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On Global Smooth Solution of A Semi-Linear System of Wave Equations in $\mathbb{R}^3$

WU Haigen*

The Graduate School of China Academy of Engineering Physics, PO Box 2101, Beijing 100088, China; and School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China.

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Abstract. In this paper we consider the Cauchy problem for a semi-linear system of wave equations with Hamilton structure. We prove the existence of global smooth solution of the system for subcritical case by using conservation of energy and Strichartz's estimate. On the basis of Morawetz-Pohožev identity, we obtain the same result for the critical case.

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1 Introduction and main results

This paper is concerned with the Cauchy problem for the non-linear system of wave equations with Hamilton structure in $\mathbb{R}^{3+1}_+$

$$\begin{cases} u_{tt}-\Delta u=-F_1(|u|^2,|v|^2)u, \\ v_{tt}-\Delta v=-F_2(|u|^2,|v|^2)v, \\ u(0)=\varphi_1(x), \quad u_t(0)=\psi_1(x), \\ v(0)=\varphi_2(x), \quad v_t(0)=\psi_2(x), \end{cases} \tag{1.1}$$

where there exists a function $F(\lambda,\mu)$ such that

$$\frac{\partial F(\lambda,\mu)}{\partial \lambda}=F_1(\lambda,\mu), \quad \frac{\partial F(\lambda,\mu)}{\partial \mu}=F_2(\lambda,\mu). \tag{1.2}$$

*Corresponding author. *Email address:* `wuhaigen@gmail.com` (H. Wu)

The linear case $F_j = m_j$, where $m_j \in \mathbb{R}$ for $j = 1,2$, corresponds to the classical Klein-Gordon system in relativistic particle physics. The constants m_j may be interpreted as masses and hence are generally assumed to be nonnegative. In order to model also non-linear phenomenon like quantization, in the 1950s systems of type (1.1) with non-linearities like $F_j = m_j + f_j$ were proposed as models in relativistic quantum mechanics with local interaction, see, e.g., [1, 2].

Various other models involving non-linearities F_j depending also on $u_t, v_t, \nabla u$ and ∇v have been studied [3]. To limit our paper to a reasonable length, we restrict our study to non-linearities depending only on u, v, i.e., the semi-linear case.

Without loss of generality, and since all important features of our problem already seem to exist in this case, we confine ourselves to real-valued solutions of (1.1). Moreover, we need to impose the following assumptions on the semi-linearities to ensure that (1.1) always has a global solution.

(H1)

$$|F_1| + |\lambda F_{11}| + |\lambda^{\frac{1}{2}} \mu^{\frac{1}{2}} F_{12}| + |F_2| + |\lambda^{\frac{1}{2}} \mu^{\frac{1}{2}} F_{21}| + |\mu F_{22}| \leq C\big(1 + \lambda^{\frac{k-1}{2}} + \mu^{\frac{k-1}{2}}\big), \tag{1.3}$$

where $F_{i1} = \partial F_i / \partial \lambda,\ \ F_{i2} = \partial F_i / \partial \mu,\ \ i = 1,2.$

(H2)

$$F(\lambda,\mu) \geq 0, \quad F(0,0) = 0, \quad \lambda^{\frac{k+1}{2}} + \mu^{\frac{k+1}{2}} \leq C_0[1 + \frac{1}{2} F(\lambda,\mu)]. \tag{1.4}$$

(H3)

$$\lambda F_1(\lambda,\mu) + \mu F_2(\lambda,\mu) \geq 2F(\lambda,\mu), \quad k = 5. \tag{1.5}$$

(H4)

$$F(\lambda,\mu) \leq C(1 + \lambda^{\frac{k+1}{2}} + \mu^{\frac{k+1}{2}}). \tag{1.6}$$

It is easy to verify that

$$F_1(\lambda,\mu) = \lambda^2 + \mu, \quad F_2(\lambda,\mu) = \mu^2 + \lambda, \quad F(\lambda,\mu) = \frac{1}{3}\lambda^3 + \frac{1}{3}\mu^3 + \lambda\mu$$

satisfy (H1)-(H4) with $k = 5$.

It is known that the energy associated with (1.1) is defined by

$$\begin{aligned} E(u,v;t) = & \frac{1}{2} \int_{\mathbb{R}^3} \big[|u_t(x,t)|^2 + |v_t(x,t)|^2 + |\nabla u(x,t)|^2 \\ & + |\nabla v(x,t)|^2 + F(|u(x,t)|^2, |v(x,t)|^2) \big] \mathrm{d}x \\ \triangleq & \frac{1}{2} \int_{\mathbb{R}^3} \big[|u'(x,t)|^2 + |v'(x,t)|^2 + F(|u(x,t)|^2, |v(x,t)|^2) \big] \mathrm{d}x. \end{aligned} \tag{1.7}$$

Notice that the above energy involves two kinds of terms: the kinetic term and the potential term involving semi-linearity $F(|u|^2, |v|^2)$. To make sure that the potential energy is controlled by the kinetic energy, we need to assume that $k\leq 5$ in view of (H2), (H4) and (1.7). This is because $F(|u|^2,|v|^2)$ behaves like $|u|^{k+1}+|v|^{k+1}$ and $H^1(\mathbb{R}^3)\hookrightarrow L^q_{\rm loc}(\mathbb{R}^3)$ only when $q\leq 6$. The case where $k=5$ is called the critical case since $q=6$ is the critical exponent for the above Sobolev embedding; while the range $1<k<5$ is called the subcritical case.

Our main result is the following theorem.

Theorem 1.1. *Let $1<k\leq 5$ and let F_1,F_2 and F satisfy* (H1) *and* (H2). *Assume also that* (H3) *is satisfied when $|u|$ and $|v|$ are larger than a constant if $k=5$. Then* (1.1) *always has a global smooth C^2 solution.*

We first point out that, in proving Theorem 1.1, we need only consider compactly supported data. More precisely, fix $\chi\in C_0^\infty(\mathbb{R}^3)$ satisfying $\chi=1$ for $|x|\leq 1$, set

$$\varphi_{jR}(x)=\chi(\frac{x}{R})\varphi_j(x),\quad \psi_{jR}(x)=\chi(\frac{x}{R})\psi_j(x),\quad j=1,2,$$

and let (u_R,v_R) be the solution of (1.1) with data (φ_{jR},ψ_{jR}). If $t_0\in\mathbb{R}_+$, denote

$$\Lambda_{0,t_0}=\{(x,t):0\leq t\leq t_0,|x|\leq t_0-t\}$$

as the backward light cone through $(0,t_0)$. Then $(u_{R_1},v_{R_1})=(u_{R_2},v_{R_2})$ in Λ_{0,t_0} if $R_1,R_2>t_0$, since (u_{R_1},v_{R_1}) and (u_{R_2},v_{R_2}) both have Cauchy data (φ_j,ψ_j) in $\Lambda_{0,t_0}\bigcap\{(x,0):x\in\mathbb{R}^3\}$. $\mathbb{R}^{3+1}_+=\bigcup_{t_0>0}\Lambda_{0,t_0}$ implies that (u_R,v_R) must converge point by point to a solution of (1.1).

The next step is to recall an important Strichartz's estimate [4–6],

$$\begin{aligned}&\|v'(\cdot,t)\|_{L^2(\mathbb{R}^3)}+\|v\|_{L_t^{\frac{2q}{q-6}}L_x^q(S_T)}\\ \leq& C_q\|v'(\cdot,0)\|_{L^2(\mathbb{R}^3)}+C_q\int_0^T\|F(\cdot,t)\|_{L^2(\mathbb{R}^3)}{\rm d}t,\quad 6\leq q<\infty,\end{aligned}$$

where $S_T=[0,T]\times\mathbb{R}^3$, and v is a solution of the problem

$$\begin{cases}v_{tt}-\Delta v=F(x,t),\\ v(0)=f(x),\quad v_t(0)=g(x).\end{cases}$$

Setting $q=12$ and $q=6$ in the above inequality respectively, we obtain

$$\|v\|_{L_t^4L_x^{12}(S_T)}\leq C\Big(\|v'(\cdot,0)\|_{L^2(\mathbb{R}^3)}+\|F\|_{L_t^1L_x^2(S_T)}\Big),\tag{1.8}$$

as well as

$$\sup_{0\leq t\leq T}\|v(\cdot,t)\|_{L^6(\mathbb{R}^3)}\leq C\big(\|v'(\cdot,0)\|_{L^2(\mathbb{R}^3)}+\|F\|_{L_t^1L_x^2(S_T)}\big).\tag{1.9}$$

It is well known that, by the local existence theorem, if (u,v) is a C^2 solution of (1.1) in a half-open strip $[0,T_*)\times\mathbb{R}^3$ with compactly supported data, then either (u,v) extends to a C^2 solution in a larger strip or else (u,v) blows up point-wise, that is, $u,v\notin L^\infty([0,T_*)\times\mathbb{R}^3)$. Our next task is to prove that we can replace L^∞ by the mixed-norm in the left-hand side of (1.8).

Proposition 1.1. *Let* $1<k\le 5$ *and* F_1,F_2 *satisfy* (H1). *Then, if* $\varphi_j\in C^3$, $\psi_j\in C^2,(j=1,2)$ *are fixed compactly supported functions, then there exists a* $T>0$ *such that* (1.1) *has a* C^2 *solution* (u,v). *Moreover, if* T_* *is the supremum of all such times, then either* $T_*=\infty$, *or*

$$u,v\notin L_t^4L_x^{12}\left([0,T_*)\times\mathbb{R}^3\right).$$

Proof. The former part of the theorem is trivial. We only need to prove the latter part. Suppose that $0<T_*<\infty$ and that (u,v) is a C^2 solution of (1.1) in $[0,T_*)\times\mathbb{R}^3$ satisfying

$$u,v\in L_t^4L_x^{12}\left([0,T_*)\times\mathbb{R}^3\right). \tag{1.10}$$

We then show that (1.10) implies

$$u,v\in L^\infty\left([0,T_*)\times\mathbb{R}^3\right). \tag{1.11}$$

Let $0<R<\infty$ be large enough so that the data vanishes for $|x|>R$. Then $u(x,t)=v(x,t)=0$ for $|x|>R+t$. Therefore (H1) implies that, if $0\le t_0<s<T_*$, then for $|\alpha|=0,1$

$$\begin{aligned}&\|\partial_x^\alpha(uF_1)\|_{L_t^1L_x^2(I)}+\|\partial_x^\alpha(vF_2)\|_{L_t^1L_x^2(I)}\\ \le&C+C\bigg(\big\||u|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}+\big\||u|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}\\ &+\big\||v|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}+\big\||v|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}\bigg),\end{aligned} \tag{1.12}$$

where $I=[t_0,s]\times\mathbb{R}^3$, the constant C may depend on R,T_* and the constant in (H1) but not on t_0 or s.

In fact, if we set $a=u,v$ for $j=1,2$ respectively, then on one hand

$$\begin{aligned}|aF_j|=|a||F_j|\le&C\left(1+|u|^{k-1}+|v|^{k-1}\right)|a|\\ =&C\left(|a|+|u|^{k-1}|a|+|v|^{k-1}|a|\right),\end{aligned}$$

which implies by Hölder inequality that

$$\begin{aligned}\|aF_j\|_{L_t^1L_x^2(I)}\le&2CT_*\left(\frac{4\pi}{3}(R+T_*)^3\right)^{\frac12}+2C\bigg(\big\||u|^{k-1}u\big\|_{L_t^1L_x^2(I)}\\ &+\big\||u|^{k-1}v\big\|_{L_t^1L_x^2(I)}+\big\||v|^{k-1}u\big\|_{L_t^1L_x^2(I)}+\big\||v|^{k-1}v\big\|_{L_t^1L_x^2(I)}\bigg).\end{aligned}$$

On the other hand

$$\begin{aligned}|\partial_x^\alpha[aF_j]|=&\left|a\frac{\partial F_j}{\partial\lambda}\frac{\partial\lambda}{\partial u}\partial_x^\alpha u+a\frac{\partial F_j}{\partial\mu}\frac{\partial\mu}{\partial v}\partial_x^\alpha v+F_j\partial_x^\alpha a\right|\\ \le&\left|2auF_{j1}\partial_x^\alpha u\right|+\left|2avF_{j2}\partial_x^\alpha v\right|+\left|F_j\partial_x^\alpha a\right|\end{aligned}$$

$$
\begin{aligned}
&\leq 3C\left(1+|u|^{k-1}+|v|^{k-1}\right)\left(|\partial_x^\alpha u|+|\partial_x^\alpha v|\right)\\
&=3C\left(|u'|+|v'|\right)+3C\Big(|u|^{k-1}|\partial_x^\alpha u|+|u|^{k-1}|\partial_x^\alpha v|\\
&\quad +|v|^{k-1}|\partial_x^\alpha u|+|v|^{k-1}|\partial_x^\alpha v|\Big),
\end{aligned}
$$

for $|\alpha|=1$, which implies that

$$
\begin{aligned}
\|\partial_x^\alpha(aF_j)\|_{L_t^1L_x^2(I)} \leq &3C\left\|1+\frac{1}{2}\|u'\|^2_{L_x^2(R^3)}+\frac{1}{2}\|v'\|^2_{L_x^2(R^3)}\right\|_{L_t^1[t_0,s]}\\
&+3C\Big(\big\||u|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}+\big\||u|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}\\
&+\big\||v|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}+\big\||v|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}\Big)\\
\leq &3CT_*[1+E(u,v;0)]+3C\Big(\big\||u|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}+\big\||u|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}\\
&+\ \big\||v|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}+\big\||v|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}\Big).
\end{aligned}
$$

If we choose

$$
C'=\max\left(4CT_*\left[\frac{4\pi}{3}(R+T_*)^3\right]^{\frac{1}{2}},\ 6CT_*[1+E(u,v;0)],\ 6C\right),
$$

then C' depends on R,T_* and the constant in (H1) but not on t_0 or s, still denoted by C in (1.12) for simplicity.

If we use (1.9), with t replaced by $t-t_0$, then

$$
\begin{aligned}
&\sup_{t_0\leq t\leq s}\sum_{|\alpha|\leq 1}\|\partial_x^\alpha u(\cdot,t)\|_{L^6(\mathbb{R}^3)}+\sup_{t_0\leq t\leq s}\sum_{|\alpha|\leq 1}\|\partial_x^\alpha v(\cdot,t)\|_{L^6(\mathbb{R}^3)}\\
\leq &C\Big(1+\sum_{|\alpha|\leq 1}\big[\|(\partial_x^\alpha u)'(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}+\|(\partial_x^\alpha v)'(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}+\big\||u|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}\\
&+\big\||u|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}+\big\||v|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}+\big\||v|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}\big]\Big)\\
\leq &C(t_0)+C\sum_{|\alpha|\leq 1}\Big(\big\||u|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}+\big\||u|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}\\
&+\big\||v|^{k-1}\partial_x^\alpha u\big\|_{L_t^1L_x^2(I)}+\big\||v|^{k-1}\partial_x^\alpha v\big\|_{L_t^1L_x^2(I)}\Big),
\end{aligned} \tag{1.13}
$$

where $C(t_0)$ is independent of s and is finite since (u,v) is a C^2 solution and vanishes for large $|x|$.

Apply Hölder's inequality to the last four terms in (1.13) and consider them in two cases. For simplicity, we take a general term $\||a|^{k-1}\partial_x^\alpha b\|_{L_t^1L_x^2(I)}$ as an example, where a,b may be u or v.

Case 1. $k=5$. In this case,

$$\begin{aligned}
&C\sum_{|\alpha|\le 1}\||a|^{k-1}\partial_x^\alpha b\|_{L_t^1L_x^2(I)}=C\sum_{|\alpha|\le 1}\||a|^4\partial_x^\alpha b\|_{L_t^1L_x^2(I)}\\
\le& C\sum_{|\alpha|\le 1}\|\partial_x^\alpha b\|_{L_t^\infty L_x^6(I)}\||a|^4\|_{L_t^1L_x^3(I)}\le C\sup_{t_0\le t\le s}\sum_{|\alpha|\le 1}\|\partial_x^\alpha b(\cdot,t)\|_{L^6(\mathbb{R}^3)}\|a\|^4_{L_t^4L_x^{12}(I)}.
\end{aligned}$$

It follows from (1.10) that the last factor must go to zero as $t_0\nearrow T_*$. By similar arguments to others three terms, we conclude that the last four terms in (1.13) are smaller than half of its left-hand side and so

$$\sup_{t_0\le t\le s}\sum_{|\alpha|\le 1}\|\partial_x^\alpha u(\cdot,t)\|_{L^6(\mathbb{R}^3)}+\sup_{t_0\le t\le s}\sum_{|\alpha|\le 1}\|\partial_x^\alpha v(\cdot,t)\|_{L^6(\mathbb{R}^3)}\le 2C(t_0).$$

Letting $s\nearrow T_*$ we conclude that

$$\sup_{0\le t\le T_*}\sum_{|\alpha|\le 1}\|\partial_x^\alpha u(\cdot,t)\|_{L^6(\mathbb{R}^3)}+\sup_{0\le t\le T_*}\sum_{|\alpha|\le 1}\|\partial_x^\alpha v(\cdot,t)\|_{L^6(\mathbb{R}^3)}<\infty. \tag{1.11'}$$

This clearly implies (1.11) by using Sobolev's embedding $W^{1,6}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$.

Case 2. $1<k<5$. In this case,

$$\begin{aligned}
C\sum_{|\alpha|\le 1}\||a|^{k-1}\partial_x^\alpha b\|_{L_t^1L_x^2(I)}\le& C\sup_{t_0\le t\le s}\sum_{|\alpha|\le 1}\|\partial_x^\alpha b(\cdot,t)\|_{L^6(\mathbb{R}^3)}\|a\|^{k-1}_{L_t^{k-1}L_x^{3(k-1)}(I)}\\
\le& C\|1\|^{k-1}_{L_t^\chi[t_0,s]L_x^{3\chi}(|x|<T_*+R)}\sup_{t_0\le t\le s}\sum_{|\alpha|\le 1}\|\partial_x^\alpha b(\cdot,t)\|_{L^6(\mathbb{R}^3)}\|a\|^{k-1}_{L_t^4L_x^{12}(I)}\\
\le& C(R,T_*)\sup_{t_0\le t\le s}\sum_{|\alpha|\le 1}\|\partial_x^\alpha b(\cdot,t)\|_{L^6(\mathbb{R}^3)}\|a\|^{k-1}_{L_t^4L_x^{12}(I)},
\end{aligned}$$

where $1/(k-1)=1/4+1/\chi$. Since $k>1$, the last part also goes to zero as $t_0\nearrow T_*$. Using similar argument used in Case 1, we conclude that (1.11) holds in this case. This completes the proof of this proposition. □

To prove Theorem 1.1, we only need to consider the following two results for subcritical case and critical case respectively.

Proposition 1.2. *Let* $1<k<5$ *and suppose that* (u,v) *is a* C^2 *solution to* (1.1) *with a compact Cauchy data supported on* $\{x\in\mathbb{R}^3:|x|\le R\}$. *If*

$$E(u,v;t)=E(u,v;0),\quad 0<t<T_*, \tag{1.14}$$

then

$$u,v\in L_t^4L_x^{12}([0,T_*)\times\mathbb{R}^3).$$

Proposition 1.3. *Let* $k=5$ *and suppose that* (u,v) *is a* C^2 *solution to* (1.1) *with a compact Cauchy data supported on* $\{x\in\mathbb{R}^3:|x|\le R\}$. *Fix* $x_0\in\mathbb{R}^3$ *and assume that*

$$\int_{|x-x_0|\le T_*-t_0}\frac{1}{2}\Big(|u'(x,t_0)|^2+|v'(x,t_0)|^2+F\big(|u(x,t_0)|^2,|v(x,t_0)|^2\big)\Big)\mathrm{d}x<\varepsilon. \tag{1.15}$$

Then there exists an $\varepsilon_0>0$ *depending only on* T_*,R *and* $E(u,v;0)$, *such that if* $0<\varepsilon<\varepsilon_0$ *and* $0\le t_0<T_*$

$$u,v\in L_t^4L_x^{12}(\Lambda(\delta;t_0,T_*)), \tag{1.16}$$

provided that $\delta>0$ *and* T_*-t_0 *are sufficiently small.*

To prove Propositions 1.2 and 1.3, we will need to use the following basic lemma.

Lemma 1.1. *Suppose that* $0\le y(s)\in C([a,b))$ *satisfies* $y(a)=0$ *and*

$$y(s)\le C_0+\varepsilon(y(s))^\sigma,$$

for some $C>0$ *and* $\sigma>0$. *Then, if* $\varepsilon<2^{-\sigma}C_0^{1-\sigma}$, *then*

$$y(s)\le 2C_0,\quad s\in[a,b).$$

Proof. Consider

$$h(x)=C_0+\varepsilon x^\sigma-x.$$

If $\varepsilon<2^{-\sigma}C_0^{1-\sigma}$ and $x_1=2C_0$, then

$$C_0+\varepsilon x_1^\sigma-x_1=h(x_1)=h(2C_0)<C_0+2^{-\sigma}C_0^{1-\sigma}(2C_0)^\sigma-2C_0=0.$$

It follows that if $h(x)=C_0+\varepsilon x^\sigma-x\ge 0,\forall x\in[0,x_0)$, then $x_0<x_1=2C_0$. Since $y(s)$ must be smaller than the supremum of such x_0, the lemma follows. □

The paper is organized as follows. In Section 2 the conservation of energy is given to prove the subcritical case of the problem. Section 3 is devoted to the critical case. Morawetz established in her seminal paper [7] Morawetz's identity for Klein-Gordon equations, and for Schrödinger equations similar identity was obtained by Lin and Strauss in [8]. As we know, Morawetz's identity, like other invariants and conservation laws, plays an important role in the scattering theory of nonlinear Klein-Gordon equations (see, e.g., [9–14]) and nonlinear Schrödinger equations (see, e.g., [8, 12, 15]). This work will be concerned with the Morawetz's identity for the wave equation. In Section 3, we use the non-concentration of potential energy to prove the global existence for the critical case of the system. Here a relevant Morawetz's identity will play an important role.

2 Proof of Proposition 1.2

To prove the subcritical case, we firstly prove the following conservation of energy.

Proposition 2.1. *Suppose that F_1,F_2,F are as in Theorem* 1.1. *Suppose also that $0<T_*<\infty$ and that (u,v) is a C^2 solution of* (1.1) *and that the Cauchy data vanish for $|x|>R$. Then*

$$E(u,v;t)=E(u,v;0),\quad 0<t<T_*. \tag{2.1}$$

Moreover, for fixed data as above, there is a constant C_{R,T_*} such that

$$\int_{\mathbb{R}^3}\big[|u'(x,t)|^2+|v'(x,t)|^2+|u(x,t)|^{k+1}+|v(x,t)|^{k+1}\big]\mathrm{d}x\le C_{R,T_*},\quad 0<t<T_*. \tag{2.1$'$}$$

Proof. It follows from $(u,v)=0$ for $|x|>t+R$ and (H2) that

$$\begin{aligned}&\int_{\mathbb{R}^3}(|u'|^2+|v'|^2+|u|^{k+1}+|v|^{k+1})\mathrm{d}x\\ \le&\int_{|x|\le T_*+R}\left(|u'|^2+|v'|^2+C_0+\frac{1}{2}C_0F\right)\mathrm{d}x\\ \le& C\left(\frac{4\pi}{3}(R+T_*)^3+E(u,v;0)\right)\triangleq C_{R,T_*}.\end{aligned}$$

Therefore, (2.1) implies (2.1′). Note that

$$E(u,v;0)=\int_{\mathbb{R}^3}\frac{1}{2}\Big[|\nabla\varphi_1|^2+|\nabla\varphi_2|^2+|\psi_1|^2+|\psi_2|^2+F(|\varphi_1|^2,|\varphi_2|^2)\Big]\mathrm{d}x<\infty,$$

in view of our assumptions on the data. Multiply $u_{tt}-\Delta u+F_1u=0$ and $v_{tt}-\Delta v+F_2v=0$ by u_t and v_t respectively. Summing the resulting equations gives

$$\begin{aligned}0=&u_t(u_{tt}-\Delta u+F_1u)+v_t(v_{tt}-\Delta v+F_2v)\\ =&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u'|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|v'|^2-\nabla\cdot(u_t\nabla u)-\nabla\cdot(v_t\nabla v)+\frac{1}{2}\frac{\mathrm{d}F}{\mathrm{d}t}\\ \triangleq&\mathrm{div}_{x,t}e(u,v),\end{aligned} \tag{2.2}$$

where, in the present context,

$$e(u,v)=\left(-u_t\nabla u-v_t\nabla v,\frac{1}{2}|u'|^2+\frac{1}{2}|v'|^2+\frac{1}{2}F\right). \tag{2.3}$$

If we fix $0<t<T_*$, then (u,v) is C^2 and has compact support in $[0,t]\times\mathbb{R}^3$. Therefore,

integrating (2.2) leads to

$$\begin{aligned}0=&\int_0^t\int_{\mathbb{R}^3}\mathrm{div}_{x,\tau}e(u,v)\mathrm{d}x\mathrm{d}\tau\\=&\int_{\mathbb{R}^3}\int_0^t\frac{\partial}{\partial\tau}\left(\frac{1}{2}|u'|^2+\frac{1}{2}|v'|^2+\frac{1}{2}F\right)\mathrm{d}\tau\mathrm{d}x\\=&\frac{1}{2}\int_{\mathbb{R}^3}\left[|u'(x,t)|^2+|v'(x,t)|^2+F\big(|u(x,t)|^2,|v(x,t)|^2\big)\right]\mathrm{d}x\\&-\frac{1}{2}\int_{\mathbb{R}^3}\left[|u'(x,0)|^2+|v'(x,0)|^2+F\big(|u(x,0)|^2,|v(x,0)|^2\big)\right]\mathrm{d}x,\end{aligned}$$

which gives (2.1). This completes the proof of this proposition. □

Proof of Proposition 1.2. It is clear we only need to show that

$$u,v\in L_t^4L_x^{12}([0,T_*)\times\mathbb{R}^3),\quad 0<T_*<\infty. \tag{2.4}$$

By (1.8) and (1.12), we have for $I=[t_0,s)\times\mathbb{R}^3$

$$\begin{aligned}&\|u\|_{L_t^4L_x^{12}(I)}+\|v\|_{L_t^4L_x^{12}(I)}\\\le&C\left(1+\|u'(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}+\|v'(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}+\big\||u|^k\big\|_{L_t^1L_x^2(I)}+\big\||v|^k\big\|_{L_t^1L_x^2(I)}\right)\\\le&C\left(2+\frac{1}{2}\|u'(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}\|v'(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}+\big\||u|^k\big\|_{L_t^1L_x^2(I)}+\big\||v|^k\big\|_{L_t^1L_x^2(I)}\right)\\\le&C\left[2+E(u,v;0)\right]+C\left(\big\||u|^k\big\|_{L_t^1L_x^2(I)}+\big\||v|^k\big\|_{L_t^1L_x^2(I)}\right),\end{aligned} \tag{2.5}$$

where we have used the conservation of energy in the last step. Note that

$$1=\frac{5-k}{4}+\frac{k-1}{4},\quad \frac{1}{2}=\frac{7-k}{12}+\frac{k-1}{12}.$$

Applying Hölder's inequality to (2.5), we obtain

$$\begin{aligned}\big\||u|^k\big\|_{L_t^1L_x^2(I)}\le&\|u\|_{L_t^{\frac{4}{5-k}}L_x^{\frac{12}{7-k}}(I)}\big\||u|^{k-1}\big\|_{L_t^{\frac{4}{k-1}}L_x^{\frac{12}{k-1}}(I)}\\=&\|u\|_{L_t^{\frac{4}{5-k}}L_x^{\frac{12}{7-k}}(I)}\|u\|^{k-1}_{L_t^4L_x^{12}(I)}.\end{aligned} \tag{2.6}$$

Note that $12/(7-k)<k+1$ for $1<k<5$. Therefore, since $u(x,t)=0$ when $|x|\ge t+R$, we can use Hölder's inequality again to obtain that

$$\begin{aligned}\|u\|_{L_t^{\frac{4}{5-k}}L_x^{\frac{12}{7-k}}(I)}\le&(T_*-t_0)^{\frac{5-k}{4}}\sup_{t_0\le t\le s}\|u(\cdot,t)\|_{L^{\frac{12}{7-k}}(\mathbb{R}^3)}\\\le&C(T_*-t_0)^{\frac{5-k}{4}}(T_*+R)^{3(\frac{7-k}{12}-\frac{1}{k+1})}\sup_{t_0\le t\le s}\|u(\cdot,t)\|_{L^{k+1}(\mathbb{R}^3)}\\\le&C'_{R,T_*}(T_*-t_0)^{\frac{5-k}{4}},\end{aligned} \tag{2.7}$$

where

$$C'_{R,T_*} = C(T_*+R)^{3(\frac{7-k}{12}-\frac{1}{k+1})} C_{R,T_*}.$$

Similarly, we have

$$\big\| |v|^k \big\|_{L^1_t L^2_x(I)} \le \|v\|_{L_t^{\frac{4}{5-k}} L_x^{\frac{12}{7-k}}(I)} \|v\|^{k-1}_{L^4_t L^{12}_x(I)}, \tag{2.6$'$}$$

$$\|v\|_{L_t^{\frac{4}{5-k}} L_x^{\frac{12}{7-k}}(I)} \le C'_{R,T_*}(T_*-t_0)^{\frac{5-k}{4}}. \tag{2.7$'$}$$

Let

$$\varepsilon(t_0) = CC'_{R,T_*}(T_*-t_0)^{\frac{5-k}{4}},$$

where C is as in (2.5). Then using (2.5)-(2.7) gives

$$\begin{aligned} &\|u\|_{L^4_t L^{12}_x(I)} + \|v\|_{L^4_t L^{12}_x(I)} \\ \le& C\big[2+E(u,v;0)\big] + \varepsilon(t_0)\Big(\|u\|_{L^4_t L^{12}_x(I)} + \|v\|_{L^4_t L^{12}_x(I)} \Big)^{k-1}. \end{aligned} \tag{2.8}$$

Note that $\varepsilon(t_0) \to 0$ as $t_0 \nearrow T_*$ since we are assuming $k<5$. Therefore, Lemma 1.1 implies that, if t_0 is sufficiently close to T_*, then

$$\|u\|_{L^4_t L^{12}_x(I)} + \|v\|_{L^4_t L^{12}_x(I)} \le 2C\big[2+E(u,v;0)\big]. \tag{2.4$'$}$$

This clearly gives (2.4), since, in $[0,t_0]\times\mathbb{R}^3$, u,v is bounded and compactly supported. $\square$

3 Proof of Proposition 1.3

To prove the critical case, one can not turn to conservation of energy anymore, but this time, we have a local version of the energy identity. Denote

$$\Lambda(\delta;t_0,s) = \{(x,t): t_0 \le t \le s,\ \ |x-x_0| \le \delta + T_* - t\}, \tag{3.1}$$

which is a portion of the backward light cone through $(x_0, T_*+\delta)$. Then the energy in the bottom ball

$$D_{t_0} = \{(x,t)\in\Lambda(\delta;t_0,s): t=t_0\},$$

equals to the energy in the top ball

$$D_s = \{(x,t)\in\Lambda(\delta;t_0,s): t=s\},$$

plus the energy flux across the rest of the boundary

$$M^s_{t_0} = \{(x,t)\in\Lambda(\delta;t_0,s): t_0\le t\le s,\ \ |x-x_0| = \delta+T_*-t\}.$$

In other words, let

$$E(u,v;D_t)=\int_{D_t}\frac{1}{2}(|u'|^2+|v'|^2+F)\mathrm{d}x, \quad 0\le t<T_*, \tag{3.2}$$

$$\mathrm{Flux}(u,v;M^s_{t_0})=\int_{M^s_{t_0}}\langle e(u,v),\vec{v}\rangle, \quad 0\le t_0<s\le T_*, \tag{3.3}$$

where $e(u,v)$ is as in (2.3), $\vec{v}$ is the outward normal through a given point on $M^s_{t_0}$. Then

$$E(u,v;D_{t_0})=E(u,v;D_s)+\mathrm{Flux}(u,v;M^s_{t_0}). \tag{3.4}$$

Firstly, we need to prove that (H2) implies that the energy flux is nonnegative. To verify this, we note that $M^s_{t_0}$ consists of points of the form $(x,\delta+T_*-|x-x_0|)$ with $\delta+T_*-|x-x_0|\in[t_0,s]$. Moreover, since the outward normal is $(-y/|y|,1)/\sqrt{2}$, where $y=x_0-x$, we have

$$\begin{aligned}\sqrt{2}\langle e(u,v),\vec{v}\rangle=&(-u_t\nabla u-v_t\nabla v)\cdot\frac{-y}{|y|}+\frac{1}{2}|u'|^2+\frac{1}{2}|v'|^2+\frac{1}{2}F\\ =&\frac{1}{2}\left|\frac{y}{|y|}u_t+\nabla u\right|^2+\frac{1}{2}\left|\frac{y}{|y|}v_t+\nabla v\right|^2+\frac{1}{2}F\ge 0.\end{aligned} \tag{3.5}$$

Since Flux ≥ 0, we conclude from (3.4) that $t\to E(u,v;D_t)$ is a non-increasing function on $[0,T_*)$. It is also bounded, as $E(u,v,D_t)\le E(u,v;t)=E(u,v;0)<\infty$. Hence, the first two terms in (3.4) must approach a common limit. This in turn gives the important fact that

$$\mathrm{Flux}(u,v;M^s_{t_0})\to 0 \quad \text{as} \quad t\to T_*.$$

Proof of Proposition 1.3. Let C_0 be the constant in (H2). Then (1.15) implies that

$$\sup_{t_0\le t<T_*}\int_{|x-x_0|\le\delta+T_*-t}\Big[|u(x,t)|^6+|v(x,t)|^6\Big]\mathrm{d}x<2C_0\varepsilon, \tag{1.15$'$}$$

provided that $\delta>0$ and T_*-t_0 are sufficiently small. In fact, for $\delta>0$ small enough, (1.15) implies that

$$\int_{|x-x_0|\le\delta+T_*-t_0}\frac{1}{2}\Big[|u'(x,t_0)|^2+|v'(x,t_0)|^2+F\big(|u(x,t_0)|^2,|v(x,t_0)|^2\big)\Big]<\frac{3}{2}\varepsilon,$$

which yields

$$\sup_{t_0\le t<T_*}\int_{|x-x_0|\le\delta+T_*-t}\frac{1}{2}\Big[|u'(x,t)|^2+|v'(x,t)|^2+F\big(|u(x,t)|^2,|v(x,t)|^2\big)\Big]<\frac{3}{2}\varepsilon,$$

where we have used the fact that $E(u,v;D_t)$ is a non-increasing function of t. It is easy to see by (H2) that

$$\begin{aligned}&\int_{|x-x_0|\le\delta+T_*-t}\Big[|u(x,t)|^6+|v(x,t)|^6\Big]\mathrm{d}x\\ \le&\frac{4\pi}{3}C_0(\delta+T_*-t_0)^3+C_0\int_{|x-x_0|\le\delta+T_*-t}\frac{1}{2}F\mathrm{d}x\\ \le&\frac{4\pi}{3}C_0(\delta+T_*-t_0)^3+\frac{3}{2}C_0\varepsilon.\end{aligned}$$

If we choose δ and t_0 small enough such that $\frac{4\pi}{3}(\delta+T_*-t_0)^3<\frac{1}{2}\varepsilon$, then (1.15′) must hold.

To prove (1.16), we need to use (1.8) where the norm on the left-hand side is only taken over $\Lambda(\delta;t_0,s)$ and the norm on the right-hand side need only be taken over the same set by Huygen's principle. Thus for $J=\Lambda(\delta;t_0,s)$,

$$\begin{aligned}&\|u\|_{L_t^4L_x^{12}(J)}+\|v\|_{L_t^4L_x^{12}(J)}\\ \le&C\Big(\|u'(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}+\|v'(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\Big)+C\|F_1u\|_{L_t^1L_x^2(J)}+C\|F_2v\|_{L_t^1L_x^2(J)}\\ \le&C\Big(\frac{1}{2}+\frac{1}{2}\|u'(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}+\frac{1}{2}+\frac{1}{2}\|v'(\cdot,t_0)\|^2_{L^2(\mathbb{R}^3)}\Big)+C\|F_1u\|_{L_t^1L_x^2(J)}+C\|F_2v\|_{L_t^1L_x^2(J)}\\ \le&C\big[1+E(u,v;0)\big]+C\|F_1u\|_{L_t^1L_x^2(J)}+C\|F_2v\|_{L_t^1L_x^2(J)}.\end{aligned}\tag{3.6}$$

On the other hand, (1.12) and Hölder's inequality imply that

$$\begin{aligned}&\|F_1u\|_{L_t^1L_x^2(J)}+\|F_2v\|_{L_t^1L_x^2(J)}\\ \le&C_1+C_1\Big(\big\||u|^4u\big\|_{L_t^1L_x^2(J)}+\big\||u|^4v\big\|_{L_t^1L_x^2(J)}+\big\||v|^4u\big\|_{L_t^1L_x^2(J)}+\big\||v|^4v\big\|_{L_t^1L_x^2(J)}\Big)\\ \le&C_1+C_1\Big(\|u\|_{L_t^\infty L_x^6(J)}\|u\|^4_{L_t^4L_x^{12}(J)}+\|u\|_{L_t^\infty L_x^6(J)}\|v\|^4_{L_t^4L_x^{12}(J)}\\ &\quad+\|v\|_{L_t^\infty L_x^6(J)}\|u\|^4_{L_t^4L_x^{12}(J)}+\|v\|_{L_t^\infty L_x^6(J)}\|v\|^4_{L_t^4L_x^{12}(J)}\Big).\end{aligned}$$

By (1.15′), the $L_t^\infty L_x^6(J)$ norm is $\le(2C_0\varepsilon)^{\frac{1}{6}}$. Therefore, if we let $C_2=C_1C$, then

$$\begin{aligned}&\|u\|_{L_t^4L_x^{12}(J)}+\|v\|_{L_t^4L_x^{12}(J)}\\ \le&\big(C[1+E(u,v;0)]+C_2\big)+2C_2(2C_0\varepsilon)^{\frac{1}{6}}\Big(\|u\|_{L_t^4L_x^{12}(J)}+\|v\|_{L_t^4L_x^{12}(J)}\Big).\end{aligned}$$

Since the constants are independent of s, an application of Lemma 1.1 gives that

$$\|u\|_{L_t^4L_x^{12}(J)}+\|v\|_{L_t^4L_x^{12}(J)}\le2\Big(C[1+E(u,v;0)]+C_2\Big),$$

provided that

$$2C_2(2C_0\varepsilon)^{\frac{1}{6}} < 2^{-4}\big(C[1+E(u,v;0)]+C_2\big)^{-3}.$$

Since ε depends only on T_*,R and $E(u,v;0)$, the proof is complete. □

Now we still need to show that the condition of Proposition 1.3 is reasonable, i.e., the energy cannot concentrate at any fixed point (x_0,T_*). To this end, it suffices to show that

$$\lim_{t\nearrow T_*}\frac{1}{2}\int_{|x-x_0|<T_*-t}(|u'|^2+|v'|^2+F)\mathrm{d}x=0. \tag{3.7}$$

Since the energy consists of two kinds of energy—the kinetic energy and the potential energy from F, to prove (3.7), we may consider them respectively. For the potential energy, we shall need a so-called Morawetz-Pohožaev identity.

A Morawetz-Pohožaev identity. Consider Lagrangian associated with (1.1):

$$L(u,v)=\frac{1}{2}(|u_t|^2+|v_t|^2-|\nabla u|^2-|\nabla v|^2-F).$$

Therefore, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_{[-T_*,0)\times\mathbb{R}^3}L(u+\varepsilon\psi,v+\varepsilon\psi)\mathrm{d}t\mathrm{d}x\bigg|_{\varepsilon=0}\\
=&\frac{1}{2}\int_{[-T_*,0)\times\mathbb{R}^3}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg(|(u+\varepsilon\psi)_t|^2+|(v+\varepsilon\psi)_t|^2-\sum_{j=1}^3|(u+\varepsilon\psi)_j|^2-\sum_{j=1}^3|(v+\varepsilon\psi)_j|^2\\
&-F(|u+\varepsilon\psi|^2,|v+\varepsilon\psi|^2)\bigg)\mathrm{d}t\mathrm{d}x\bigg|_{\varepsilon=0}\\
=&\int_{[-T_*,0)\times\mathbb{R}^3}\bigg((u+\varepsilon\psi)_t\psi_t+(v+\varepsilon\psi)_t\psi_t-\sum_{j=1}^3(u+\varepsilon\psi)_j\psi_j-\sum_{j=1}^3(v+\varepsilon\psi)_j\psi_j\\
&-\frac{1}{2}\frac{\partial F}{\partial\lambda}\frac{\partial\lambda}{\partial(u+\varepsilon\psi)}\frac{\partial(u+\varepsilon\psi)}{\partial\varepsilon}-\frac{1}{2}\frac{\partial F}{\partial\mu}\frac{\partial\mu}{\partial(v+\varepsilon\psi)}\frac{\partial(v+\varepsilon\psi)}{\partial\varepsilon}\bigg)\mathrm{d}t\mathrm{d}x\bigg|_{\varepsilon=0}\\
=&\int_{[-T_*,0)\times\mathbb{R}^3}\bigg(u_t\psi_t+v_t\psi_t-\sum_{j=1}^3u_j\psi_j-\sum_{j=1}^3v_j\psi_j-F_1u\psi-F_2v\psi\bigg)\mathrm{d}t\mathrm{d}x\\
=&-\int_{[-T_*,0)\times\mathbb{R}^3}\Big[(u_{tt}-\Delta u+F_1u)\psi+(v_{tt}-\Delta v+F_2v)\psi\Big]\mathrm{d}t\mathrm{d}x=0,
\end{aligned}$$

whenever $\psi\in C_0^\infty([-T_*,0)\times\mathbb{R}^3)$. Thus, (u,v) must satisfy the Euler-Langrange equations associated with (1.1)

$$\begin{aligned}
&\frac{\partial L(u,v)}{\partial u}-\sum_{j=0}^3\partial_j\left(\frac{\partial L(u,v)}{\partial u_j}\right)=0,\\
&\frac{\partial L(u,v)}{\partial v}-\sum_{j=0}^3\partial_j\left(\frac{\partial L(u,v)}{\partial v_j}\right)=0.
\end{aligned}$$

If u_r,v_r are one-parameter C^1 deformations of u,v respectively, then

$$\begin{aligned}\partial_r L(u_r,v_r)=&\frac{\partial L(u_r,v_r)}{\partial u_r}\partial_r u_r+\frac{\partial L(u_r,v_r)}{\partial v_r}\partial_r v_r\\&+\sum_{j=0}^{3}\frac{\partial L(u_r,v_r)}{\partial u_{rj}}\partial_j\partial_r u_r+\sum_{j=0}^{3}\frac{\partial L(u_r,v_r)}{\partial v_{rj}}\partial_j\partial_r v_r.\end{aligned}$$

If we assume that $u_{r_0}=u$ and $v_{r_0}=v$, then

$$\partial_r L(u_r,v_r)\big|_{r=r_0}=\sum_{j=0}^{3}\partial_j\Big(\frac{\partial L(u,v)}{\partial u_j}\partial_r u+\frac{\partial L(u,v)}{\partial v_j}\partial_r v\Big). \tag{3.8}$$

For the Morawetz identity we need arise from the deformations

$$u_r(x,t)=ru(rx,rt),\quad v_r(x,t)=rv(rx,rt)$$

with $r_0=1$. In this case,

$$\begin{aligned}&\partial_r u_r\big|_{r=1}=u(rx,rt)+\sum_{j=0}^{3}ru_j(rx,rt)x_j\big|_{r=1}=u(x,t)+\sum_{j=0}^{3}u_j(x,t)x_j,\\&\partial_r v_r\big|_{r=1}=v(x,t)+\sum_{j=0}^{3}v_j(x,t)x_j.\end{aligned}$$

Note also that

$$\begin{aligned}&\partial_j u_r=\partial_j\big[ru(rx,rt)\big]=r\partial_j u(rx,rt)r=r^2\partial_j u(rx,rt),\\&\partial_j v_r=\partial_j\big[rv(rx,rt)\big]=r\partial_j v(rx,rt)r=r^2\partial_j v(rx,rt).\end{aligned}$$

Consequently,

$$L(u_r,v_r)=r^4\big[L(u,v)\big](rx,rt)+\frac{1}{2}r^4F\big(|u(rx,rt)|^2,|v(rx,rt)|^2\big)-\frac{1}{2}F(|u_r|^2,|v_r|^2),$$

and hence

$$\begin{aligned}&\partial_r L(u_r,v_r)\big|_{r=1}\\=&\Big\{r^4\partial_r\big(\big[L(u,v)\big](rx,rt)\big)+4r^3\big[L(u,v)\big](rx,rt)+2r^3F\big(|u(rx,rt)|^2,|v(rx,rt)|^2\big)\\&+\frac{1}{2}r^4\partial_r\big[F(|u(rx,rt)|^2,|v(rx,rt)|^2)\big]-\frac{1}{2}\partial_r\big[F(|u_r|^2,|v_r|^2)\big]\Big\}\Big|_{r=1}\\\triangleq& I+II+III+IV+V.\end{aligned}$$

It can be verified that

$$
\begin{aligned}
IV=&\frac{1}{2}r^4\left(\frac{\partial F}{\partial\lambda}\frac{\partial\lambda}{\partial u(rx,rt)}\frac{\partial u(rx,rt)}{\partial r}+\frac{\partial F}{\partial\mu}\frac{\partial\mu}{\partial v(rx,rt)}\frac{\partial v(rx,rt)}{\partial r}\right)\Big|_{r=1}\\
=&F_1u\frac{\partial u}{\partial r}+F_2v\frac{\partial v}{\partial r}
\end{aligned}
$$

and

$$
\begin{aligned}
V=&-\frac{1}{2}\left(\frac{\partial F}{\partial\lambda}\frac{\partial\lambda}{\partial u_r}\frac{\partial u_r}{\partial r}+\frac{\partial F}{\partial\mu}\frac{\partial\mu}{\partial v_r}\frac{\partial v_r}{\partial r}\right)\Big|_{r=1}\\
=&-\left\{F_1u_r\left[u(rx,rt)+r\frac{\partial u(rx,rt)}{\partial r}\right]+F_2v_r\left[v(rx,rt)+r\frac{\partial v(rx,rt)}{\partial r}\right]\right\}\Big|_{r=1}\\
=&-F_1|u|^2-F_1u\frac{\partial u}{\partial r}-F_2|v|^2-F_2v\frac{\partial v}{\partial r}.
\end{aligned}
$$

Consequently,

$$
I+II+III+IV+V=\sum_{j=0}^{3}x_j\partial_jL(u,v)+4L(u,v)+2F-F_1|u|^2-F_2|v|^2.
$$

Combing this with (3.8) leads to

$$
\begin{aligned}
&\sum_{j=0}^{3}\partial_j\left\{\frac{\partial L(u,v)}{\partial u_j}\left[u+\sum_{k=0}^{3}x_ku_k\right]+\frac{\partial L(u,v)}{\partial v_j}\left[v+\sum_{k=0}^{3}x_kv_k\right]-x_jL(u,v)\right\}\\
=&2F-F_1|u|^2-F_2|v|^2.
\end{aligned}
$$

Using the definition of our Lagrangian, we can rewrite the above equation as

$$
\mathrm{div}_{x,t}(-tP,tQ+u_tu+v_tv)=2F-F_1|u|^2-F_2|v|^2, \tag{3.9}
$$

where

$$
\begin{aligned}
Q=&\frac{1}{2}|u'|^2+\frac{1}{2}|v'|^2+\frac{1}{2}F+u_t\frac{x\cdot\nabla u}{t}+v_t\frac{x\cdot\nabla v}{t},\\
P=&\left(\frac{1}{2}|u_t|^2+\frac{1}{2}|v_t|^2-\frac{1}{2}|\nabla u|^2-\frac{1}{2}|\nabla v|^2-\frac{1}{2}F\right)\frac{x}{t}+\left(\frac{u}{t}+u_t+\frac{x\cdot\nabla u}{t}\right)\nabla u\\
&+\left(\frac{v}{t}+v_t+\frac{x\cdot\nabla v}{t}\right)\nabla v.
\end{aligned}
$$

In fact, note that

$$\begin{aligned}
&\frac{\partial L(u,v)}{\partial u_t}\Big[u+\sum_{k=0}^{3}x_k u_k\Big]+\frac{\partial L(u,v)}{\partial v_t}\Big[v+\sum_{k=0}^{3}x_k v_k\Big]-tL(u,v)\\
=&u_t(u+tu_t+x\cdot\nabla u)+v_t(v+tv_t+x\cdot\nabla v)\\
&-\frac{t}{2}\Big(|u_t|^2+|v_t|^2-|\nabla u|^2-|\nabla v|^2-F\Big)\\
=&u_t u+t|u_t|^2+u_t x\cdot\nabla u+v_t v+t|v_t|^2+v_t x\cdot\nabla v\\
&-\frac{t}{2}\Big(|u_t|^2+|v_t|^2-|\nabla u|^2-|\nabla v|^2-F\Big)\\
=&u_t u+v_t v+\frac{t}{2}|u'|^2+\frac{t}{2}|v'|^2+\frac{t}{2}F+u_t x\cdot\nabla u+v_t x\cdot\nabla v\\
\triangleq& I,
\end{aligned}$$

which implies that

$$\frac{(I-u_t u-v_t v)}{t}=\frac{1}{2}|u'|^2+\frac{1}{2}|v'|^2+\frac{1}{2}F+u_t\frac{x\cdot\nabla u}{t}+v_t\frac{x\cdot\nabla v}{t}=Q.$$

On the other hand,

$$\begin{aligned}
&\frac{\partial L(u,v)}{\partial u_j}\Big[u+\sum_{k=0}^{3}x_k u_k\Big]+\frac{\partial L(u,v)}{\partial v_j}\Big[v+\sum_{k=0}^{3}x_k v_k\Big]-x_jL(u,v)\\
=&-u_j(u+tu_t+x\cdot\nabla u)-v_j(v+tv_t+x\cdot\nabla v)\\
&-\frac{x_j}{2}\Big(|u_t|^2+|v_t|^2-|\nabla u|^2-|\nabla v|^2-F\Big)\\
\triangleq& II_j,
\end{aligned}$$

which yields that

$$\begin{aligned}
P_j=\frac{II_j}{-t}=&u_j\left(\frac{u}{t}+u_t+\frac{x\cdot\nabla u}{t}\right)+v_j\left(\frac{v}{t}+v_t+\frac{x\cdot\nabla v}{t}\right)\\
&+\frac{1}{2}\left(|u_t|^2+|v_t|^2-|\nabla u|^2-|\nabla v|^2-F\right)\frac{x_j}{t}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
P=&\left(\frac{1}{2}|u_t|^2+\frac{1}{2}|v_t|^2-\frac{1}{2}|\nabla u|^2-\frac{1}{2}|\nabla v|^2-\frac{1}{2}F\right)\frac{x}{t}+\left(\frac{u}{t}+u_t+\frac{x\cdot\nabla u}{t}\right)\nabla u\\
&+\left(\frac{v}{t}+v_t+\frac{x\cdot\nabla v}{t}\right)\nabla v.
\end{aligned}$$

Now we shall use the so-called Morawetz-Pohožaev identity we just obtained above to prove the non-concentration of the energy from F. We shift (x_0,T_*) to the origin again for simplicity.

Proposition 3.1. *Let* $k=5$ *and let* (u,v) *be as in Proposition* 1.3. *Then*

$$\lim_{t\nearrow T_*}\frac{1}{2}\int_{|x|<t}F\mathrm{d}x=0.$$

Proof. If $T_*<T<S\le 0$, we set

$$\begin{aligned}&D_T=\{(x,T):|x|\le -T\},\\&\Lambda(T,S)=\{(x,t):T\le t\le S,\quad |x|\le -t\},\\&M_T^S=\{(x,t):T\le t\le S,\quad |x|=-t\}.\end{aligned}$$

If we integrate (3.9) and apply the divergence theorem, we obtain

$$\begin{aligned}&\int_{D_S}(SQ+u_tu+v_tv)\mathrm{d}x+J+JJ\\=&\int\int_{\Lambda(T,S)}(2F-F_1|u|^2-F_2|v|^2)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where

$$\begin{aligned}&J=-\int_{D_T}(TQ+u_tu+v_tv)\mathrm{d}x,\\&JJ=\frac{1}{\sqrt{2}}\int_{M_T^0}(tQ+u_tu+v_tv+x\cdot P)\mathrm{d}\sigma.\end{aligned}$$

Using Proposition 2.1 and Höder's inequality one finds that the first term goes to zero as $S\nearrow 0$. Thus

$$J+JJ=\int\int_{\Lambda(T,0)}(2F-F_1|u|^2-F_2|v|^2)\mathrm{d}x\mathrm{d}t.$$

Using the assumption (H3), i.e., $F_1|u|^2+F_2|v|^2\ge 2F$ when $|u|,|v|$ are larger than a fixed constant, we conclude that

$$J+JJ\le CT^4. \tag{3.10}$$

In fact, $E(u,v;t)=E(u,v;0)<\infty$, implies that

$$\int_{\mathbb{R}^3}F\mathrm{d}x<\infty.$$

On the other hand, the fact $F\ge 0$ and F is continuous implies that F is bounded by some constant M, while (H1) leads to

$$\begin{aligned}&\big|F_1|u|^2+F_2|v|^2\big|\\\le& C(1+|u|^{k-1}+|v|^{k-1})(|u|^2+|v|^2)\le C(1+|u|^{k+1}+|v|^{k+1}),\end{aligned}$$

where we have used the facts that

$$|u|^{k-1}|v|^2,|v|^{k-1}|u|^2\le\big(\max(|u|,|v|)\big)^{k+1}\le |u|^{k+1}+|v|^{k+1}$$

and $|\cdot|^2 \le C(1+|\cdot|^{k+1})$ for $k>1$. Therefore,

$$\begin{aligned}
J+JJ=&\int\int_{\Lambda(T,0)} \big(2F-F_1|u|^2-F_2|v|^2\big)\mathrm{d}x\mathrm{d}t\\
\le&\int\int_{\Lambda(T,0)} \Big(2F+C[1+|u|^{k+1}+|v|^{k+1}]\Big)\mathrm{d}x\mathrm{d}t\\
\le&\int\int_{\Lambda(T,0)} \Big(2F+C+CC_0[1+\frac{F}{2}]\Big)\mathrm{d}x\mathrm{d}t\\
=&\int\int_{\Lambda(T,0)} \Big[(2+\frac{CC_0}{2})F+(C+CC_0)\Big]\mathrm{d}x\mathrm{d}t\\
\le&\int\int_{\Lambda(T,0)} \Big[(2+\frac{CC_0}{2})M+(C+CC_0)\Big]\mathrm{d}x\mathrm{d}t\\
\le&\Big[(2+\frac{CC_0}{2})M+(C+CC_0)\Big]\int_0^{|T|}\frac{4\pi}{3}|t|^3\mathrm{d}t\le CT^4.
\end{aligned}$$

To exploit (3.10), we need to obtain a lower bound for J and JJ. Note that $t=-|x|$ on M_T^0. We rewrite JJ as

$$\begin{aligned}
JJ=&\frac{1}{\sqrt{2}}\int_{M_T^0}(tQ+u_tu+v_tv+x\cdot P)\mathrm{d}\sigma\\
=&\frac{1}{\sqrt{2}}\int_{M_T^0}\Big(-\frac{|x|}{2}|u'|^2-\frac{|x|}{2}|v'|^2-\frac{|x|}{2}F+u_tx\cdot\nabla u+v_tx\cdot\nabla v+u_tu+v_tv\\
&-\frac{u}{|x|}x\cdot\nabla u+u_tx\cdot\nabla u-\frac{1}{|x|}(x\cdot\nabla u)^2-\frac{v}{|x|}x\cdot\nabla v+v_tx\cdot\nabla v-\frac{1}{|x|}(x\cdot\nabla v)^2\\
&+\frac{|x|}{2}|\nabla u|^2+\frac{|x|}{2}|\nabla v|^2-\frac{|x|}{2}|u_t|^2-\frac{|x|}{2}|v_t|^2+\frac{|x|}{2}F\Big)\mathrm{d}\sigma\\
=&\frac{1}{\sqrt{2}}\int_{M_T^0}\Big(-|x||u_t|^2-|x||v_t|^2+2u_tx\cdot\nabla u+2v_tx\cdot\nabla v+u_tu+v_tv\\
&-\frac{1}{|x|}(x\cdot\nabla u)^2-\frac{1}{|x|}(x\cdot\nabla v)^2-\frac{u}{|x|}x\cdot\nabla u-\frac{v}{|x|}x\cdot\nabla v\Big)\mathrm{d}\sigma\\
=&-\frac{1}{\sqrt{2}}\int_{M_T^0}\Big(|x|(\frac{x\cdot\nabla u}{|x|}-u_t)^2+u(\frac{x\cdot\nabla u}{|x|}-u_t)\\
&+|x|(\frac{x\cdot\nabla v}{|x|}-v_t)^2+v(\frac{x\cdot\nabla v}{|x|}-v_t)\Big)\mathrm{d}\sigma.
\end{aligned}$$

If we parameterize M_T^0 by

$$y\longrightarrow(y,-|y|),\quad |y|\le -T,$$

then $d\sigma=\sqrt{2}dy$. If we set $\bar{u}(y)=u(y,-|y|)$ and $\bar{v}(y)=v(y,-|y|)$, then

$$y\cdot\frac{\nabla\bar{u}}{|y|}=\frac{x\cdot\nabla u}{|x|}-u_t,$$
$$y\cdot\frac{\nabla\bar{v}}{|y|}=\frac{x\cdot\nabla v}{|x|}-v_t,$$

where $\nabla\bar{u}=\sum_{j=0}^{3}\partial_j\bar{u}$ and $\nabla u=\sum_{j=1}^{3}\partial_j u$. Therefore

$$\begin{aligned}JJ=&-\int_{|y|\le|T|}\left(\frac{|y\cdot\nabla\bar{u}|^2}{|y|}+\bar{u}\frac{y\cdot\nabla\bar{u}}{|y|}+\frac{|y\cdot\nabla\bar{v}|^2}{|y|}+\bar{v}\frac{y\cdot\nabla\bar{v}}{|y|}\right)dy\\ =&-\int_{|y|\le|T|}\frac{1}{|y|}\left(|y\cdot\nabla\bar{u}+\bar{u}|^2+|y\cdot\nabla\bar{v}+\bar{v}|^2\right)dy\\ &+\int_{|y|\le|T|}\frac{1}{|y|}\left(|\bar{u}|^2+\bar{u}y\cdot\nabla\bar{u}+|\bar{v}|^2+\bar{v}y\cdot\nabla\bar{v}\right)dy.\end{aligned}$$

To evaluate the second term, note that, if we use polar coordinates $y=r\omega$, then

$$\bar{u}y\cdot\nabla\bar{u}/|y|=\bar{u}\partial_r\bar{u}=\frac{1}{2}\partial_r(\bar{u}^2).$$

Hence, integration by parts gives

$$\begin{aligned}\int_{|y|\le|T|}\bar{u}\frac{y\cdot\nabla\bar{u}}{|y|}dy=&\frac{1}{2}\int_{S^2}\int_0^{|T|}\partial_r\left[\bar{u}^2(r\omega)\right]r^2drd\sigma(\omega)\\ =&\frac{1}{2}\int_{S^2}\bar{u}^2(|T|\omega)|T|^2d\sigma(\omega)-\int_{S^2}\int_0^{|T|}\bar{u}^2(r\omega)rdrd\sigma(\omega)\\ =&\frac{1}{2}\int_{\partial D_T}u^2d\sigma-\int_{|y|\le|T|}\frac{\bar{u}^2}{|y|}dy,\end{aligned}$$

and

$$\int_{|y|\le|T|}\bar{v}\frac{y\cdot\nabla\bar{v}}{|y|}dy=\frac{1}{2}\int_{\partial D_T}v^2d\sigma-\int_{|y|\le|T|}\frac{\bar{v}^2}{|y|}dy.$$

Combing these with the earlier formula and switching back to the original coordinates gives

$$\begin{aligned}JJ=&\frac{1}{\sqrt{2}}\int_{M_T^0}\left(t\left|\frac{x}{|x|}\cdot\nabla u-u_t+\frac{u}{|x|}\right|^2+t\left|\frac{x}{|x|}\cdot\nabla v-v_t+\frac{v}{|x|}\right|^2\right)d\sigma\\ &+\frac{1}{2}\int_{\partial D_T}(u^2+v^2)d\sigma.\end{aligned}\tag{3.11}$$

When it comes to J, we first notice that

$$\begin{aligned}TQ+u_tu+v_tv=&T\left(\frac{1}{2}|u'|^2+\frac{1}{2}|v'|^2+\frac{1}{2}F+\frac{u_t}{T}x\cdot\nabla u+\frac{v_t}{T}x\cdot\nabla v\right)+u_tu+v_tv\\ =&T\left(\frac{1}{2}|u'|^2+\frac{1}{2}|v'|^2+\frac{1}{2}F\right)+u_t(u+x\cdot\nabla u)+v_t(v+x\cdot\nabla v),\end{aligned}$$

while

$$\begin{aligned}|u_t(u+x\cdot\nabla u)|=&\left|u_t\left(\frac{x\cdot x}{|x|^2}u+x\cdot\nabla u\right)\right|=\left|u_tx\cdot\left(\frac{x}{|x|}u+\nabla u\right)\right|\\ =&\left|x\cdot u_t\left(\frac{x}{|x|}u+\nabla u\right)\right|\leq-T\left(\frac{1}{2}|u_t|^2+\frac{1}{2}\left|\frac{x}{|x|}u+\nabla u\right|^2\right),\\ |v_t(v+x\cdot\nabla v)|\leq&-T\left(\frac{1}{2}|v_t|^2+\frac{1}{2}|\frac{x}{|x|}v+\nabla v|^2\right).\end{aligned}$$

Consequently,

$$\begin{aligned}J\geq&-T\int_{D_T}\frac{1}{2}F\mathrm{d}x-T\int_{D_T}\left(\frac{1}{2}|\nabla u|^2-\frac{1}{2}|\nabla u+\frac{x}{|x|^2}u|^2+\frac{1}{2}|\nabla v|^2-\frac{1}{2}|\nabla v+\frac{x}{|x|^2}v|^2\right)\mathrm{d}x\\ =&|T|\int_{D_T}\frac{1}{2}F\mathrm{d}x+T\left\{\int_{D_T}\left(u\frac{x}{|x|^2}\cdot\nabla u+v\frac{x}{|x|^2}\cdot\nabla v\right)\mathrm{d}x+\frac{1}{2}\int_{D_T}\left(\frac{u^2}{|x|^2}+\frac{v^2}{|x|^2}\right)\mathrm{d}x\right\}.\end{aligned}$$

Using integration by parts as before, we find that

$$\begin{aligned}\int_{D_T}u\frac{x}{|x|^2}\cdot\nabla u\mathrm{d}x+\frac{1}{2}\int_{D_T}\frac{u^2}{|x|^2}\mathrm{d}x=\frac{1}{2}\int_{\partial D_T}\frac{u^2}{-T}\mathrm{d}\sigma,\\ \int_{D_T}v\frac{x}{|x|^2}\cdot\nabla v\mathrm{d}x+\frac{1}{2}\int_{D_T}\frac{v^2}{|x|^2}\mathrm{d}x=\frac{1}{2}\int_{\partial D_T}\frac{v^2}{-T}\mathrm{d}\sigma,\end{aligned}$$

which yields

$$J>|T|\int_{D_T}\frac{1}{2}F\mathrm{d}x-\frac{1}{2}\int_{\partial D_T}(u^2+v^2)\mathrm{d}\sigma.$$

Combining this with (3.10) and (3.11), we see that

$$\begin{aligned}&|T|\int_{D_T}\frac{1}{2}F\mathrm{d}x\\ \leq&CT^4+\frac{1}{\sqrt{2}}\int_{M_T^0}|t|\left(\left|\frac{x}{|x|}\cdot\nabla u-u_t+\frac{u}{|x|}\right|^2+\left|\frac{x}{|x|}\cdot\nabla v-v_t+\frac{v}{|x|}\right|^2\right)\mathrm{d}\sigma\\ \leq&CT^4+|T|\int_{M_T^0}\left(|\frac{x}{|x|}\cdot\nabla u-u_t|^2+|\frac{x}{|x|}\cdot\nabla v-v_t|^2\right)\mathrm{d}\sigma+\int_{M_T^0}\frac{u^2+v^2}{|t|}\mathrm{d}\sigma.\end{aligned}\tag{3.12}$$

By (3.5), the second last term can be bounded by $|T|\text{Flux}(u,v;M_T^0)$, and the last term can also be controlled by the energy flux. In fact, if we use (H2) and Hölder's inequality, then we have

$$\begin{aligned}\int_{M_T^0}\frac{u^2+v^2}{|t|}\mathrm{d}\sigma=&\int_{M_T^0}\frac{u^2}{|t|}\mathrm{d}\sigma+\int_{M_T^0}\frac{v^2}{|t|}\mathrm{d}\sigma\\ \leq&\Big(\int_{M_T^0}|t|^{-\frac{3}{2}}\mathrm{d}\sigma\Big)^{\frac{2}{3}}\Big[\Big(\int_{M_T^0}u^6\mathrm{d}\sigma\Big)^{\frac{1}{3}}+\Big(\int_{M_T^0}v^6\mathrm{d}\sigma\Big)^{\frac{1}{3}}\Big]\\ \leq&2|T|\Big[\int_{M_T^0}C_0(1+\frac{1}{2}F)\mathrm{d}\sigma\Big]^{\frac{1}{3}}\\ \leq&2|T|[C_0\text{Flux}(u,v;M_T^0)]^{\frac{1}{3}}+2|T|\Big(\frac{4\pi}{3}C_0|T|^3\Big)^{\frac{1}{3}}.\end{aligned}$$

If we plug in our estimates for the last two terms into (3.12), we conclude for small $|T|$ that

$$\begin{aligned}\int_{D_T}\frac{1}{2}F\mathrm{d}x\leq& C|T|^3+\text{Flux}(u,v;M_T^0)+2[C_0\text{Flux}(u,v;M_T^0)]^{\frac{1}{3}}+2\Big(\frac{4\pi}{3}C_0\Big)^{\frac{1}{3}}|T|\\ \leq& C\big[|T|+\text{Flux}(u,v;M_T^0)+(\text{Flux}(u,v;M_T^0))^{\frac{1}{3}}\big].\end{aligned}$$

This finally gives us the result since $\text{Flux}(u,v;M_T^0)\to 0$ as $T\nearrow 0$. □

To finish the proof of the global existence theorem, we are just left with showing that the kinetic energy can not concentrate.

Proposition 3.2. *Let $k=5$ and let (u,v) be as in Proposition* 1.3. *Then*

$$\lim_{t\nearrow T_*}\frac{1}{2}\int_{|x-x_0|<T_*-t}(|u'|^2+|v'|^2)\mathrm{d}x=0. \tag{3.13}$$

Proof. From the assumptions (H2) and (H4), the non-concentrate of the potential energy from semi-linearity F

$$\lim_{t\nearrow T_*}\frac{1}{2}\int_{|x-x_0|<T_*-t}F\mathrm{d}x=0.$$

is equivalent to

$$\lim_{t\nearrow T_*}\int_{|x-x_0|<T_*-t}(|u|^6+|v|^6)\mathrm{d}x=0.$$

The proof of Proposition 1.3 shows that this in turn implies that, for backward light cone through (x_0,T_*) we have

$$u,v\in L_t^4L_x^{12}(\Lambda(0;0,T_*)). \tag{3.14}$$

Applying (1.9) to

$$\begin{cases} u'_{tt}-\Delta u'=-(F_1u)' \\ v'_{tt}-\Delta v'=-(F_2v)' \end{cases}$$

and arguing as in the proof of Proposition 1.3 gives for $J_0=\Lambda(0;t_0,s)$

$$\begin{aligned}
&\sup_{t_0\le t\le T_*}\left(\int_{|x-x_0|<T_*-t}|u|^6\mathrm{d}x\right)^{\frac{1}{6}}+\sup_{t_0\le t\le T_*}\left(\int_{|x-x_0|<T_*-t}|v|^6\mathrm{d}x\right)^{\frac{1}{6}}\\
=&\|u'\|_{L_t^\infty L_x^6(J_0)}+\|v'\|_{L_t^\infty L_x^6(J_0)}\\
\le& C\sum_{|\alpha|=2}\|\partial^\alpha u(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}+C\|(F_1u)'\|_{L_t^1L_x^2(J_0)}\\
&+C\sum_{|\alpha|=2}\|\partial^\alpha v(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}+C\|(F_2v)'\|_{L_t^1L_x^2(J_0)}\\
\le& C\sum_{|\alpha|=2}\Big(\|\partial^\alpha u(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}+\|\partial^\alpha v(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\Big)\\
&+C'\Big(1+\big\||u|^4u'\big\|_{L_t^1L_x^2(J_0)}+\big\||u|^4v'\big\|_{L_t^1L_x^2(J_0)}+\big\||v|^4u'\big\|_{L_t^1L_x^2(J_0)}+\big\||v|^4v'\big\|_{L_t^1L_x^2(J_0)}\Big)\\
\le& C\sum_{|\alpha|=2}\Big(\|\partial^\alpha u(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}+\|\partial^\alpha v(\cdot,t_0)\|_{L^2(\mathbb{R}^3)}\Big)\\
&+C'+C'\Big(\|u\|^4_{L_t^4L_x^{12}(J_0)}\|u'\|_{L_t^\infty L_x^6(J_0)}+\|u\|^4_{L_t^4L_x^{12}(J_0)}\|v'\|_{L_t^\infty L_x^6(J_0)}\\
&+\|v\|^4_{L_t^4L_x^{12}(J_0)}\|u'\|_{L_t^\infty L_x^6(J_0)}+\|v\|^4_{L_t^4L_x^{12}(J_0)}\|v'\|_{L_t^\infty L_x^6(J_0)}\Big)\\
=&C(t_0)+C'\Big(\|u\|^4_{L_t^4L_x^{12}(J_0)}\|u'\|_{L_t^\infty L_x^6(J_0)}+\|u\|^4_{L_t^4L_x^{12}(J_0)}\|v'\|_{L_t^\infty L_x^6(J_0)}\\
&+\|v\|^4_{L_t^4L_x^{12}(J_0)}\|u'\|_{L_t^\infty L_x^6(J_0)}+\|v\|^4_{L_t^4L_x^{12}(J_0)}\|v'\|_{L_t^\infty L_x^6(J_0)}\Big),
\end{aligned}$$

The result (3.14) implies that the $L_t^4L_x^{12}(J_0)$ norm goes to zero as $t_0\to T_*$. We therefore conclude that, if t_0 is close to T_*,

$$\sup_{t_0\le t<T_*}\left(\int_{|x-x_0|<T_*-t}|u'|^6\mathrm{d}x\right)^{\frac{1}{6}}+\sup_{t_0\le t<T_*}\left(\int_{|x-x_0|<T_*-t}|v'|^6\mathrm{d}x\right)^{\frac{1}{6}}\le 2C(t_0).$$

But the application of Hölder's inequality shows that this leads to

$$\begin{aligned}
&\left(\int_{|x-x_0|<T_*-t}|u'|^2\mathrm{d}x\right)^{\frac{1}{2}}\le 2C(t_0)\left(\frac{4\pi}{3}(T_*-t)^3\right)^{\frac{1}{3}},\\
&\left(\int_{|x-x_0|<T_*-t}|v'|^2\mathrm{d}x\right)^{\frac{1}{2}}\le 2C(t_0)\left(\frac{4\pi}{3}(T_*-t)^3\right)^{\frac{1}{3}}.
\end{aligned}$$

Hence we obtained the desired result (3.13). This completes the proof of Proposition 3.2. □

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References

[1] Schiff L I, Non-linear meson theory of nuclear forces I. *Phys Rev*, 1951, **84**: 1–9.
[2] Segal I E, The global Cauchy problem for a relativistic scalar field with power interaction. *Bull Soc Math France*, 1963, **91**: 129–135.
[3] Shatah J, Weak solutions and the development of singularities in the SU(2) σ-model. *Comm Pure Appl Math*, 1988, **41**: 459–469.
[4] Miao C X, Hamonic Analysis and Applications to PDE. Monographs on Modern Pure Mathematics, No.89 (2nd ed.). Beijing: Science Press, 2004.
[5] Sogge C D, Lectures on Nonlinear Wave Equation, Monographs in Analysis Vol. II. Cambridge: Cambridge University Press, 1995.
[6] Strichartz W, Restrictions of Fourier transforms to quadratic surface and decay of solutions of wave equations. *Duke Math J*, 1977, **44**: 705–714.
[7] Morawetz C, Time decay for nonlinear Klein-Gordon equations. *Proc Royal Soc London A*, 1968, **306**: 503–518.
[8] Lin J E and Strauss W, Decay and scattering of Schödinger equation. *J Func Anal*, 1978, **30**: 245–263.
[9] Brenner P, On scattering and every defined scattering operator for nonlinear Klein-Gordon equations. *J Differial Equations*, 1985, **56**: 310–344.
[10] Ginibre J and Velo G, Conformal invariance and time decay for nonlinear wave equations. *Ann Inst Henri Poincaré Phys Theérique*, 1987, **47**: 221–276.
[11] Morawetz C S and Strauss W, Decay and scattering of solutions of a nonlinear relativistic wave equation. *Comm Pure Appl Math*, 1972, **25**: 1–31.
[12] Nakanish K, Remarks on the energy scattering for nonlinear Klein-Gordon and Schrödinger equations. *Tohoku Math J*, 2001, **53**: 285–303.
[13] Strauss W, Nonlinear Invariant Wave Equations. Lecture Note in Physics, Vol 78, Berlin: Springer-Verlag, 1978, 197–249.
[14] Strauss W, Nonlinear Wave Equation. Regional Conference Series in Mathematics, Vol 73, Providence, Rhode Island: Amer Math Soc, 1989.
[15] Ginibre J and Velo G, Scattering theory in energy space for a class of nonlinear Schrödinger equations. *J Math Pure Appl*, 1985, **64**: 363–401.
[16] Lin J E, Local time decay for a non-linear beam equations. *Methods Appl Anal*, 2004, **11**: 065–068.
[17] Miao C X, Modern Method for Nonlinear Wave Equations. Lectures in Contemporary Mathematics No.2, Beijing: Science Press, 2005.