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Boundary Homogenization in the Spontaneous Potential Well-Logging

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Abstract. Motivated by the study on the spontaneous potential well-logging, this paper deals with the homogenization of boundary conditions for a class of elliptic problems with jump interface conditions.

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Key Words: Jump interface condition; boundary condition with equivalued surface; homogenization of boundary condition; elliptic problem.

1 Introduction

In petroleum exploitation one often uses various methods of well-logging, among which the spontaneous potential (SP) well-logging is one of the most common and important techniques. Since positive ions and negative ions have different diffusion speeds in a solution, and the grains of mud-stone often absorb positive ions, there is a steady potential difference called the *spontaneous potential difference* on any interface of different formations. These potential differences cause a spontaneous potential field in the earth. After a well has been drilled, one puts a log-tool with a measuring electrode into the well and then measures the SP on the electrode. Raising the tool along the well-bore one gets the corresponding SP curve, as shown in Fig. 1. The SP on the electrode varies with the change of the rock formation, and it shows the osmotic formation clearly (cf. [1–4]).

As usual, we suppose that the formation is symmetric about the well axis and the central plane (see Fig. 2), then the SP field has the same symmetry. Therefore, we consider only the corresponding two-dimensional problem instead of the three-dimensional one. In addition, since the influence of the electric field is very little far apart from the

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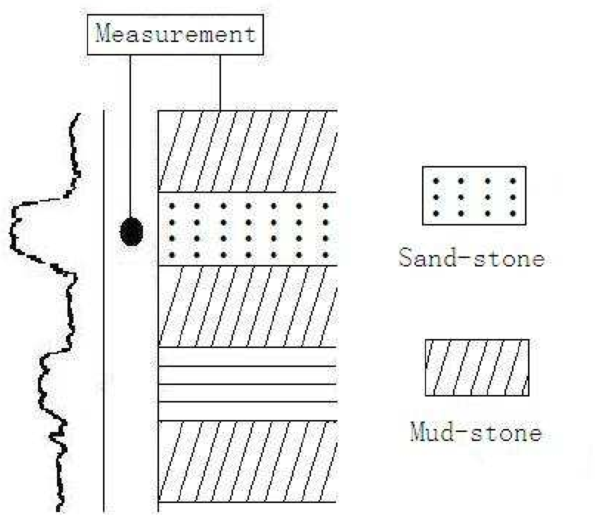

Figure 1: Illustration of measurement for the well-logging.

electrode, we may suppose that the field exists only on a finite but quite large region (cf. [5,6]).

Taking the center of the electrode as the origin and the well axis as the z axis (see Fig. 2), we consider the problem on the domain

$$\Omega \triangleq \{(r,z)|0\leq r\triangleq \sqrt{x^2+y^2}\leq R, 0\leq z\leq Z\},$$

where R and Z are suitably large positive numbers. Suppose furthermore that the resistivity of the earth is piecewise constant:

$$Re=\begin{cases} R_m\triangleq R_1 & \text{in } \Omega_m\triangleq\Omega_1,\\ R_s\triangleq R_2 & \text{in } \Omega_s\triangleq\Omega_2,\\ R_{x_0}\triangleq R_3 & \text{in } \Omega_{x_0}\triangleq\Omega_3,\\ R_t\triangleq R_4 & \text{in } \Omega_t\triangleq\Omega_4. \end{cases}$$

In Fig. 2, the shaded part is the area occupied by the log-tool, whose top surface is insulated; Ω_m is the well-bore filled by mud; Ω_s is the enclosing rock; Ω_{x_0} and Ω_t are two parts of the objective formation, which is the main object to be measured. Since the objective formation is usually composed of porous sand-stone, the mud filtrate penetrates into the porous region and changes the resistivity in the domain Ω_{x_0}, which is then called the invaded zone.

If the geometrical structure of the formation, the resistivity in each subdomain and the SP difference on each interface are all known, as a direct problem, the spontaneous potential $u(r,z)$ satisfies the following quasi-harmonic equation in each subdomain Ω_i $(1\leq i\leq 4)$:

$$Lu=0,$$

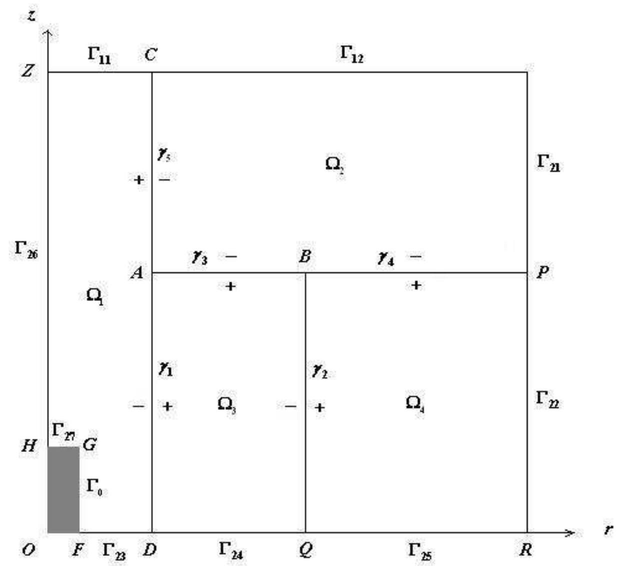

Figure 2: Illustration of a symmetric SP field.

where

$$L=-\left(\frac{\partial}{\partial r}\left(\frac{r}{\mathrm{Re}}\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{r}{\mathrm{Re}}\frac{\partial}{\partial z}\right)\right)$$

is the quasi-harmonic operator.

Owing to SP difference E_j $(1\leq j\leq 5)$ on each segment γ_j $(1\leq j\leq 5)$ of the interface between formations with different resistivity, the electric current is continuous but the potential has a jump. Thus we have the interface conditions

$$\begin{cases} u^+-u^-=E_j, \\ \left(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}\right)^+=\left(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}\right)^-, \end{cases}$$

where the superscripts '$+$' and '$-$' stand for the values on both sides of γ_j, respectively, as prescribed in Fig. 2. The unite normal vector $\mathbf{n}$ takes the same direction on both sides of γ_j.

Furthermore, since the electrode does not discharge any electric current, on the surface Γ_0 of the measuring electrode, we have the following boundary condition with equivalued surface

$$\begin{cases} u=C \ \ (\text{unknown constant}), \\ \displaystyle\int_{\Gamma_0}\frac{r}{\mathrm{Re}}\frac{\partial u}{\partial n}\mathrm{d}s=0, \end{cases}$$

where $\mathbf{n}$ is the unit outward normal vector to Γ_0.

On the well axis, the plane of symmetry, the insulated surface of the log-tool and the distant boundary $r=R$, the normal derivative of the SP vanishes. Denoting these parts of the boundary by $\Gamma_2 \triangleq \bigcup_{j=1}^{7} \Gamma_{2j}$, we have

$$\frac{r}{\mathrm{Re}}\frac{\partial u}{\partial n}=0 \quad \text{on } \Gamma_2.$$

Moreover, on the surface of the earth $\Gamma_1 \triangleq \Gamma_{11} \bigcup \Gamma_{12}$, in order to be compatible with the jump condition on γ_5, we have

$$u=E_5(C) \ \text{ on } \ \Gamma_{11}; \quad u=0 \ \text{ on } \ \Gamma_{12}.$$

Without loss of generality, we may suppose $E_5(C)=0$. Otherwise, by making a translation $\tilde{u}=u-v$, where

$$v=\begin{cases} E_5(C) & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_i, \quad i=2,3,4, \end{cases}$$

this condition can be always satisfied. Consequently, the boundary conditions on Γ_1 can be unified to $u=0$. Thus, on the domain Ω, the SP potential $u=u(r,z)$ satisfies the following problem (cf. [1–4]):

$$(\mathrm{I}) \quad \begin{cases} Lu=0 & \text{in } \Omega_i, \quad 1\le i\le 4, \\ u=0, \quad \text{on } \Gamma_1, \qquad \dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}=0, \quad \text{on } \Gamma_2, & \\ u=C \ \ (\text{unknown constant}) & \text{on } \Gamma_0, \\ \displaystyle\int_{\Gamma_0}\frac{r}{\mathrm{Re}}\frac{\partial u}{\partial n}\mathrm{d}s=0, & \\ u^+-u^-=E_j, \quad \Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}\Big)^+=\Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}\Big)^-, & \text{on } \gamma_j, \quad 1\le j\le 5. \end{cases}$$

It is known that the problem (I) is always well-posed (cf. [1–4]). More precisely, when the compatibility condition

$$\begin{cases} \Delta_A \triangleq E_1(A)+E_5(A)-E_3(A)=0, \\ \Delta_B \triangleq E_2(B)+E_3(B)-E_4(B)=0 \end{cases} \tag{1.1}$$

is satisfied, there exists a unique piecewise H^1_* weak solution. On the other hand, when the compatibility condition (1.1) fails, the problem has a unique piecewise $W^{1,p}_*(1<p_0<p<2)$ weak solution (cf. [3,4]).

With the purpose of making a perfect contact between the electrode and the well-wall, engineers have designed a so-called patched electrode system by separating the electrode into many small cells, embedding them into a rubber and then connecting them with wire behind the rubber. This electrode system is much more flexible than the original one and fits the shape of the well wall more easily.

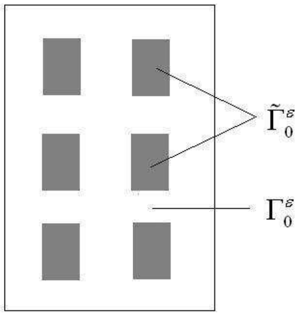

Figure 3: A partition of Γ_0.

In the patched electrode system we partition Γ_0 into two parts Γ_0^ε and $\widetilde{\Gamma}_0^\varepsilon$ (see Fig. 3), where Γ_0^ε denotes the rubber surface inside Γ_0, while $\widetilde{\Gamma}_0^\varepsilon$ denotes the union of the connected surfaces of all the metal cells $\widetilde{\Gamma}_{0,i}^\varepsilon$ $(i=1,\cdots,m(\varepsilon))$, namely,

$$\widetilde{\Gamma}_0^\varepsilon=\bigcup_{i=1}^{m(\varepsilon)}\widetilde{\Gamma}_{0,i}^\varepsilon.$$

We denote the potential corresponding to the patched electrode system by u_ε. Since there is a short-circuit between the connected pieces of the electrode, the potential u_ε must be still a constant (unknown) on $\widetilde{\Gamma}_0^\varepsilon$. Therefore, the boundary condition on Γ_0 should be changed to

$$\begin{cases} \dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}=0 & \text{on } \Gamma_0^\varepsilon,\\ u_\varepsilon=C_\varepsilon \ \ (\text{unknown constant}) & \text{on } \widetilde{\Gamma}_0^\varepsilon,\\ \displaystyle\int_{\widetilde{\Gamma}_0^\varepsilon}\frac{r}{\mathrm{Re}}\frac{\partial u_\varepsilon}{\partial n}\mathrm{d}s=0. \end{cases}$$

Then u_ε satisfies the following problem:

$$(\mathrm{I}_\varepsilon)\quad \begin{cases} Lu_\varepsilon=0 \qquad \text{in } \Omega_i, \quad 1\le i\le 4,\\ u_\varepsilon=0, \quad \text{on } \Gamma_1, \qquad \dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}=0, \quad \text{on } \Gamma_2,\\ \dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}=0, \quad \text{on } \Gamma_0^\varepsilon, \qquad u_\varepsilon=C_\varepsilon, \quad \text{on } \widetilde{\Gamma}_0^\varepsilon,\\ \displaystyle\int_{\widetilde{\Gamma}_0^\varepsilon}\frac{r}{\mathrm{Re}}\frac{\partial u_\varepsilon}{\partial n}\mathrm{d}s=0,\\ u_\varepsilon^+-u_\varepsilon^-=E_j, \quad \Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}\Big)^+=\Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}\Big)^- \quad \text{on } \gamma_j, \quad 1\le j\le 5, \end{cases}$$

where C_ε is a constant to be determined.

The problem (I_ε) is still well-posed, however, it is quite difficult to solve this problem by a numerical method, for example, by the finite element method. In fact, if the electrode is composed of plenty of metal cells, the boundary condition on Γ_0 rapidly changes its type, and then we need a great many nodes in a neighbourhood of each $\widetilde{\Gamma}^\varepsilon_{0,i}$. This increases the amount of computation greatly, and does not guarantee the precision in calculation. Furthermore, the situation will become more difficult along with the increment of mental cells; the problem may even not be solved numerically.

In order to reduce this complexity of computation it is natural to ask if this complicated boundary condition on $\Gamma_0=\Gamma_0^\varepsilon\cup\widetilde{\Gamma}_0^\varepsilon$ can be replaced approximately by a much simpler and unified boundary condition. Moreover, if it exists, it will be interesting to know the reduced boundary condition. This kind of problem was first discussed by Damlamian and Li for the resistivity well-logging (cf. [6–9]), characterized by emitting an electric current from the electrode, but not possessing spontaneous potential difference. In their work, the concept of homogenization of boundary condition and the corresponding theories were presented.

In this paper we shall consider the boundary homogenization to the problem with the jump interface condition for the SP well-logging. We prove that, when the compatibility condition (1.1) is satisfied, the solution u_ε of the problem (I_ε) converges strongly in piecewise H^1_* to the solution u of the problem (I) as ε goes to zero. When the compatibility condition (1.1) fails, the convergence is in piecewise $W^{1,p}_*$.

2 Preliminaries

To study the problem of homogenization of boundary condition, we give a restriction on the geometrical structure of $\widetilde{\Gamma}_0^\varepsilon$ as $\varepsilon\to 0$. We make the following hypothesis:

(H) For any weak * convergent subsequence $\{\chi_{\varepsilon'}\}$ of $\{\chi_\varepsilon\}$ in $L^\infty(\Gamma_0)$, its limit function is always different from zero a.e. on Γ_0; namely, if $\chi_{\varepsilon'}\to\chi$ weak * in $L^\infty(\Gamma_0)$, then $\chi\neq 0$, a.e. on Γ_0, where

$$\chi_\varepsilon=\begin{cases} 1 & \text{on } \widetilde{\Gamma}_0^\varepsilon, \\ 0 & \text{on } \Gamma_0^\varepsilon \end{cases}$$

is the characteristic function of $\widetilde{\Gamma}_0^\varepsilon$ on Γ_0.

Lemma 2.1. *Under the hypothesis* (H)*, let A be the set of $L^\infty(\Gamma_0)$ weak * cluster-points of characteristic functions $\{\chi_\varepsilon\}$. We have*

$$\inf_{\chi\in A}\int_{\Gamma_0}\chi\mathrm{d}S>0.$$

Proof. If the conclusion is not true, then

$$\inf_{\chi\in A}\int_{\Gamma_0}\chi\mathrm{d}S=0.$$

Note $\|\chi_\varepsilon\|_{L^\infty}=1$, $A\neq\varnothing$. If A is a finite set, it is easy to obtain a contradiction. Now we consider the case that A is an infinite set. Then for any given $m>0$, there exists $\chi_m\in A$ such that

$$\int_{\Gamma_0}\chi_m \mathrm{d}S<\frac{1}{m}.$$

Owing to $\chi_m\in A$, we can find $\chi_{\varepsilon_m}\in\{\chi_\varepsilon\}$ such that

$$\left|\int_{\Gamma_0}(\chi_{\varepsilon_m}-\chi_m)\mathrm{d}S\right|<\frac{1}{m}.$$

Consequently,

$$\int_{\Gamma_0}\chi_{\varepsilon_m}\mathrm{d}S<\frac{2}{m},$$

which implies

$$\mathrm{mes}(\widetilde{\Gamma}_0^{\varepsilon_m})<\frac{2}{m}\to 0,\quad m\to\infty.$$

Thus, for any given region D contained in Γ_0, we have

$$\int_D\chi_{\varepsilon_m}\mathrm{d}S=\int_{D\cap\widetilde{\Gamma}_0^{\varepsilon_m}}\mathrm{d}S=\mathrm{mes}(D\cap\widetilde{\Gamma}_0^{\varepsilon_m})\to 0,\quad m\to\infty.$$

Hence, noting $\|\chi_{\varepsilon_m}\|_{L^\infty(\Gamma_0)}=1$ and using the generalized Riemann-Lebesgue theorem (cf. [10]), we get

$$\chi_{\varepsilon_m}\to 0 \text{ weak * } \quad\text{in } L^\infty(\Gamma_0)$$

as $m\to\infty$. This is a contradiction to the hypothesis (H). □

Lemma 2.2. *Under the hypothesis* (H)*, for any given* $\varepsilon>0$*, there exists a positive constant* α *independent of* ε*, such that*

$$\mathrm{mes}(\widetilde{\Gamma}_0^{\varepsilon})\geq\alpha. \tag{2.1}$$

Proof. If (2.1) is not true, then, for any given $m>0$, there exists an ε_m such that

$$\mathrm{mes}(\widetilde{\Gamma}_0^{\varepsilon_m})<\frac{1}{m}\to 0,\quad m\to\infty.$$

Therefore, similar to the proof of Lemma 2.1, we can obtain a contradiction to the hypothesis (H). □

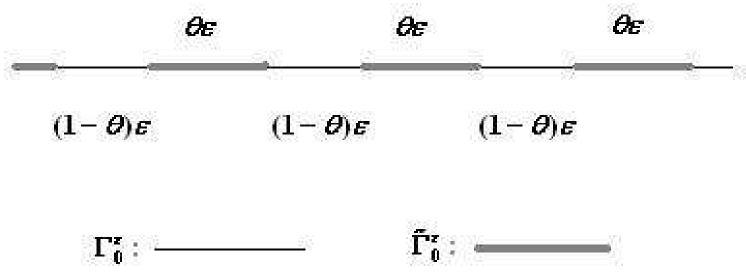

Figure 4: The sets and notations stated in Lemma 2.3.

Lemma 2.3. *Consider a one-dimensional periodic structure on interval* $\Gamma_0=[0,1]=\widetilde{\Gamma}_0^\varepsilon\cup\Gamma_0^\varepsilon$, *where*

$$\widetilde{\Gamma}_0^\varepsilon=\bigcup_{i=-\infty}^{+\infty}(a_i,b_i]\bigcap[0,1],\quad \Gamma_0^\varepsilon=\bigcup_{i=-\infty}^{+\infty}(b_i,a_{i+1}]\bigcap[0,1],$$

with $a_i=i\varepsilon$, $b_i=(i+\theta)\varepsilon$, $(0<\theta<1)$ (*see Fig.* 4). *Then the characteristic function* χ_ε *of* $\widetilde{\Gamma}_0^\varepsilon$ *on* Γ_0 *satisfies*

$$\chi_\varepsilon\to\theta \text{ weak}^* \text{ in } L^\infty([0,1]). \tag{2.2}$$

Proof. For any given $[\tilde{a},\tilde{b}]\subseteq[0,1]$, let $i_0=[\tilde{a}/\varepsilon]+1$, $j_0=[\tilde{b}/\varepsilon]-1$, where $[\cdot]$ stands for the integer part. Then, for $i_0\le i\le j_0$, we have $(a_i,a_{i+1}]\subset[\tilde{a},\tilde{b}]$, and therefore

$$\begin{aligned}\int_{[\tilde{a},\tilde{b}]}\chi_\varepsilon \mathrm{d}t&=\int_{\cup_{i=-\infty}^{+\infty}(a_i,a_{i+1}]\cap[\tilde{a},\tilde{b}]}\chi_\varepsilon \mathrm{d}t\\
&=\int_{\cup_{i=i_0}^{j_0}(a_i,a_{i+1}]}\chi_\varepsilon \mathrm{d}t+\int_{(\tilde{a},a_{i_0}]\cup(a_{j_0+1},\tilde{b}]}\chi_\varepsilon \mathrm{d}t\\
&=\int_{\cup_{i=i_0}^{j_0}(a_i,b_i]}\chi_\varepsilon \mathrm{d}t+\int_{(\tilde{a},a_{i_0}]\cup(a_{j_0+1},\tilde{b}]}\chi_\varepsilon \mathrm{d}t\\
&=(j_0-i_0+1)\theta\varepsilon+\int_{(\tilde{a},a_{i_0}]\cup(a_{j_0+1},\tilde{b}]}\chi_\varepsilon \mathrm{d}t\to\theta(\tilde{b}-\tilde{a}),\quad \text{as } \varepsilon\to0.\end{aligned}$$

Consequently, we have

$$\lim_{\varepsilon\to0}\int_{[\tilde{a},\tilde{b}]}(\chi_\varepsilon-\theta)\mathrm{d}t=0.$$

Thus, for any given ε, noting $\|\chi_\varepsilon\|_{L^\infty(\Gamma_0)}=1$, (2.2) follows directly from the generalized Riemann-Lebesgue theorem. □

Remark 2.1. χ_ε in Lemma 2.3 can be constructed as follows: Extend the function

$$\chi(t)=\begin{cases}1 & t\in[0,\theta],\\ 0 & t\in(\theta,1]\end{cases}$$

to a periodic function with period 1 on R. Then take $\chi_\varepsilon(t)=\chi(t/\varepsilon)$.

Remark 2.2. According to Lemma 2.3, in the SP well-logging, the partition of the electrode should follow the following principle: The total area of the surfaces of all the metal cells keeps to be a constant or of the same order of magnitude, when the electrode is divided into finer and finer, at the same time, more and more cells. According to this principle, the hypothesis (H) is automatically satisfied.

We outline a brief proof for the above principle. If it is not true, then there exists a subsequence $\{\chi_{\varepsilon_m}\}$ of $\{\chi_\varepsilon\}$, such that

$$\chi_{\varepsilon_m} \to 0 \text{ weak * in } L^\infty(\Gamma_0)$$

as $m\to\infty$. Then

$$\int_{\Gamma_0} \chi_{\varepsilon_m} \mathrm{d}S \to 0, \quad m\to\infty.$$

Consequently, $\mathrm{mes}(\widetilde{\Gamma}_0^{\varepsilon_m})\to 0$, $m\to\infty$. This gives a contradiction to the suggested principle.

3 The case with the compatibility condition satisfied

We first discuss the homogenization of boundary condition for the SP well-logging in the case that the jump condition satisfies the compatibility condition (1.1). In this case, the corresponding variational problem of the problem (I) is to seek $u\in V_2$ such that

$$a(u,\phi)=0, \quad \forall \phi\in V_2^0, \tag{3.1}$$

where

$$\begin{aligned}
&a(u,v)=\sum_{i=1}^{4}\frac{1}{R_i}\int_{\Omega_i}\Big(\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\Big) r\mathrm{d}r\mathrm{d}z,\\
&V_2=\{v|v\in H^1_*(\Omega_i), 1\le i\le 4; v^+-v^-|_{\gamma_j}=E_j, 1\le j\le 5; v|_{\Gamma_1}=0; v|_{\Gamma_0}=C\},\\
&V_2^0=\{v|v\in H^1_*(\Omega_i), 1\le i\le 4; v^+-v^-|_{\gamma_j}=0, 1\le j\le 5; v|_{\Gamma_1}=0; v|_{\Gamma_0}=C\}\\
&\quad\;=\{v|v\in H^1_*(\Omega), v|_{\Gamma_1}=0; v|_{\Gamma_0}=C\}.
\end{aligned}$$

Similarly, the corresponding variational problem of the problem (I_ε) is to seek $u_\varepsilon\in V_{\varepsilon,2}$ such that

$$a(u_\varepsilon,\phi_\varepsilon)=0, \quad \forall \phi_\varepsilon\in V_{\varepsilon,2}^0, \tag{3.2}$$

where

$$\begin{aligned}
&V_{\varepsilon,2}=\{v|v\in H^1_*(\Omega_i), 1\le i\le 4; v^+-v^-|_{\gamma_j}=E_j, 1\le j\le 5; v|_{\Gamma_1}=0; v|_{\widetilde{\Gamma}_0^\varepsilon}=C_\varepsilon\},\\
&V_{\varepsilon,2}^0=\{v|v\in H^1_*(\Omega_i), 1\le i\le 4; v^+-v^-|_{\gamma_j}=0, 1\le j\le 5; v|_{\Gamma_1}=0; v|_{\widetilde{\Gamma}_0^\varepsilon}=C_\varepsilon\}\\
&\quad\;=\{v|v\in H^1_*(\Omega), v|_{\Gamma_1}=0; v|_{\widetilde{\Gamma}_0^\varepsilon}=C_\varepsilon\}.
\end{aligned}$$

Here

$$H^1_* = \left\{ v | r^{1/2} v \in L^2, \quad r^{1/2} \nabla v \in L^2 \right\}, \tag{3.3}$$

equipped with the norm

$$\|v\|_{H^1_*} = \left(\|r^{1/2} v\|^2_{L^2} + \|r^{1/2} \nabla v\|^2_{L^2} \right)^{1/2}.$$

Moreover, C and C_ε are the constants to be determined.

When the jump condition satisfies the compatibility condition (1.1), for the problem (I) (resp. the problem (I_ε)) we have the following conclusions (cf. [1,3,4]).

Lemma 3.1. *Suppose that the compatibility condition* (1.1) *is satisfied, then* $V_2 \neq \varnothing$.

Lemma 3.2. *The problem* (I) *(resp. the problem* (I_ε)*) admits a unique piecewise* H^1_* *weak solution if and only if* E_j $(1 \le j \le 5)$ *satisfies the compatibility condition* (1.1).

Theorem 3.1. *Under the hypothesis* (H), *suppose that* E_j $(1 \le j \le 5)$ *satisfies the compatibility condition* (1.1). *Then for the unique piecewise* H^1_* *weak solution* u_ε *to the problem* (I_ε), *we have*

$$u_\varepsilon \to u \text{ strongly in } H^1_*(\Omega_i), \quad 1 \le i \le 4, \tag{3.4}$$

as $\varepsilon \to 0$, *where* $u \in V_2$ *is the unique piecewise* H^1_* *weak solution to the problem* (I).

Proof. By Lemma 3.2, there exists a unique $u_\varepsilon \in V_{\varepsilon,2}$ satisfying (3.2). Furthermore, owing to Lemma 3.1, for any given $v \in V_2$, we have $w_\varepsilon = u_\varepsilon - v \in V^0_{\varepsilon,2}$, and

$$a(w_\varepsilon, \phi_\varepsilon) = -a(v, \phi_\varepsilon), \quad \forall \phi_\varepsilon \in V^0_{\varepsilon,2}.$$

Then, specifically choosing $\phi_\varepsilon = w_\varepsilon$, we obtain

$$a(w_\varepsilon, w_\varepsilon) = -a(v, w_\varepsilon). \tag{3.5}$$

Thus, by Poincaré's inequality, we get

$$\|w_\varepsilon\|_{H^1_*(\Omega)} \le M, \tag{3.6}$$

where M is a positive constant independent of ε. Therefore, there exists a subsequence $\{w_{\varepsilon'}\}$ of $\{w_\varepsilon\}$ and $w \in H^1_*(\Omega)$, such that

$$w_{\varepsilon'} \to w \text{ weakly in } H^1_*(\Omega) \tag{3.7}$$

as $\varepsilon' \to 0$. Noting that

$$w_{\varepsilon'}|_{\Gamma_1} = u_{\varepsilon'}|_{\Gamma_1} - v|_{\Gamma_1} = 0,$$

by the trace theorem and (3.7), we have

$$w|_{\Gamma_1}=0. \tag{3.8}$$

Furthermore, (2.1) and (3.6) yield

$$\begin{aligned}|C_{\varepsilon'}|=|w_{\varepsilon'}|_{\widetilde{\Gamma}_0^{\varepsilon'}}|=&\Big(\frac{1}{\mathrm{mes}(\widetilde{\Gamma}_0^{\varepsilon'})}\int_{\widetilde{\Gamma}_0^{\varepsilon'}}w_{\varepsilon'}^2\mathrm{d}S\Big)^{1/2}\\ \leq&\frac{1}{\alpha^{1/2}}\|w_{\varepsilon'}\|_{H^{1/2}(\widetilde{\Gamma}_0^{\varepsilon'})}\leq M\|w_{\varepsilon'}\|_{H^1_*(\Omega)}\leq\widetilde{M},\end{aligned}$$

where $\widetilde{M}$ is a positive constant independent of ε. Consequently, there exists a subsequence of $\{C_{\varepsilon'}\}$, still denoted by $\{C_{\varepsilon'}\}$, and a constant C, such that $C_{\varepsilon'}\to C$ as $\varepsilon'\to 0$. Moreover, by the trace theorem and the imbedding theorem, we have

$$w_{\varepsilon'}\to w \text{ strongly in } L^2(\Gamma_0),\quad \text{as } \varepsilon'\to 0.$$

In addition, from the hypotheses (H), we can suppose (if necessarily, we take a subsequence)

$$\chi_{\varepsilon'}\to\chi \text{ weak * in } L^\infty(\Gamma_0),\quad \varepsilon'\to 0,$$

and $\chi\neq 0, a.e.$ on Γ_0. Thus, for any given $f\in L^2(\Gamma_0)$, we get

$$\begin{aligned}&\int_{\Gamma_0}(\chi_{\varepsilon'}w_{\varepsilon'}-\chi w)f\mathrm{d}S=\int_{\Gamma_0}[\chi_{\varepsilon'}(w_{\varepsilon'}-w)+(\chi_{\varepsilon'}-\chi)w]f\mathrm{d}S\\ =&\int_{\Gamma_0}\chi_{\varepsilon'}(w_{\varepsilon'}-w)f\mathrm{d}S+\int_{\Gamma_0}(\chi_{\varepsilon'}-\chi)(wf)\mathrm{d}S\to 0,\quad \text{as } \varepsilon'\to 0.\end{aligned}$$

Consequently,

$$\chi_{\varepsilon'}w_{\varepsilon'}\to\chi w \text{ weak * in } L^2(\Gamma_0),\quad \varepsilon'\to 0.$$

On the other hand, from the above discussion, it is easy to see that

$$\chi_{\varepsilon'}w_{\varepsilon'}=\chi_{\varepsilon'}C_{\varepsilon'}\to\chi C \text{ weak * in } L^\infty(\Gamma_0),\quad \varepsilon'\to 0,$$

which yields

$$\chi w=\chi C\quad \text{a.e. on } \Gamma_0.$$

Since $\chi\neq 0$, a.e. on Γ_0, we then obtain

$$w|_{\Gamma_0}=C. \tag{3.9}$$

Thus, (3.8) and (3.9) yield $w\in V_2^0$. Let $u=w+v$. Since $w\in V_2^0$ and $v\in V_2$, we conclude that u satisfies the Dirichlet boundary condition, the equivalued surface boundary condition and the jump interface condition in the problem (I). Then, $u\in V_2$. In addition, by (3.7), we have

$$u_{\varepsilon'}\to u \text{ weakly in } H^1_*(\Omega_i),\quad 1\le i\le 4,\quad \text{as } \varepsilon'\to 0. \tag{3.10}$$

For any given $\varepsilon>0$, $V_2^0\subset V_{\varepsilon,2}^0$. Therefore, for any given $\varphi\in V_2^0$, especially setting $\varphi_\varepsilon=\varphi$ and replacing ε by ε' in (3.2), we get (3.1). Namely, u is the unique piecewise H^1_* weak solution to the problem (I).

Thus, taking $\varphi=w$ in (3.1) yields

$$a(v,w)=-a(w,w).$$

Furthermore, the combination of (3.5) with (3.7) gives

$$a(w_{\varepsilon'},w_{\varepsilon'})=-a(v,w_{\varepsilon'})\to -a(v,w)=a(w,w).$$

Then, as $\varepsilon'\to 0$, we get

$$\|w_{\varepsilon'}\|_{H^1_*(\Omega)}\to \|w\|_{H^1_*(\Omega)}.$$

Thus, it follows from (3.7) that

$$w_{\varepsilon'}\to w \text{ strongly in } H^1_*(\Omega),\quad \varepsilon'\to 0.$$

Accordingly,

$$u_{\varepsilon'}\to u \text{ strongly in } H^1_*(\Omega_i)\quad 1\le i\le 4,\quad \varepsilon'\to 0.$$

Moreover, by Lemma 3.2, the uniqueness of the problem (I) guarantees that the whole sequence $u_\varepsilon\to u$ strongly in $H^1_*(\Omega_i)$ $(1\le i\le 4)$. Hence, (3.4) holds. □

4 The case with the compatibility condition failed

In general, E_j $(1\le j\le 5)$ do not satisfy the compatibility condition (1.1). Owing to Lemma 3.2, in this case, for the problem (I) or problem (I_ε), it is impossible to get a piecewise H^1_* weak solution. So we have to seek a solution in a larger class of functions. For any given p $(1<p<2)$, introduce

$$\begin{aligned}
V_p&=\{v|v\in W^{1,p}_*(\Omega_i),1\le i\le 4;v^+-v^-|_{\gamma_j}=E_j,1\le j\le 5;v|_{\Gamma_1}=0;v|_{\Gamma_0}=C\},\\
V_p^0&=\{v|v\in W^{1,p}_*(\Omega_i),1\le i\le 4;v^+-v^-|_{\gamma_j}=0,1\le j\le 5;v|_{\Gamma_1}=0;v|_{\Gamma_0}=C\}\\
&=\{v|v\in W^{1,p}_*(\Omega),v|_{\Gamma_1}=0;v|_{\Gamma_0}=C\},\\
V_{\varepsilon,p}&=\{v|v\in W^{1,p}_*(\Omega_i),1\le i\le 4;v^+-v^-|_{\gamma_j}=E_j,1\le j\le 5;v|_{\Gamma_1}=0;v|_{\tilde{\Gamma}_0^\varepsilon}=C_\varepsilon\},
\end{aligned}$$

$$\begin{aligned}V^0_{\varepsilon,p} &= \{v|v\in W^{1,p}_*(\Omega_i), 1\le i\le 4; v^+-v^-|_{\gamma_j}=0, 1\le j\le 5; v|_{\Gamma_1}=0; v|_{\widetilde{\Gamma}^\varepsilon_0}=C_\varepsilon\}\\ &= \{v|v\in W^{1,p}_*(\Omega), v|_{\Gamma_1}=0; v|_{\widetilde{\Gamma}^\varepsilon_0}=C_\varepsilon\},\end{aligned}$$

where

$$W^{1,p}_* = \{v|r^{1/p}v\in L^p,\ r^{1/p}\nabla v\in L^p\}, \tag{4.1}$$

equipped with the norm

$$\|v\|_{W^{1,p}_*} = (\|r^{1/p}v\|^p_{L^p} + \|r^{1/p}\nabla v\|^p_{L^p})^{1/p}.$$

Moreover, C and C_ε are the constants to be determined. Obviously, if p=2, then $W^{1,2}_* = H^1_*$.

Now, the corresponding variational problem of the problem (I) becomes to seek $u\in V_p$, such that

$$a(u,\phi)=0, \quad \forall \phi\in V^0_{p'}, \tag{4.2}$$

and the corresponding variational problem of the problem (I_ε) becomes to seek $u_\varepsilon\in V_{\varepsilon,p}$, such that

$$a(u_\varepsilon,\phi_\varepsilon)=0, \quad \forall \phi_\varepsilon\in V^0_{\varepsilon,p'}, \tag{4.3}$$

where p' is the dual number of p: $1/p'+1/p$=1, and the definition of $a(u,v)$ is the same as in Section 3.

When the compatibility condition (1.1) fails, for the problem (I) (resp. problem (I_ε)), we have the following conclusions (cf. [3,4]).

Lemma 4.1. *There exists $\beta>0$ such that for any given p $(2-\beta<p<2)$, the problem* (I) (*resp. problem* (I_ε)) *admits a unique piecewise $W^{1,p}_*$ weak solution $u\in V_p$* (*resp.* $u_\varepsilon\in V_{\varepsilon,p}$), *satisfying* (4.2) (*resp.* (4.3)).

Lemma 4.2. *There exists $v_A, v_B\in W^{1,p}_*(\Omega_i)$ (p is the same as that in Lemma* 4.1; $1\le i\le 4$) *such that*

(1) *The support of v_A and v_B is in a neighbourhood of A and B respectively;*

(2) *The functionals of*

$$l_A(\phi) \triangleq \sum_{i=1}^4\int_{\Omega_i}(Lv_A)\phi \mathrm{d}r\mathrm{d}z, \quad l_B(\phi) \triangleq \sum_{i=1}^4\int_{\Omega_i}(Lv_B)\phi \mathrm{d}r\mathrm{d}z$$

are continuous and linear on V^0_2 and on $V^0_{\varepsilon,2}$, respectively.

In addition, for the unique piecewise $W^{1,p}_*$ weak solution u (*resp.* u_ε) *to the problem* (I) (*resp. the problem* (I_ε)), *let $w=u-v_A-v_B$* (*resp.* $w_\varepsilon=u_\varepsilon-v_A-v_B$). *We have*

(3) *$w \in H^1_*\ (\Omega_i)(1 \le i \le 4)$, $w|_{\Gamma_1} = 0$, $w|_{\Gamma_0}$ is a constant to be determined (resp. $w_\varepsilon \in H^1_*(\Omega_i)$ $(1 \le i \le 4)$, $w_\varepsilon|_{\Gamma_1} = 0, w_\varepsilon|_{\widetilde{\Gamma}_0^\varepsilon}$ is a constant to be determined), and satisfies the corresponding compatibility condition at A and B;*

(4) *w satisfies that*

$$a(w,\phi) = -l_A(\phi) - l_B(\phi), \quad \forall \phi \in V_2^0 \tag{4.4}$$

and w_ε satisfies

$$a(w_\varepsilon,\phi_\varepsilon) = -l_A(\phi_\varepsilon) - l_B(\phi_\varepsilon), \quad \forall \phi_\varepsilon \in V_{\varepsilon,2}^0. \tag{4.5}$$

Theorem 4.1. *Under the hypothesis* (H), *for the unique piecewise $W_*^{1,p}$ weak solution u_ε to the problem* (I_ε), *we have*

$$u_\varepsilon \to u \text{ strongly in } W_*^{1,p}(\Omega_i), \quad 1 \le i \le 4, \tag{4.6}$$

as $\varepsilon \to 0$, where $u \in V_p$ is the unique piecewise $W_*^{1,p}$ weak solution to the problem (I).

Proof. Similar to Lemma 4.2, we construct $v_A, v_B \in W_*^{1,p}(\Omega_i)(1 \le i \le 4)$ and $l_A(\phi)$ and $l_B(\phi)$. Then, v_A, v_B have a compact support in a neighbourhood of A and B respectively, and are independent of ε. Furthermore, $l_A(\phi)$ and $l_B(\phi)$ are continuous linear functionals on $V_{\varepsilon,2}^0$. Let

$$w_\varepsilon = u_\varepsilon - v_A - v_B.$$

The jump $\widetilde{E}_j$ $(1 \le j \le 5)$ of w_ε on the interfaces are then

$$\widetilde{E}_j = E_j - [(v_A + v_B)^+ - (v_A + v_B)^-]|\gamma_j, \quad 1 \le j \le 5.$$

By Lemma 4.2, $w_\varepsilon \in \widetilde{V}_{\varepsilon,2}$ and (4.5) holds, where

$$\widetilde{V}_{\varepsilon,2} = \Big\{ v | v \in H^1_*(\Omega_i), 1 \le i \le 4; v^+ - v^-|_{\gamma_j} = \widetilde{E}_j, 1 \le j \le 5; \\ v|_{\Gamma_1} = 0; v|_{\widetilde{\Gamma}_0^\varepsilon} = C_\varepsilon \text{ (a constant to be determined)} \Big\}.$$

Furthermore, by Lemma 4.2, $\widetilde{E}_j$ $(1 \le j \le 5)$ satisfies the corresponding compatibility condition. Then, from the proof of Theorem 3.1, we have

$$w_\varepsilon \to w \text{ strongly in } H^1_*(\Omega_i), \quad 1 \le i \le 4, \tag{4.7}$$

as $\varepsilon \to 0$. Moreover, $w \in \widetilde{V}_2$ and (4.4) holds, where

$$\widetilde{V}_2 = \Big\{ v | v \in H^1_*(\Omega_i), 1 \le i \le 4; v^+ - v^-|_{\gamma_j} = \widetilde{E}_j, 1 \le j \le 5; \\ v|_{\Gamma_1} = 0; v|_{\Gamma_0} = C \text{ (a constant to be determined)} \Big\}.$$

Let $u = w + v_A + v_B$. It is easy to see that $u \in V_p$, and (4.6) follows from (4.7).

Noting $V_{p'}^0 \subset V_{\varepsilon,p'}^0$ and choosing $\varphi_\varepsilon = \varphi$ in (4.3), for any given $\varphi \in V_{p'}^0$, we obtain (4.2). Therefore, $u \in V_p$ is the unique piecewise $W_*^{1,p}$ weak solution to the problem (I). $\square$

5 Some remarks

Consider

$$
(\mathrm{II})\qquad \begin{cases} Lu=0, \quad \text{in } \Omega_i, \quad 1\leq i\leq 4,\\ u=0, \quad \text{on } \Gamma_1, \qquad \dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}=0, \quad \text{on } \Gamma_2,\\ u=C, \quad \text{on } \Gamma_0, \qquad \displaystyle\int_{\Gamma_0}\dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}\mathrm{d}s=I_0,\\ u^+=u^-, \quad \Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}\Big)^+=\Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}\Big)^-, \quad \text{on } \gamma_j, \quad 1\leq j\leq 5, \end{cases}
$$

where C is a constant to be determined, and

$$
(\mathrm{II}_\varepsilon)\qquad \begin{cases} Lu_\varepsilon=0, \quad \text{in } \Omega_i, \quad 1\leq i\leq 4,\\ u_\varepsilon=0, \quad \text{on } \Gamma_1, \qquad \dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}=0, \quad \text{on } \Gamma_2,\\ \dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}=0, \quad \text{on } \Gamma_0^\varepsilon, \qquad u_\varepsilon=C_\varepsilon, \quad \text{on } \widetilde{\Gamma}_0^\varepsilon,\\ \displaystyle\int_{\widetilde{\Gamma}_0^\varepsilon}\dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}\mathrm{d}s=I_0,\\ u_\varepsilon^+=u_\varepsilon^-, \quad \Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}\Big)^+=\Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}\Big)^-, \quad \text{on } \gamma_j, \quad 1\leq j\leq 5, \end{cases}
$$

where C_ε is a constant to be determined, the total electric current I_0 is a positive constant. For the homogenization of boundary condition for the resistivity well-logging, the following results are proved in [7,8].

Lemma 5.1. *Under the hypothesis* (H), *the problem* $(\mathrm{II}_\varepsilon)$ *admits a unique* H_*^1 *weak solution* $u_\varepsilon\in V_{\varepsilon,2}^0$ *such that*

$$
a(u_\varepsilon,\phi_\varepsilon)=I_0\phi_\varepsilon\big|_{\widetilde{\Gamma}_0^\varepsilon}, \quad \forall\phi_\varepsilon\in V_{\varepsilon,2}^0,
$$

and

$$
u_\varepsilon\rightarrow u \text{ strongly in } H_*^1(\Omega), \quad \textit{as } \ \varepsilon\rightarrow 0,
$$

where $u\in V_2^0$ *is the unique* H_*^1 *weak solution to the problem* (II) *satisfying*

$$
a(u,\phi)=I_0\phi\big|_{\Gamma_0}, \quad \forall\phi\in V_2^0.
$$

We now discuss the homogenization of boundary condition in the case that there are both an electric current emitted from the electrode and the jump interface conditions, namely, we consider the following boundary problems respectively:

$$
\text{(III)}\quad \begin{cases} Lu=0, \quad \text{in } \Omega_i, \quad 1\le i\le 4, \\ u=0, \quad \text{on } \Gamma_1, \qquad \dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}=0, \quad \text{on } \Gamma_2, \\ u=C, \quad \text{on } \Gamma_0, \quad \displaystyle\int_{\Gamma_0}\frac{r}{Re}\frac{\partial u}{\partial n}\mathrm{d}s=I_0, \\ u^+-u^-=E_j, \quad \Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}\Big)^+=\Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u}{\partial n}\Big)^-, \quad \text{on } \gamma_j, \quad 1\le j\le 5, \end{cases}
$$

where C is a constant to be determined, and

$$
(\mathrm{III}_\varepsilon)\quad \begin{cases} Lu_\varepsilon=0, \quad \text{in } \Omega_i, \quad 1\le i\le 4, \\ u_\varepsilon=0, \quad \text{on } \Gamma_1, \qquad \dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}=0, \quad \text{on } \Gamma_2, \\ \dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}=0, \quad \text{on } \Gamma_0^\varepsilon, \qquad u_\varepsilon=C_\varepsilon, \quad \text{on } \widetilde{\Gamma}_0^\varepsilon, \\ \displaystyle\int_{\widetilde{\Gamma}_0^\varepsilon}\frac{r}{\mathrm{Re}}\frac{\partial u_\varepsilon}{\partial n}\mathrm{d}s=I_0, \\ u_\varepsilon^+-u_\varepsilon^-=E_j, \quad \Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}\Big)^+=\Big(\dfrac{r}{\mathrm{Re}}\dfrac{\partial u_\varepsilon}{\partial n}\Big)^-, \quad \text{on } \gamma_j, \quad 1\le j\le 5, \end{cases}
$$

where C_ε is a constant to be determined, the total electric current I_0 is a positive constant. According to Theorem 3.1, Theorem 4.1 and Lemma 5.1, by superposition it is easy to get the following two theorems.

Theorem 5.1. *Under the hypothesis* (H), *suppose that* E_j $(1\le j\le 5)$ *satisfies the compatibility condition* (1.1), *then the problem* $(\mathrm{III}_\varepsilon)$ *admits a unique piecewise* H^1_* *weak solution* $u_\varepsilon\in V_{\varepsilon,2}$ *such that*

$$
a(u_\varepsilon,\phi_\varepsilon)=I_0\phi_\varepsilon\big|_{\widetilde{\Gamma}_0^\varepsilon}, \quad \forall\phi_\varepsilon\in V^0_{\varepsilon,2},
$$

and

$$
u_\varepsilon\to u \ \text{ strongly in } H^1_*(\Omega_i), \quad 1\le i\le 4, \quad as \ \ \varepsilon\to 0,
$$

where $u\in V_2$ *is the unique piecewise* H^1_* *weak solution to the problem* (III) *satisfying*

$$
a(u,\phi)=I_0\phi\big|_{\Gamma_0}, \quad \forall\phi\in V^0_2.
$$

Theorem 5.2. *Under the hypothesis* (H), *there exists* $\beta>0$ *such that, for any given* p $(2-\beta<p<2)$, *the problem* $(\mathrm{III}_\varepsilon)$ *admits a unique piecewise* $W^{1,p}_*$ *weak solution* $u_\varepsilon\in V_{\varepsilon,p}$ *such that*

$$
a(u_\varepsilon,\phi_\varepsilon)=I_0\phi_\varepsilon\big|_{\widetilde{\Gamma}_0^\varepsilon}, \quad \forall\phi_\varepsilon\in V^0_{\varepsilon,p'},
$$

and

$$u_\varepsilon \to u \ \text{ strongly in } W_*^{1,p}(\Omega_i), \quad 1 \le i \le 4,$$

as $\varepsilon \to 0$*, where* $u \in V_p$ *is the unique piecewise* $W_*^{1,p}$ *weak solution to the problem* (III) *satisfying*

$$a(u,\phi) = I_0 \phi\big|_{\Gamma_0}, \quad \forall \phi \in V_{p'}^0.$$

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