega) \cap \{\frac{\partial u}{\partial n} = 0 \quad \text{on} \ \ \partial\Omega\}, \qquad \psi \in H^1(\Omega), \qquad f(u) \in C([0,T]; H^1(\Omega)),$$

then (29) *has a unique solution $v \in Y_T \cap W_T$ and satisfies*

$$\begin{aligned} &v \in C([0,T]; H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \quad \text{on} \ \ \partial\Omega\}), \\ &v_t \in C([0,T]; H^1(\Omega)), \quad v_{tt} \in C([0,T]; L^2(\Omega)), \end{aligned}$$

and

$$\begin{aligned} &\|v(t,\cdot)\|_{H^2(\Omega)} + \|v_t(t,\cdot)\|_{H^1(\Omega)} \\ &\quad \le e^{cT}(\|\varphi\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)} + \int_0^T \|f(u(\tau,\cdot))\|_{H^1(\Omega)}\,\mathrm{d}\tau), \quad \forall t \in [0,T], \end{aligned} \tag{30}$$

where $C > 0$ is a constant which is independent of T.

Proof Set $v_t = w$, we introduce following system

$$\begin{cases} v_t = w, \\ w_t = \triangle v + h(v^2)v + f(u), \end{cases} \tag{31}$$

then we obtain a abstract form, for $U = (v,w)$:

$$U_t = LU + F(U) \ \ in \ \ X = H^1(\Omega) \times L^2(\Omega), \tag{32}$$

where $D(L) = H^2(\Omega) \cap \{\frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\} \times H^1(\Omega)$ and $L(v,w) = (w, \triangle v - v)$ for $(v,w) \in D(L)$, $F(v,w) = (0, (h(v^2)+1)v + f(u))$.

We know that L is a generator of a unitary group. If we can prove that $F(U) : X \to X$, and $F(U)$ is locally Lipschitz from X to X, then we can obtain the existence of solution of (29).

Firstly we prove that

$$F(U):\ X\to X.$$

For $N=1$ and $v\in H^1(\Omega)\hookrightarrow L^\infty(\Omega)$, the condition (a) gives $h(v^2)v\in L^2(\Omega)$, so $F(v,w)\in X$. If $N=2$, then for each $p>0$, $v\in H^1(\Omega)\hookrightarrow L^p(\Omega)$, so we can also deduce that $h(v^2)v\in L^2(\Omega)$ by using the condition (b) and Cauchy inequality, which implies $F(v,w)\in X$. If $N=3$, then $v\in H^1(\Omega)\hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$ ($\frac{2N}{N-2}=6$), similarly we have, from Cauchy inequality, that

$$\left\|h(v^2)v\right\|_2\le\left\|h(v^2)\right\|_N\cdot\|v\|_{\frac{2N}{N-2}},$$

thus the condition (c) implies that $h(v^2)v\in L^2(\Omega)$ and $F(v,w)\in X$.

Secondly, we prove that $F(U)$ is locally Lipschitz from X to X. Since

$$\begin{aligned}F(U_1)-F(U_2)&=F(v_1,w_1)-F(v_2,w_2)\\&=(0,h(v_1^2)v_1-h(v_2^2)v_2+v_1-v_2),\end{aligned}$$

and

$$\begin{aligned}\|F(U_1)-F(U_2)\|_X&=\left\|h(v_1^2)v_1-h(v_2^2)v_2+v_1-v_2\right\|_{L^2}\\&\le\left\|h(v_1^2)v_1-h(v_2^2)v_2\right\|_{L^2}+\|v_1-v_2\|_{L^2}.\end{aligned}$$

Observe that

$$\begin{aligned}\left\|h(v_1^2)v_1-h(v_2^2)v_2\right\|_{L^2}&=\left\|\int_{v_2}^{v_1}(h(\xi^2)+2\xi^2h'(\xi^2))\mathrm{d}\xi\right\|_{L^2}\\&\le\left\|\int_{v_2}^{v_1}h(\xi^2)\mathrm{d}\xi\right\|_{L^2}+\left\|\int_{v_2}^{v_1}2\xi^2h'(\xi^2))\mathrm{d}\xi\right\|_{L^2}.\end{aligned}$$

Similar to the process above, we have that

$$\|F(U_1)-F(U_2)\|_X\le C\|v_1-v_2\|_{H^1}\le C\|U_1-U_2\|_X,$$

thus F is locally Lipschitz function, which implies the existence of the solution of (29).

Now we prove the estimate (30), which implies the uniqueness of the solution. From the equation (32), we have

$$\begin{aligned}\|U(t)\|_{H^2\times H^1}&\le\|T(t)U_0\|_{H^2\times H^1}+\int_0^t\|T(t-s)F(U(s))\|_{H^2\times H^1}\,\mathrm{d}s\\&\le\|U_0\|_{H^2\times H^1}+\int_0^t\|F(U)\|_{H^2\times H^1}\,\mathrm{d}s\\&=\|\varphi\|_{H^2}+\|\psi\|_{H^1}+\int_0^t\left\|(h(v^2)+1)v+f(u)\right\|_{H^1}\mathrm{d}s\\&\le\|\varphi\|_{H^2}+\|\psi\|_{H^1}+\int_0^t\left\|(h(v^2)+1)v\right\|_{H^1}\mathrm{d}s\\&\quad+\int_0^t\|f(u)\|_{H^1}\mathrm{d}s,\quad 0\le t\le T,\end{aligned}\tag{33}$$

where $T(t) = e^{tL}$. From the conditions (a), (b) and (c), we know that $\|h(v^2)v\|_{H^1} \leq C\|v\|_{H^2}$, so

$$\begin{aligned}
&\|U(t)\|_{H^2\times H^1}\\
&\quad\leq \|\varphi\|_{H^2} + \|\psi\|_{H^1} + c\int_0^t \|v\|_{H^2}\mathrm{d}s + \int_0^T \|f(u)\|_{H^1}\mathrm{d}s\\
&\quad\leq \|\varphi\|_{H^2} + \|\psi\|_{H^1} + c\int_0^t \|U(s)\|_{H^2\times H^1}\mathrm{d}s + \int_0^T \|f(u)\|_{H^1}\mathrm{d}s, \quad 0\leq t\leq T.
\end{aligned}$$

By using the Gronwall inequality, we have that

$$\begin{aligned}
\|U\|_{H^2\times H^1} &\leq e^{ct}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} + \int_0^T \|f(u)\|_{H^1}\mathrm{d}s)\\
&\leq e^{cT}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} + \int_0^T \|f(u)\|_{H^1}\mathrm{d}s), \quad 0\leq t\leq T, \qquad (34)
\end{aligned}$$

which is the estimate (30).

The proof of Theorem 3.1 Consider $g \in X_{t_0}$, $g(0) = u_0$ and let $v = v(g)$ denote the corresponding solution of following equation

$$\begin{cases}
v_{tt} = \Delta v + h(v^2)v + f(g), \quad \text{in} \quad \Omega\times(0,T),\\
\dfrac{\partial v}{\partial n} = 0, \quad \text{on} \quad \partial\Omega\times(0,T),\\
v(0,\cdot) = \varphi, \quad v_t(0,\cdot) = \psi, \quad \text{in} \quad \Omega.
\end{cases} \qquad (35)$$

In terms of Lemma 3.3, we can solve the problem (35) to get the solution $v \in Y_{t_0}$ and

$$\|v\|_{Y_{t_0}} \leq e^{ct_0}(\|\varphi\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)} + \int_0^{t_0} \|f(g(\tau,\cdot))\|_{H^1(\Omega)}\,\mathrm{d}\tau). \qquad (36)$$

For above v, we define $u = u(v(g))$ to be the corresponding solution of

$$\begin{cases}
u_t = \nabla(\nabla u - \chi u\nabla v), \quad \text{in} \quad \Omega\times(0,T),\\
\dfrac{\partial v}{\partial n} = 0, \quad \text{on} \quad \partial\Omega\times(0,T),\\
u(0,\cdot) = u_0 = g(0), \quad \text{in} \quad \Omega.
\end{cases} \qquad (37)$$

Thus from Lemma 2.2, we know the solution $u \in X_{t_0}$, which implies that we have a mapping $G : X_{t_0} \to X_{t_0}$, as defined by $Gg = u(v(g))$. Let $B_M = \{g \in X_{t_0} \mid g(0) = u_0$, $\|g(t,\cdot)\|_{H^\sigma} \leq M,\ 0\leq t\leq t_0\}$, where $M = 2C\|u_0\|_{H^\sigma}$ and the constant $C \geq 1$ is given by (11), then from the estimates (11) and (36), we have

$$\begin{aligned}
\|Gg\|_{X_{t_0}} &\leq C\|u_0\|_{H^\sigma} + Ct_0^{1-\frac{\sigma}{2}}\|v\|_{Y_{t_0}}\cdot\|Gg\|_{X_{t_0}}\\
&\leq C\|u_0\|_{H^\sigma} + Ct_0^{1-\frac{\sigma}{2}}e^{ct_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1}\\
&\quad + \int_0^{t_0} \|f(g(\tau,\cdot))\|_{H^1}\mathrm{d}\tau)\cdot\|Gg\|_{X_{t_0}}. \qquad (38)
\end{aligned}$$

If $g \in B_M$, $\|g\|_{H^1} \le \|g\|_{H^\sigma} \le M$, and $f \in C^2(\mathbf{R})$, we can deduce that

$$\|f(g(\tau,\cdot))\|_{H^1} \le \|f\|_{C^2[-M,M]} \cdot M + \|f(0)\|_{L^2},$$

thus the estimate (38) gives that $\|Gg\|_{X_{t_0}} \le 2C\|u_0\|_{H^\sigma}$ if $t_0 > 0$ is small enough. Hence we have proved that, for $t_0 > 0$ small enough, G maps B_M into B_M itself. It is similar to the proof of Theorem 2.1, we can prove that, if t_0 is small enough, the mapping G is a contract mapping, that means the problem (27) has a unique local solution $(u,v) \in (X_{t_0} \cap Z_{t_0}) \times (Y_{t_0} \cap W_{t_0})$.

Next, similar to the estimate (30), we can prove that, for $s \le 2$, the solution (u,v) will satisfy

$$\|v(t,\cdot)\|_{H^s} \le e^{ct_0}(c_0 + \int_0^{t_0} \|f(u(\tau,\cdot))\|_{H^{s-1}} \mathrm{d}\tau), \quad 0 \le t \le t_0, \tag{39}$$

where $c_0 = \|\varphi\|_{H^2} + \|\psi\|_{H^1}$ and c is independent of t_0. In fact, from the equation (32) we know that $U = (v,w)$ and $F(U) = (0, h(v^2)+1)v + f(u))$. Then for $s > 1$ we have $H^s \times H^{s-1} \subset H^1 \times L^2$. If we denote $T(t)\mid_{H^s \times H^{s-1}}$ as the restriction of $T(t) = e^{tL}$ on $H^s \times H^{s-1}$, thus, by similar process of (33) and (34), we can deduce that

$$\|U(t)\|_{H^s \times H^{s-1}} \le e^{ct_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} + \int_0^{t_0} \|f(u)\|_{H^{s-1}} \mathrm{d}\tau), \ \ 0 \le t \le t_0. \tag{40}$$

If $s < 1$, then $H^1 \times L^2 \subset H^s \times H^{s-1}$, we use Hahn-Banach theorem to get the operator $T(t)$ can be continuously extended on $H^s \times H^{s-1}$ and the norm of $T(t)$ is invariable. Thus for $s < 1$, we have also that

$$\|U(t)\|_{H^s \times H^{s-1}} \le e^{ct_0}(\|\varphi\|_{H^2} + \|\psi\|_{H^1} + \int_0^{t_0} \|f(u)\|_{H^{s-1}} \mathrm{d}\tau), \ \ 0 \le t \le t_0. \tag{41}$$

The estimate (39) can be deduced directly by (40) and (41).

In the case of $N = 1$, we know the problem (27) has a unique local solution $(u,v) \in (X_{t_0} \times Y_{t_0}) \cap (Z_{t_0} \times W_{t_0})$. If we take s=1/2 in (39), then

$$\|v(t,\cdot)\|^2_{H^{\frac{1}{2}}} \le ce^{2ct_0}(c_0 + \int_0^{t_0} \|f(u(\tau,\cdot))\|^2_{H^{-\frac{1}{2}}} \mathrm{d}\tau), \quad 0 \le t \le t_0. \tag{42}$$

Since $u_0 \ge 0$ and from the first equation of (27), we can deduce that $\|u(t,\cdot)\|_{L^1} = \|u_0\|_{L^1}$, also Sobolev imbedding theorem implies that $W^{0,1}(\Omega) \hookrightarrow H^{-\frac{1}{2}}(\Omega)$, hence we have

$$\begin{aligned}\|v(t,\cdot)\|^2_{H^{\frac{1}{2}}} &\le ce^{2ct_0}(c_0 + \int_0^{t_0} \|f(u(\tau,\cdot))\|^2_{H^{-\frac{1}{2}}} \mathrm{d}\tau) \\ &\le ce^{2ct_0}(c_0 + \int_0^{t_0} \|f(u(\tau,\cdot))\|^2_{L^1} \mathrm{d}\tau) \\ &\le ce^{2ct_0}(c_0 + \int_0^{t_0} (M_1\|u\|_{L^1} + \|f(0)\|_{L^1})^2 \mathrm{d}\tau) \\ &= ce^{2ct_0}(c_0 + t_0(M_1\|u_0\|_{L^1} + \|f(0)\|_{L^1})^2), \quad 0 \le t \le t_0, \end{aligned} \tag{43}$$

where $M_1 = \|f\|_{C^2}$.

On the other hand, for each $s \leq \sigma$ and $0 \leq \sigma_0 < 2$,

$$\begin{aligned}\|u(t,\cdot)\|_{H^s} &\leq c\,\|u_0\|_{H^\sigma} + ct_0^{1-\frac{\sigma_0}{2}} \sup_{0\leq t\leq t_0} \|\nabla(u\nabla v)\|_{H^{s-\sigma_0}} \\ &\leq c\,\|u_0\|_{H^\sigma} + ct_0^{1-\frac{\sigma_0}{2}} \sup_{0\leq t\leq t_0} \|u\nabla v\|_{H^{s-\sigma_0+1}}, \quad 0 \leq t \leq t_0. \qquad (44)\end{aligned}$$

Especially for $s = -\frac{1}{2} + \frac{1}{4}$ and $\sigma_0 = 2 - \frac{1}{8}$, we have

$$\|u(t,\cdot)\|_{H^{-\frac{1}{2}+\frac{1}{4}}} \leq c\,\|u_0\|_{H^\sigma} + ct_0^{\frac{1}{16}} \sup_{0\leq t\leq t_0} \|u\nabla v\|_{H^{-1-\frac{1}{8}}}, \quad 0 \leq t \leq t_0. \qquad (45)$$

By Sobolev imbedding theorem and (43),

$$\begin{aligned}\|u\nabla v\|_{H^{-1-\frac{1}{8}}} &\leq c\,\|u\|_{H^{-1-\frac{1}{8}}} \cdot \|\nabla v\|_{W^{-1-\frac{1}{8},\infty}} \\ &\leq c\,\|u\|_{H^{-1}} \cdot \|\nabla v\|_{H^{-\frac{1}{2}}} \\ &\leq c\,\|u\|_{L^1} \cdot \|v\|_{H^{\frac{1}{2}}} \\ &\leq c\,\|u_0\|_{L^1} \cdot e^{ct_0}(c_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}(M_1\,\|u_0\|_{L^1} + \|f(0)\|_{L^1})), \quad 0 \leq t \leq t_0. \qquad (46)\end{aligned}$$

Thus from (45) we have

$$\begin{aligned}\|u(t,\cdot)\|_{H^{-\frac{1}{4}}} &\leq c\,\|u_0\|_{H^\sigma} + ct_0^{\frac{1}{16}} \sup_{0\leq t\leq t_0} \|u\nabla v\|_{H^{-1-\frac{1}{8}}} \\ &\leq c\,\|u_0\|_{H^\sigma} + ct_0^{\frac{1}{16}}\,\|u_0\|_{L^1} \cdot e^{ct_0}(c_0^{\frac{1}{2}} + t_0^{\frac{1}{2}}(M_1\,\|u_0\|_{L^1} + \|f(0)\|_{L^1})). \qquad (47)\end{aligned}$$

Take $s = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ in (39), then from (47) we have

$$\begin{aligned}\|v(t,\cdot)\|^2_{H^{\frac{3}{4}}} &\leq ce^{2ct_0}(c_0 + \int_0^{t_0} \|f(u(\tau,\cdot))\|^2_{H^{\frac{3}{4}-1}} \mathrm{d}\tau) \\ &\leq ce^{2ct_0}(c_0 + t_0(M_1 \sup_{0\leq\tau\leq t_0} \|u(\tau,\cdot)\|_{H^{-\frac{1}{4}}} + \|f(0)\|_{H^{-\frac{1}{4}}})^2) \\ &\leq ce^{2ct_0}\{c_0 + t_0[M_1[c\,\|u_0\|_{H^\sigma} + ct_0^{\frac{1}{16}}\,\|u_0\|_{L^1} \cdot e^{ct_0}(c_0^{\frac{1}{2}} \\ &\quad + t_0^{\frac{1}{2}}(M_1\,\|u_0\|_{L^1} + \|f(0)\|_{L^1}))] + \|f(0)\|_{H^{-\frac{1}{4}}}]^2\}, \quad 0 \leq t \leq t_0. \qquad (48)\end{aligned}$$

Take $s = -\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 0$ and $\sigma_0 = 2 - \frac{1}{8}$ in (44) again, we obtain

$$\begin{aligned}\|u(t,\cdot)\|_{L^2} &\leq c\,\|u_0\|_{H^\sigma} + ct_0^{1-\frac{\sigma_0}{2}} \sup_{0\leq t\leq t_0} \|\nabla(u\nabla v)\|_{H^{-\sigma_0}} \\ &\leq c\,\|u_0\|_{H^\sigma} + ct_0^{\frac{1}{16}} \sup_{0\leq t\leq t_0} \|u\nabla v\|_{H^{-\sigma_0+1}} \\ &\leq c\,\|u_0\|_{H^\sigma} + ct_0^{\frac{1}{16}} \sup_{0\leq t\leq t_0} \|u\nabla v\|_{H^{-1+\frac{1}{8}}}, \quad 0 \leq t \leq t_0. \qquad (49)\end{aligned}$$

Here

$$\begin{aligned}\|u\nabla v\|_{H^{-1+\frac{1}{8}}} &\le c\,\|u\|_{H^{-1+\frac{1}{8}}}\cdot\|\nabla v\|_{W^{-1+\frac{1}{8},\infty}}\\ &\le c\,\|u\|_{H^{-\frac{1}{4}}}\cdot\|\nabla v\|_{H^{-\frac{1}{2}+\frac{1}{4}}}\\ &\le c\,\|u\|_{H^{-\frac{1}{4}}}\cdot\|v\|_{H^{\frac{3}{4}}},\quad 0\le t\le t_0,\end{aligned}\tag{50}$$

which implies, from the estimate (49), that

$$\begin{aligned}\|u(t,\cdot)\|_{L^2} &\le c\,\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma_0}{2}}\sup_{0\le t\le t_0}\|\nabla(u\nabla v)\|_{H^{-\sigma_0}}\\ &\le c\,\|u_0\|_{H^\sigma}+ct_0^{\frac{1}{16}}\sup_{0\le t\le t_0}\|u\nabla v\|_{H^{-1+\frac{1}{8}}}\\ &\le c\,\|u_0\|_{H^\sigma}+ct_0^{\frac{1}{16}}\cdot\sup_{0\le t\le t_0}\|u\|_{H^{-\frac{1}{4}}}\cdot\|v\|_{H^{\frac{3}{4}}},\quad 0\le t\le t_0.\end{aligned}\tag{51}$$

From the estimates (47) and (48) above, we have obtained that $\|u(t,\cdot)\|_{L^2}$ grows by a bounded manner in time.

Again we take $s=\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$ in (39), then the estimates (39) and (51) imply that $\|v(t,\cdot)\|_{H^1}$ grows also by a bounded manner in time.

Taking $s=-\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{1}{4}$ and $\sigma_0=2-\frac{1}{8}$ in (44) once more, since $\|v(t,\cdot)\|_{H^1}$ grows by a bounded manner in time, similar to which we have done in (49), (50) and (51), we can deduce that $\|u(t,\cdot)\|_{H^{\frac{1}{4}}}$ grows by a bounded manner in time.

Let us repeat processes above four times, we can prove that $\|u(t,\cdot)\|_{H^{\frac{5}{4}}}$ $(\sigma=\frac{5}{4})$ and $\|v(t,\cdot)\|_{H^2}$ grow by a bounded manner in time, that means the solution of (27) is global.

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