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A Nonlinear Diffusion System with Coupled Nonlinear Boundary Flux

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Abstract. This paper studies a nonlinear diffusion system with coupled nonlinear boundary flux and two kinds of inner sources (positive for the first and negative for the second), where the four nonlinear mechanisms are described by eight nonlinear parameters. The critical exponent of the system is determined by a complete classification of the eight nonlinear parameters, which is represented via the *characteristic algebraic system* introduced to the problem.

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1 Introduction

In this paper we consider coupled nonlinear diffusion equations of the form

$$(u^m)_t=\Delta u+a_1u^{\alpha_1},\quad (v^n)_t=\Delta v-a_2v^{\beta_1},\qquad (x,t)\in\Omega\times(0,T), \tag{1.1}$$

$$\frac{\partial u}{\partial\eta}=u^{\alpha_2}v^p,\quad \frac{\partial v}{\partial\eta}=u^qv^{\beta_2},\qquad (x,t)\in\partial\Omega\times(0,T), \tag{1.2}$$

$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\qquad x\in\bar{\Omega}, \tag{1.3}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, the parameters m,n,p,q,a_1,a_2 are positive, $\alpha_i,\beta_i\geq 0$ $(i=1,2)$, u_0 and v_0 are positive functions satisfying

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the compatibility conditions on $\bar{\Omega}$. The diffusion in (1.1) may be fast or slow, e.g., when $m>1$ or $0<m<1$ for the component u. There are positive and negative sources for u and v respectively, together with coupled nonlinear boundary condition (1.2) describing the nonlinear radiation laws in heat propagations. The critical exponent to the semilinear case $(m=n=1)$ of (1.1)-(1.3) was studied by Zheng, Liang and Song [1]. They introduced the matrix equation

$$\begin{pmatrix} \alpha_2-\mu & p \\ q & \beta_2-\gamma \end{pmatrix}\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with

$$\mu=1-\Big(\frac{\alpha_1-1}{2}\Big)_+,\quad \gamma=1+\Big(\frac{\beta_1-1}{2}\Big)_+,$$

and obtained that the critical exponent of (1.1)-(1.3) is just $(1/\rho_1,1/\rho_2)=(0,0)$. We refer also to, e.g., Zheng *et al.* [2, 3], Bedjaouit and Souplet [4] for the results on parabolic equations with inner absorptions.

The scalar case was studied by Filo [5], and Deng *et al.* [6]. For the scalar nonlinear diffusion equation with positive source

$$(u^m)_t=\Delta u+a_1u^\alpha, \qquad (x,t)\in\Omega\times(0,T), \tag{1.4}$$

$$\frac{\partial u}{\partial\eta}=b_1u^\beta, \qquad (x,t)\in\partial\Omega\times(0,T), \tag{1.5}$$

$$u(x,0)=u_0(x), \qquad x\in\bar{\Omega}, \tag{1.6}$$

with $m>0$, $\alpha,\beta\geq 0$, $a_1,b_1>0$, Song and Zheng [7] proved the following result:

Proposition 1.1. *The solutions of* (1.4)-(1.6) *blow up in a finite time for large initial value provided* (i) $\alpha>m$, $a_1>0$, *or* (ii) $0<m\leq 1$, $\beta>m$, $b_1>0$, *or* (iii) $m>1$, $\beta>(m+1)/2$, $b_1>0$.

Andreu *et al.* [8] and Li *et al.* [9] studied the scalar case with absorption, i.e.,

$$(u^n)_t=\Delta u-a_2u^{\beta_1}, \qquad (x,t)\in\Omega\times(0,T), \tag{1.7}$$

$$\frac{\partial u}{\partial\eta}=u^{\beta_2}, \qquad (x,t)\in\partial\Omega\times(0,T), \tag{1.8}$$

$$u(x,0)=u_0(x), \qquad x\in\bar{\Omega}, \tag{1.9}$$

with $a_2,n>0$, $\beta_i\geq 0$ $(i=1,2)$, and obtained the blow-up criterion:

Proposition 1.2. *The solutions of* (1.7)-(1.9) *blow up in a finite time for large initial data if*

(i) $n\geq 1$ *with* $\beta_1\leq n$, $\beta_2>(n+1)/2$ *or* $\beta_1>n$, $\beta_2>(\beta_1+1)/2$; *or*

(ii) $0<n<1$ *with* $\beta_2>1$, $\beta_2>(\beta_1+1)/2$, *or* $n<\beta_2\leq 1$, $\beta_2>\beta_1$.

Recently, Zheng and Wang [10] established the critical exponent for the nonlinear diffusion system with inner absorptions and nonlinear boundary conditions.

Definition 1.1. *A vector function* (u,v)*, defined everywhere on* $\bar{\Omega}\times(0,T]$ *and a.e. on* $\bar{\Omega}\times\{0\}$*, is called a subsolution (supersolution) of* (1.1)-(1.3) *in* $Q_T=\Omega\times(0,T)$ *if it satisfies:*

(i) $u,v\in L^\infty(Q_T)$;

(ii) $u(x,0)\le(\ge)u_0(x), v(x,0)\le(\ge)v_0(x)$ *a.e. on* $\bar{\Omega}$;

(iii) *for every* $t\in[0,T]$ *and every* $\psi_1,\psi_2\in C(\bar{Q}_T)\cap C^{2,1}(\bar{\Omega}\times(0,T))\cap V(Q_T)$,

$$\int_\Omega u^m\psi_1-u_0^m\psi_{10}\mathrm{d}x\le(\ge)\int_0^t\int_\Omega(u^m(\psi_1)_\tau+u\Delta\psi_1+a_1u^{\alpha_1}\psi_1)\mathrm{d}x\mathrm{d}\tau$$
$$+\int_0^t\int_{\partial\Omega}(u^{\alpha_2}v^p\psi_1-\frac{\partial\psi_1}{\partial\eta}u)\mathrm{d}S\mathrm{d}\tau, \tag{1.10}$$
$$\int_\Omega v^n\psi_2-v_0^n\psi_{20}\mathrm{d}x\le(\ge)\int_0^t\int_\Omega(v^n(\psi_2)_\tau+v\Delta\psi_2-a_2v^{\beta_1}\psi_2)\mathrm{d}x\mathrm{d}\tau$$
$$+\int_0^t\int_{\partial\Omega}(v^{\beta_2}u^q\psi_2-\frac{\partial\psi_2}{\partial\eta}v)\mathrm{d}S\mathrm{d}\tau, \tag{1.11}$$

where $V(Q_T)=\{\psi:\psi_t,|\nabla\psi|,\Delta\psi\in L^2(Q_T),\psi\ge0\}$*. Furthermore,* (u,v) *is called a weak solution of* (1.1)-(1.3) *in* Q_T *if it is both a subsolution and a supersolution of* (1.1)-(1.3) *in* Q_T*.*

By the standard technique (see, e.g., Theorems 2.1 and 3.1, pp. 118-123 in [11]), we have the following proposition:

Proposition 1.3. (i) (*Local existence and continuation*) *There is some* $T^*>0$ *such that there exists a nonnegative weak solution* (u,v) *of* (1.1)-(1.3) *in* Q_T *for each* $T<T^*$*. Moreover, if* $T^*<+\infty$*, then*

$$\limsup_{t\to T^*}(\|u(\cdot,t)\|_\infty+\|v(\cdot,t)\|_\infty)=\infty.$$

(ii) (*Comparison principle*) *Let* $(\underline{u},\underline{v})$ *and* $(\bar{u},\bar{v})$ *be nonnegative sub and supersolutions of* (1.1)-(1.3) *in* Q_T *respectively. Then* $(\underline{u},\underline{v})\le(\bar{u},\bar{v})$ *a.e. on* $\bar{Q}_T$ *if*

$$(\underline{u}(x,0),\underline{v}(x,0))\le(\bar{u}(x,0),\bar{v}(x,0))$$

with $\underline{u},\underline{v}\ge\delta$(or $\bar{u},\bar{v}\ge\delta$) *for some* $\delta>0$.

Remark 1.1. It is observed that every classical subsolution (supersolution) of (1.1)-(1.3) is also a weak subsolution (supersolution). In the sequel, we will use explicit classical positive sub and supersolutions instead of those in Definition 1.1.

There are four kinds of nonlinear mechanisms in the model (1.1)-(1.3): nonlinear diffusion, nonlinear absorption, nonlinear reaction, and nonlinear boundary flux. We are interested in the interactions among them. To represent the critical exponent for (1.1)-(1.3), we introduce the following *characteristic algebraic system* [2,3,10]:

$$\begin{pmatrix}\alpha_2-\mu & p\\ q & \beta_2-\gamma\end{pmatrix}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}, \tag{1.12}$$

namely,

$$\rho_1=\frac{p-\beta_2+\gamma}{pq-(\mu-\alpha_2)(\gamma-\beta_2)},\quad \rho_2=\frac{q-\alpha_2+\mu}{pq-(\mu-\alpha_2)(\gamma-\beta_2)} \tag{1.13}$$

with

$$\mu=\frac{1+m}{2},\ \gamma=\frac{1+n}{2}+\Big(\frac{\beta_1-n}{2}\Big)_+ \qquad \text{for } m,n\geq 1; \tag{1.14}$$

$$\mu=m,\ \gamma=n+\Big(\beta_1-n\Big)_+-\Big(\frac{\beta_1-1}{2}\Big)_+ \qquad \text{for } 0<m,n<1; \tag{1.15}$$

$$\mu=m,\ \gamma=\frac{1+n}{2}+\Big(\frac{\beta_1-n}{2}\Big)_+ \qquad \text{for } 0<m<1,\ n\geq 1; \tag{1.16}$$

$$\mu=\frac{1+m}{2},\ \gamma=n+\Big(\beta_1-n\Big)_+-\Big(\frac{\beta_1-1}{2}\Big)_+ \qquad \text{for } m\geq 1,\ 0<n<1. \tag{1.17}$$

Remark 1.2. Clearly, all the eight exponents m, n, p, q, α_i, β_i $(i=1,2)$ from the nonlinear terms of (1.1)-(1.3) are included in (1.12), and the classification for $m,n>0$ is complete. We will use the signs of $1/\rho_1,1/\rho_2$ (rather than those of ρ_1,ρ_2 themselves) to describe the critical properties of solutions. Here, we have to define $(1/\rho_1,1/\rho_2)=(0,0)$ by the limit of $(1/\rho_1,1/\rho_2)$ as

$$pq-(\mu-\alpha_2)(\gamma-\beta_2)\rightarrow 0 \quad \text{with } p\neq\beta_2-\gamma,\quad q\neq\alpha_2-\mu.$$

Clearly, if $q=\alpha_2-\mu$, $p\neq\beta_2-\gamma$, then $\rho_1=1/q$, $\rho_2=0$. For $p=\beta_2-\gamma$, $q=\alpha_2-\mu$, define, e.g.,

$$\rho_1=\rho_2=\frac{1}{p+q}$$

to satisfy (1.12).

Now state the main results of the paper.

Theorem 1.1. *If $\alpha_1>m$, then the solutions of* (1.1)-(1.3) *will blow up in a finite time for large initial data.*

Theorem 1.2. *If $1/\rho_1>0$ or $1/\rho_2>0$ with $\alpha_1\leq m$, then the solutions of* (1.1)-(1.3) *will blow up in finite time for large initial data.*

Theorem 1.3. *If $1/\rho_1,1/\rho_2<0$ with $\alpha_1\leq m$, then the solutions of* (1.1)-(1.3) *are global.*

Theorem 1.4. *Assume $(1/\rho_1,1/\rho_2)=(0,0)$ with $\alpha_1\leq m$.* (i) *If $\alpha_2>\mu$, then the solutions of* (1.1)-(1.3) *will blow up in finite time for large initial data.* (ii) *If $\alpha_2<\mu$, then the solutions of* (1.1)-(1.3) *are global.*

Remark 1.3. Notice from (1.13) and Remark 1.2 that $1/\rho_1=0$ is equivalent to $1/\rho_2=0$ and that $(1/\rho_1,1/\rho_2)=(0,0)$ with $pq>0$ excludes the possibility of $\alpha_2=\mu$. Therefore, in Theorems 1.1-1.4, the classification for the nonlinear parameters is complete. We know from Theorems 1.2-1.4 that the critical exponent of (1.1)-(1.3) can be simply stated as $(1/\rho_1,1/\rho_2)=(0,0)$ under the nontrivial case of $\alpha_1\leq m$.

Remark 1.4. The previous results for nonlinear parabolic equations [1,12,13] are covered by the above theorems when taking special parameters. For example, let $\alpha_1=\beta_1=0$, $a_1=a_2=0$ to get the results in [12,13], and take $m=n=1$ to reach those in [1].

Remark 1.5. Since the source is positive for u and negative for v in (1.1)-(1.3), the roles of the two components are unequal. This is quite different from that considered in [10], where both of the two components u and v possess negative source terms. In addition, it was known that blow-up or not of the solutions in the critical case to that problem will be determined by the coefficients of absorption terms [10]. However, this does not occur for our problem (1.1)-(1.3) by Theorem 1.4 due to the effects of the positive source for the component u.

This paper is arranged as follows. In the next section, we will verify the blow-up conditions in Theorems 1.1 and 1.2. Then Theorem 1.3 will be proven in Section 3 to fix the conditions for global solutions. Section 4 deals with the critical situation stated in Theorem 1.4, for which the arguments for Theorems 1.2 and 1.3 will be applied directly.

2 Blowing up of solutions

In this section, we will prove the blow-up of solutions under the conditions of Theorems 1.1 and 1.2. We first treat the more trivial blow-up case of $\alpha_1>m$.

Proof of Theorem 1.1. Introduce a scalar problem of the form

$$(U^m)_t=\Delta U+a_1U^{\alpha_1},\qquad (x,t)\in\Omega\times(0,T), \tag{2.1}$$
$$\frac{\partial U}{\partial\eta}=0,\qquad (x,t)\in\partial\Omega\times(0,T), \tag{2.2}$$
$$U(x,0)=u_0(x),\qquad x\in\bar{\Omega} \tag{2.3}$$

with $a_1>0$, $\alpha_1>m$. By Proposition 1.1 (i), U blows up in a finite time for large initial data, while $(U,0)$ is just a pair of subsolutions to the problem (1.1)-(1.3). □

The assumption $\alpha_1>m$ implies that the component u in (1.1)-(1.3) can blow up in a finite time without the help of v. So, we will always assume $\alpha_1\le m$ in the sequel.

We need the eigenvalue problem related to (1.1)-(1.3) throughout the rest of the paper. Let φ_0 be the first eigenfunction of

$$\Delta\varphi+\lambda\varphi=0\quad\text{in }\Omega,\qquad \varphi=0\quad\text{on }\partial\Omega \tag{2.4}$$

with the first eigenvalue λ_0, normalized by $\|\varphi_0\|_\infty=1$, $\varphi_0>0$ in Ω. It is well known that there exist positive constants c_i $(i=1,\,2,\,3)$ such that

$$|\nabla\varphi_0|\le c_1\quad\text{on }\bar{\Omega},\qquad c_2\le-\frac{\partial\varphi_0}{\partial\eta}\le c_3\quad\text{on }\partial\Omega. \tag{2.5}$$

Denote

$$\Omega_1=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)<\varepsilon\},\qquad \Omega_2=\Omega\backslash\bar{\Omega}_1. \tag{2.6}$$

It is easy to see that there exist some ε, $c_4\in(0,1)$ such that

$$|\nabla\varphi_0|\ge c_2/2\quad \text{on } \bar{\Omega}_1,\qquad \varphi_0\ge c_4\quad \text{on } \bar{\Omega}_2. \tag{2.7}$$

Furthermore, let h_0 be the solution of the linear problem

$$\Delta h=\lambda_h=|\partial\Omega|/|\Omega|\quad \text{in } \Omega,\qquad \partial h/\partial\eta=1\quad \text{on } \partial\Omega \tag{2.8}$$

with $\inf_{\bar{\Omega}}h_0(x)=0$, $\sup_{\bar{\Omega}}h_0(x)=c_5$, $|\Omega|$ and $|\partial\Omega|$ being the Lebesgue measures of Ω and $\partial\Omega$, respectively.

It is observed that the assumption $1/\rho_1$ or $1/\rho_2>0$ with $\alpha_1\le m$ in Theorem 1.2 comes from the following three cases only:

(a) $\beta_2>\gamma$;

(b) $\alpha_2>\mu$, $\beta_2\le\gamma$;

(c) $\alpha_2\le\mu,\beta_2\le\gamma$ with $pq>(\mu-\alpha_2)(\gamma-\beta_2)$.

We will prove Theorem 1.2 via six lemmas. The first one deals with the case (a), and the other five will treat both (b) and (c).

Lemma 2.1. *Under the condition* (a)*, the solutions of* (1.1)-(1.3) *blow up in a finite time for large initial data.*

Proof. Let (u,v) be a solution of (1.1)-(1.3), U solve (2.1)-(2.3). Due to Theorem 1.1, we can assume $u\ge U\ge\delta>0$. Consider the scalar problem

$$(V^n)_t=\Delta V-a_2V^{\beta_1},\qquad (x,t)\in\Omega\times(0,T), \tag{2.9}$$

$$\frac{\partial V}{\partial\eta}=\delta^qV^{\beta_2},\qquad (x,t)\in\partial\Omega\times(0,T), \tag{2.10}$$

$$V(x,0)=v_0(x),\qquad x\in\bar{\Omega}. \tag{2.11}$$

We know that $v\ge V$ by the comparison principle.

Clearly, the condition $\beta_2>\gamma$ implies that at least one of the following holds:

(i) $n\ge1$ with $\beta_1\le n$, $\beta_2>(n+1)/2$ or $\beta_1>n$, $\beta_2>(\beta_1+1)/2$;

(ii) (ii) $0<n<1$ with $\beta_2>1$, $\beta_2>(\beta_1+1)/2$, or $n<\beta_2\le1$, $\beta_2>\beta_1$.

By Proposition 1.2, V blows up in a finite time with large initial data. □

Lemma 2.2. *Assume $m,n\ge1$, $\beta_1>n$ with one of* (b) *and* (c)*. Then the solutions of* (1.1)-(1.3) *will blow up in a finite time for large initial data.*

Proof. Notice $\alpha_1 \le m$, $\beta_1 > n \ge 1$ implies $\mu = (1+m)/2$, $\gamma = (1+\beta_1)/2$ by (1.14). Construct

$$\underline{u}(x,t) = \frac{A}{(1-ct)^K} \mathrm{e}^{-\frac{a\varphi(x)}{(1-ct)^{K_1}}}, \quad \underline{v}(x,t) = B\Big[\varphi B^{\frac{\beta_1-1}{2}} + (1-ct)^L\Big]^{-\frac{2}{\beta_1-1}}$$

for $(x,t) \in \bar{\Omega} \times [0,1/c)$, where $\varphi = M\varphi_0$, $a = A^{\frac{m-1}{2}}$, $K_1 = 1$ for $m=1$, $K_1 = ((m-1)K+1)/2$ for $m>1$, and A, B, M, K, L, c are positive constants to be determined. We have

$$\begin{aligned}
(\underline{u}^m)_t &\le \frac{A^m Kmc}{(1-ct)^{Km+1}} \mathrm{e}^{-\frac{ma\varphi(x)}{(1-ct)^{K_1}}}, \\
\Delta \underline{u} &= \frac{Aa\lambda_0\varphi}{(1-ct)^{K+K_1}} \mathrm{e}^{-\frac{a\varphi(x)}{(1-ct)^{K_1}}} + \frac{Aa^2|\nabla\varphi|^2}{(1-ct)^{K+2K_1}} \mathrm{e}^{-\frac{a\varphi(x)}{(1-ct)^{K_1}}}.
\end{aligned}$$

We know from (2.5)-(2.7) that for $x \in \Omega_1$,

$$\Delta \underline{u} \ge \frac{Aa^2M^2c_2^2}{4(1-ct)^{K+2K_1}} \mathrm{e}^{-\frac{a\varphi(x)}{(1-ct)^{K_1}}},$$

and for $x \in \bar{\Omega}_2$,

$$\begin{aligned}
\Delta \underline{u} &\ge \frac{Aa\lambda_0 Mc_4}{(1-ct)^{K+K_1}} \mathrm{e}^{-\frac{a\varphi(x)}{(1-ct)^{K_1}}}, \\
(\underline{u}^m)_t &\le \frac{A^m Kmc}{(m-1)ac_4(1-ct)^{Km+1-K_1}} \mathrm{e}^{-\frac{a\varphi(x)}{(1-ct)^{K_1}}}, \quad m>1,
\end{aligned}$$

where the fact $y\mathrm{e}^{-y} \le \mathrm{e}^{-1}$ for $y>0$ is used. By choosing

$$\begin{aligned}
c &\le \min\Big\{\frac{\lambda_0 Mc_4}{K}, \frac{M^2c_2^2}{4K}\Big\} \quad \text{for } m=1; \\
c &\le \min\Big\{\frac{(m-1)\lambda_0 Mc_4^2}{Km}, \frac{M^2c_2^2}{4Km}\Big\} \quad \text{for } m>1,
\end{aligned}$$

we get

$$(\underline{u}^m)_t \le \Delta \underline{u} + a_1 \underline{u}^{\alpha_1} \quad \text{in } \Omega \times (0,1/c). \tag{2.12}$$

Similarly, for $\underline{v}$ with $L \ge \max\{(\beta_1-1)/2(\beta_1-n), 1\}$, we have

$$\begin{aligned}
(\underline{v}^n)_t &\le \frac{2B^n Lnc}{\beta_1-1}\Big[\varphi B^{\frac{\beta_1-1}{2}} + (1-ct)^L\Big]^{\frac{-2\beta_1}{\beta_1-1}} \Big[MB^{\frac{\beta_1-1}{2}} + 1\Big]^{\frac{2(\beta_1-n)}{\beta_1-1} - \frac{1}{L}}, \\
\Delta \underline{v} &= \frac{2B^{\frac{\beta_1+1}{2}}\lambda_0\varphi}{(\beta_1-1)[\varphi B^{\frac{\beta_1-1}{2}} + (1-ct)^L]^{\frac{\beta_1+1}{\beta_1-1}}} + \frac{2(\beta_1+1)B^{\beta_1}|\nabla\varphi|^2}{(\beta_1-1)^2[\varphi B^{\frac{\beta_1-1}{2}} + (1-ct)^L]^{\frac{2\beta_1}{\beta_1-1}}}, \\
a_2 \underline{v}^{\beta_1} &= \frac{a_2 B^{\beta_1}}{[\varphi B^{\frac{\beta_1-1}{2}} + (1-ct)^L]^{\frac{2\beta_1}{\beta_1-1}}}.
\end{aligned}$$

With (2.5)-(2.7), we know for $x\in\Omega_1$ that

$$\frac{1}{2}\Delta\underline{v}-a_2\underline{v}^{\beta_1}\ge\frac{B^{\beta_1}}{[\varphi B^{\frac{\beta_1-1}{2}}+(1-ct)^L]^{\frac{2\beta_1}{\beta_1-1}}}\Big[\frac{(\beta_1+1)M^2c_2^2}{4(\beta_1-1)^2}-a_2\Big],$$

and for $x\in\bar{\Omega}_2$ that

$$\frac{1}{2}\Delta\underline{v}-a_2\underline{v}^{\beta_1}\ge\frac{B^{\beta_1}}{[\varphi B^{\frac{\beta_1-1}{2}}+(1-ct)^L]^{\frac{2\beta_1}{\beta_1-1}}}\Big(\frac{\lambda_0M^2c_4^2}{\beta_1-1}-a_2\Big).$$

By choosing

$$\begin{aligned}&M^2=\max\{a_2(\beta_1-1)/\lambda_0c_4^2,4a_2(\beta_1-1)^2/(\beta_1+1)c_2^2\},\\&c\le\frac{(\beta_1-1)B^{\beta_1-n}a_2}{2Ln\big[MB^{\frac{\beta_1-1}{2}}+1\big]^{\frac{2(\beta_1-n)}{\beta_1-1}-\frac{1}{L}}},\end{aligned}$$

we get

$$(\underline{v}^n)_t\le\Delta\underline{v}-a_2\underline{v}^{\beta_1}\quad\text{in }\Omega\times(0,1/c).\tag{2.13}$$

On the boundary $\partial\Omega\times(0,1/c)$, we find

$$\frac{\partial\underline{u}}{\partial\eta}\le\frac{aAMc_3}{(1-ct)^{K+K_1}},\quad\frac{\partial\underline{v}}{\partial\eta}\le\frac{2B^{\frac{\beta_1+1}{2}}Mc_3}{(\beta_1-1)(1-ct)^{\frac{L(\beta_1+1)}{\beta_1-1}}},\tag{2.14}$$

$$\underline{u}^{\alpha_2}\underline{v}^p=\frac{A^{\alpha_2}B^p}{(1-ct)^{K\alpha_2+\frac{2Lp}{\beta_1-1}}},\quad\underline{u}^q\underline{v}^{\beta_2}=\frac{A^qB^{\beta_2}}{(1-ct)^{Kq+\frac{2L\beta_2}{\beta_1-1}}}.\tag{2.15}$$

Since the condition (b) implies $\alpha_2>(m+1)/2\ge1$, $\beta_2\le(\beta_1+1)/2$, we have

$$K+K_1<\alpha_2K+\frac{2Lp}{\beta_1-1},\quad\frac{L(\beta_1+1-2\beta_2)}{\beta_1-1}<Kq$$

for large K, and

$$A>\max\Big\{(aMc_3)^{\frac{1}{\alpha_2-1}},\Big(\frac{2Mc_3}{\beta_1-1}B^{\frac{\beta_1+1}{2}-\beta_2}\Big)^{1/q}\Big\},\quad B\ge1$$

for large A.

For the condition (c), due to the fact that

$$pq>\Big(\frac{m+1}{2}-\alpha_2\Big)\Big(\frac{\beta_1+1}{2}-\beta_2\Big)\ge\Big(1-\alpha_2\Big)\Big(\frac{\beta_1+1}{2}-\beta_2\Big)\quad\text{with }\alpha_2\le\frac{m+1}{2},\ \beta_2\le\frac{\beta_1+1}{2},$$

we know

$$\begin{aligned}&(aMc_3)^{\frac{1}{p}}A^{\frac{1-\alpha_2}{p}}<B<\Big(\frac{\beta_1-1}{2Mc_3}\Big)^{\frac{1}{\frac{\beta_1+1}{2}-\beta_2}}A^{\frac{q}{\frac{\beta_1+1}{2}-\beta_2}},\\&K+K_1<\alpha_2K+\frac{2Lp}{\beta_1-1},\quad \frac{L(\beta_1+1-2\beta_2)}{\beta_1-1}<Kq,\end{aligned}$$

provided that A, B, K, L are sufficiently large. In summary, for both (b) and (c), we obtain from (2.14)-(2.15) that

$$\frac{\partial \underline{u}}{\partial\eta}\le \underline{u}^{\alpha_2}\underline{v}^{p},\quad \frac{\partial \underline{v}}{\partial\eta}\le \underline{u}^{q}\underline{v}^{\beta_2}\quad \text{on } \partial\Omega\times(0,1/c). \tag{2.16}$$

Letting $u_0(x)\ge\underline{u}(x,0)$, $v_0(x)\ge\underline{v}(x,0)$, we understand from (2.12), (2.13) and (2.16) that $(\underline{u},\underline{v})$ is a blow-up subsolution of (1.1)-(1.3). □

Lemma 2.3. *Assume* $0<m$, $n<1$, $\beta_1>n$ *with one of* (b) *and* (c). *Then the solutions of* (1.1)-(1.3) *will blow up in a finite time for large initial data.*

Proof. By (1.15), divide the condition $0<m$, $n<1$, $\beta_1>n$ into two subcases: (i) $\beta_1>1$ with $\mu=m$, $\gamma=(\beta_1+1)/2$; (ii) $n<\beta_1\le 1$ with $\mu=m$, $\gamma=\beta_1$. Construct

$$\begin{aligned}&\underline{u}(x,t)=\frac{A}{(1-ct)^K}e^{hA^{m-1}(1-ct)^{K(1-m)-1}},\\&\underline{v}(x,t)=B\Big[\varphi B^{\frac{\beta_1-1}{2}}+(1-ct)^L\Big]^{-\frac{2}{\beta_1-1}}\quad\text{for (i)},\\&\underline{v}(x,t)=\frac{B}{(1-ct)^L}e^{hB^{\beta_1-1}(1-ct)^{-L(\beta_1-1)}}\quad\text{for (ii)},\end{aligned}$$

with $\varphi=M\varphi_0$, $h=Mh_0$, and A, B, M, K, L, $c>0$ to be determined. We first consider the subcase (i). We have with $\beta_1>1$, $K>1/(1-m)$ that

$$\begin{aligned}&(\underline{u}^m)_t\le\frac{A^mKmc}{(1-ct)^{Km+1}}e^{hA^{m-1}(1-ct)^{K(1-m)-1}},\\&\Delta\underline{u}\ge\frac{A^m\lambda_hM}{(1-ct)^{Km+1}}e^{hA^{m-1}(1-ct)^{K(1-m)-1}},\end{aligned}$$

and consequently,

$$(\underline{u}^m)_t\le\Delta\underline{u}+a_1\underline{u}^{\alpha_1}\quad\text{in }\Omega\times(0,1/c) \tag{2.17}$$

if $0<c<\lambda_hM/(Km)$. Similar to Lemma 2.2, we have

$$(\underline{v}^n)_t\le\Delta\underline{v}-a_2\underline{v}^{\beta_1}\quad\text{in }\Omega\times(0,1/c) \tag{2.18}$$

with

$$\begin{aligned} &M^2=\max\{a_2(\beta_1-1)/\lambda_0c_4^2,4a_2(\beta_1-1)^2/(\beta_1+1)c_2^2\},\\ &c\le\frac{(\beta_1-1)B^{\beta_1-n}a_2}{2Ln[MB^{\frac{\beta_1-1}{2}}+1]^{\frac{2(\beta_1-n)}{\beta_1-1}-\frac{1}{L}}},\quad L\ge\max\{(\beta_1-1)/2(\beta_1-n),1\}. \end{aligned}$$

Moreover, for $(x,t)\in\partial\Omega\times(0,1/c)$,

$$\frac{\partial\underline{u}}{\partial\eta}\le\frac{MA^m\mathrm{e}^{Mc_5}}{(1-ct)^{Km+1}},\quad \frac{\partial\underline{v}}{\partial\eta}\le\frac{2B^{\frac{\beta_1+1}{2}}Mc_3}{(\beta_1-1)(1-ct)^{\frac{L(\beta_1+1)}{\beta_1-1}}}, \tag{2.19}$$

$$\underline{u}^{\alpha_2}\underline{v}^p\ge\frac{A^{\alpha_2}B^p}{(1-ct)^{K\alpha_2+\frac{2Lp}{\beta_1-1}}},\quad \underline{u}^q\underline{v}^{\beta_2}\ge\frac{A^qB^{\beta_2}}{(1-ct)^{Kq+\frac{2L\beta_2}{\beta_1-1}}}. \tag{2.20}$$

Under the condition (b) with $\alpha_2>m$, $\beta_2\le(\beta_1+1)/2$, or the condition (c) with $\alpha_2\le m$, $\beta_2\le(\beta_1+1)/2$ by $pq>(m-\alpha_2)(\frac{\beta_1+1}{2}-\beta_2)$, it holds that

$$\begin{aligned} &(M\mathrm{e}^{Mc_5})^{\frac{1}{p}}A^{\frac{m-\alpha_2}{p}}<B<\Big(\frac{\beta_1-1}{2Mc_3}\Big)^{\frac{1}{\frac{\beta_1+1}{2}-\beta_2}}A^{\frac{q}{\frac{\beta_1+1}{2}-\beta_2}},\\ &K>\frac{\beta_1+1-2\beta_2}{q(\beta_1-1)}L,\quad L>\frac{(\beta_1-1)q}{2pq-(m-\alpha_2)(\beta_1+1-2\beta_2)} \end{aligned}$$

for A, B, K, L sufficiently large. Consequently, we obtain from (2.19)-(2.20) that

$$\frac{\partial\underline{u}}{\partial\eta}\le\underline{u}^{\alpha_2}\underline{v}^p,\quad \frac{\partial\underline{v}}{\partial\eta}\le\underline{u}^q\underline{v}^{\beta_2}\quad \text{on } \partial\Omega\times(0,1/c). \tag{2.21}$$

With $u_0(x)\ge\underline{u}(x,0)$, $v_0(x)\ge\underline{v}(x,0)$, we know from (2.17), (2.18) and (2.21) that $(\underline{u},\underline{v})$ is a blowing up subsolution of (1.1)-(1.3).

Next consider the subcase (ii), where $n<\beta_1\le1$ with $\mu=m$, $\gamma=\beta_1$. Similarly to subcase (i), we can get for u that

$$(\underline{u}^m)_t\le\Delta\underline{u}+a_1\underline{u}^{\alpha_1}\quad \text{in } \Omega\times(0,1/c). \tag{2.22}$$

For $\underline{v}$, a simple calculation shows that

$$\begin{aligned} &(\underline{v}^n)_t\le\frac{B^nLnc}{(1-ct)^{Ln+1}}\mathrm{e}^{hB^{\beta_1-1}(1-ct)^{-L(\beta_1-1)}},\\ &\Delta\underline{v}\ge\frac{B^{\beta_1}\lambda_hM}{(1-ct)^{L\beta_1}}\mathrm{e}^{hB^{\beta_1-1}(1-ct)^{-L(\beta_1-1)}},\\ &\Delta\underline{v}-a_2\underline{v}^{\beta_1}\ge\frac{B^{\beta_1}}{(1-ct)^{L\beta_1}}(\lambda_hM-a_2)\mathrm{e}^{hB^{\beta_1-1}(1-ct)^{-L(\beta_1-1)}}, \end{aligned}$$

and thus,

$$(\underline{v}^n)_t \le \Delta \underline{v} - a_2 \underline{v}^{\beta_1} \quad \text{in } \Omega\times(0,1/c) \tag{2.23}$$

provided that

$$L > \frac{1}{\beta_1 - n}, \quad 0 < c \le \frac{B^{\beta_1 - n}}{Ln}(\lambda_h M - a_2).$$

Moreover, on $\partial\Omega\times(0,1/c)$,

$$\frac{\partial \underline{u}}{\partial \eta} \le \frac{A^m \mathrm{e}^{Mc_5} M}{(1-ct)^{Km+1}}, \quad \frac{\partial \underline{v}}{\partial \eta} \le \frac{B^{\beta_1} M \mathrm{e}^{Mc_5}}{(1-ct)^{L\beta_1}}, \tag{2.24}$$

$$\underline{u}^{\alpha_2} \underline{v}^p \ge \frac{A^{\alpha_2} B^p}{(1-ct)^{K\alpha_2 + Lp}}, \quad \underline{u}^q \underline{v}^{\beta_2} \ge \frac{A^q B^{\beta_2}}{(1-ct)^{Kq + L\beta_2}}. \tag{2.25}$$

For (b) with $\alpha_2 > m$, $\beta_2 \le \beta_1$, or (c) with $pq > (m-\alpha_2)(\beta_1-\beta_2)$, $\alpha_2 \le m$, and $\beta_2 \le \beta_1$, let A, B, K, L be sufficiently large to get

$$(M\mathrm{e}^{Mc_5})^{\frac{1}{p}} A^{\frac{m-\alpha_2}{p}} < B < \Big(\frac{1}{M\mathrm{e}^{Mc_5}}\Big)^{\frac{1}{\beta_1-\beta_2}} A^{\frac{q}{\beta_1-\beta_2}},$$
$$K > \frac{\beta_1-\beta_2}{q} L, \quad L > \frac{q}{pq-(m-\alpha_2)(\beta_1-\beta_2)},$$

and hence

$$\frac{\partial \underline{u}}{\partial \eta} \le \underline{u}^{\alpha_2} \underline{v}^p, \quad \frac{\partial \underline{v}}{\partial \eta} \le \underline{u}^q \underline{v}^{\beta_2} \quad \text{on } \partial\Omega\times(0,1/c) \tag{2.26}$$

by (2.24)-(2.25). With $u_0(x) \ge \underline{u}(x,0)$, $v_0(x) \ge \underline{v}(x,0)$, it follows from (2.22), (2.23) and (2.26) that $(\underline{u},\underline{v})$ is a subsolution of (1.1)-(1.3). □

Lemma 2.4. *Assume* $0 < m < 1$, $n \ge 1$, $\beta_1 > n$ *with one of* (b) *and* (c). *Then the solutions of* (1.1)-(1.3) *will blow up in a finite time for large initial data.*

Proof. Notice $0 < m < 1$, $n \ge 1$, $\beta_1 > n$ imply $\mu = m$, $\gamma = \frac{1+\beta_1}{2}$ by (1.16). Construct

$$\underline{u}(x,t) = \frac{A}{(1-ct)^K} \mathrm{e}^{hA^{m-1}(1-ct)^{K(1-m)-1}},$$
$$\underline{v}(x,t) = B\Big[\varphi B^{\frac{\beta_1-1}{2}} + (1-ct)^L\Big]^{-\frac{2}{\beta_1-1}}$$

with $\varphi = M\varphi_0$, $h = Mh_0$. Similar to the proof for Lemmas 2.2 and 2.3, we can show that $(\underline{u},\underline{v})$ is a subsolution of (1.1)-(1.3). □

Lemma 2.5. *Assume* $m \ge 1$, $0 < n < 1$, $\beta_1 > n$ *with one of* (b) *and* (c). *Then the solutions of* (1.1)-(1.3) *will blow up in a finite time for large initial data.*

Proof. Clearly, it follows from (1.17) that $m\geq 1$, $0<n<1$, $\beta_1>n$ imply $\mu=(m+1)/2$, $\gamma=(1+\beta_1)/2$ for $\beta_1>1$, and $\mu=(m+1)/2$, $\gamma=\beta_1$ for $n<\beta_1\leq 1$. Construct

$$\begin{aligned}
&\underline{u}(x,t)=\frac{A}{(1-ct)^K}\mathrm{e}^{-\frac{\varphi(x)}{1-ct}}, && m=1,\quad (x,t)\in\bar{\Omega}\times[0,1/c),\\
&\underline{u}(x,t)=\frac{A}{(1-ct)^K}\mathrm{e}^{-\frac{a\varphi(x)}{(1-ct)^{K_1}}}, && m>1,\quad (x,t)\in\bar{\Omega}\times[0,1/c),\\
&\underline{v}(x,t)=B\Big[\varphi B^{\frac{\beta_1-1}{2}}+(1-ct)^L\Big]^{-\frac{2}{\beta_1-1}}, && \beta_1>1,\quad (x,t)\in\bar{\Omega}\times[0,1/c),\\
&\underline{v}(x,t)=\frac{B}{(1-ct)^L}\mathrm{e}^{hB^{\beta_1-1}(1-ct)^{-L(\beta_1-1)}}, && n<\beta_1\leq 1,\quad (x,t)\in\bar{\Omega}\times[0,1/c)
\end{aligned}$$

with $\varphi=M\varphi_0$, $h=Mh_0$. According to the arguments used for Lemmas 2.2 and 2.3, we know that $(\underline{u},\underline{v})$ is a blowing up subsolution of (1.1)-(1.3). □

Lemma 2.6. *Assume $\beta_1\leq n$ with one of* (b) *and* (c)*. Then the solutions of* (1.1)-(1.3) *will blow up in finite time for large initial data.*

Proof. We know $\mu=(m+1)/2$, $\gamma=(n+1)/2$ for m, $n\geq 1$ by (1.14); $\mu=m$, $\gamma=n$ for $0<m$, $n<1$; $\mu=m$, $\gamma=(n+1)/2$ for $0<m<1$, $n\geq 1$ by (1.15); $\mu=(m+1)/2$, $\gamma=n$ for $m\geq 1$, $0<n<1$ by (1.17) due to $\alpha_1\leq m$, $\beta_1\leq n$.

We still deal with the two subcases (b) and (c). Clearly, we can choose $\beta_0>n\geq\beta_1$ for (b) such that $\alpha_2>\mu_0$, $\beta_2\leq\gamma_0$, where $\mu_0=\mu$, $\gamma_0=(\beta_0+1)/2$ for $n\geq 1$, $\gamma_0=\beta_0$ for $0<n<1$. Similarly, choose $\beta_0>n\geq\beta_1$ with μ_0,γ_0 defined as above for (c) such that $pq>(\mu_0-\alpha_2)(\gamma_0-\beta_2)$ with $\alpha_2\leq\mu_0$, $\beta_2\leq\gamma_0$. Consider the auxiliary problem

$$(w^m)_t=\Delta w+a_1w^{\alpha_1},\quad (z^n)_t=\Delta z-a_2z^{\beta_0},\qquad (x,t)\in\Omega\times(0,T), \tag{2.27}$$

$$\frac{\partial w}{\partial\eta}=w^{\alpha_2}z^p,\quad \frac{\partial z}{\partial\eta}=w^qz^{\beta_2},\qquad (x,t)\in\partial\Omega\times(0,T), \tag{2.28}$$

$$w(x,0)=w_0(x),\quad z(x,0)=z_0(x),\qquad x\in\bar{\Omega}. \tag{2.29}$$

By Lemmas 2.2-2.5, the solutions of (2.27)-(2.29) with $\beta_0>n\geq\beta_1$ blow up in a finite time for large initial data. Let (w,z) be such a solution with blow-up time T'. Take T_1 satisfying

$$\{n/[a_2(\beta_0-n)T_1]\}^{1/(\beta_0-n)}>1, \tag{2.30}$$

and let w_0,z_0 be large such that $T'\leq T_1$. It can be verified that the ODE problem

$$(\eta^n)_t=-a_2\eta^{\beta_0}(t),\quad \eta(0)=M$$

has a solution of the form

$$\eta(t)=\Big(\frac{n}{a_2(\beta_0-n)t+nM^{n-\beta_0}}\Big)^{\frac{1}{\beta_0-n}}.$$

By the comparison principle, we have that

$$z(x,t)\geq\eta(t)\geq\Big(\frac{n}{a_2(\beta_0-n)T_1+nM^{n-\beta_0}}\Big)^{\frac{1}{\beta_0-n}}>1,\quad (x,t)\in\bar{\Omega}\times[0,T')$$

provided M large enough due to (2.30). Consequently, (w,z) can be a subsolution to (1.1)-(1.3) with $w_0,z_0\geq M$. Theorem 1.2 is obtained from Lemmas 2.1-2.6. ☐

3 Global solutions

We will prove that the case of $1/\rho_1,1/\rho_2<0$ corresponds to the global existence of solutions stated in Theorem 1.3. Firstly, we claim that the assumption $1/\rho_1,1/\rho_2<0$ is equivalent to

(d) $pq<(\mu-\alpha_2)(\gamma-\beta_2)$ with $\alpha_2<\mu$, $\beta_2<\gamma$.

In fact, $1/\rho_1$, $1/\rho_2<0$ implies $(p-\beta_2+\gamma)(q-\alpha_2+\mu)>0$. If $p<\beta_2-\gamma$, $q<\alpha_2-\mu$, then $pq<(\beta_2-\gamma)(\alpha_2-\mu)$, and thus $1/\rho_1$, $1/\rho_2>0$, a contradiction. Consequently, it has to be satisfied that $p>\beta_2-\gamma$, $q>\alpha_2-\mu$ and $pq<(\beta_2-\gamma)(\alpha_2-\mu)$. If $\alpha_2\geq\mu$ or $\beta_2\geq\gamma$, then $pq>(\beta_2-\gamma)(\alpha_2-\mu)$, also a contradiction.

We will prove that the solutions are global even under the following more general assumption:

(e) $pq\leq(\mu-\alpha_2)(\gamma-\beta_2)$ with $\alpha_2<\mu$, $\beta_2<\gamma$.

Notice that the assumption (e) covers the critical case $1/\rho_1=1/\rho_2=0$ with $\alpha_2<\mu$, $\beta_2<\gamma$, corresponding to Theorem 1.4 (ii).

It suffices to treat the following eight situations:

1. m, $n\geq1$ with $\beta_1>n$;
2. $0<m$, $n<1$ with $\beta_1>n$;
3. $0<m<1$, $n\geq1$ with $\beta_1>n$;
4. $m\geq1$, $0<n<1$ with $\beta_1>n$;
5. m, $n\geq1$ with $\beta_1\leq n$;
6. $0<m$, $n<1$ with $\beta_1\leq n$;
7. $0<m<1$, $n\geq1$ with $\beta_1\leq n$;
8. $m\geq1$, $0<n<1$ with $\beta_1\leq n$.

Correspondingly, we will introduce eight lemmas in the sequel, where it is always assumed that $\alpha_1\leq m$ due to Theorem 1.1.

Lemma 3.1. *Assume $m,n\geq1$ with $\beta_1>n$, and the condition* (e) *holds. Then the solutions of* (1.1)-(1.3) *are global.*

Proof. The assumption of the lemma implies

$$\mu=\frac{1+m}{2},\quad \gamma=\frac{1+n}{2}+\Big(\frac{\beta_1-n}{2}\Big)_+=\frac{1+\beta_1}{2}$$

with m, $n\geq 1$ and $\beta_1>n$ by (1.14), and thus

$$pq\leq\Big(\frac{1+m}{2}-\alpha_2\Big)\Big(\frac{1+\beta_1}{2}-\beta_2\Big).$$

Construct

$$\bar{u}(x,t)=\mathrm{e}^{Kt}\big(M+A^{-1}\mathrm{e}^{-A^2\varphi_0\mathrm{e}^{K(m-1)t/2}}\big),\quad \bar{v}(x,t)=B\mathrm{e}^{Lt}\big(M+\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)/2}}\big)$$

with positive constants K, L, A, B, M to be determined. Due to $-y\mathrm{e}^{-y}\geq-\mathrm{e}^{-1}$ for $y\geq 0$,

$$\begin{aligned}
(\bar{u}^m)_t=&Km\mathrm{e}^{Kmt}\big(M+A^{-1}\mathrm{e}^{-A^2\varphi_0\mathrm{e}^{K(m-1)t/2}}\big)^{m-1}\Big(M+A^{-1}\mathrm{e}^{-A^2\varphi_0\mathrm{e}^{K(m-1)t/2}}\\
&-\frac{(m-1)A^2\varphi_0\mathrm{e}^{K(m-1)t/2}}{2}A^{-1}\mathrm{e}^{-A^2\varphi_0\mathrm{e}^{K(m-1)t/2}}\Big)\geq Km\mathrm{e}^{Kmt},\\
\Delta\bar{u}=&\Big(A\mathrm{e}^{K(m+1)t/2}\lambda_0\varphi_0+A^3|\nabla\varphi_0|^2\mathrm{e}^{Kmt}\Big)\mathrm{e}^{-A^2\varphi_0\mathrm{e}^{K(m-1)t/2}}\\
\leq& A(\lambda_0+A^2c_1^2)\mathrm{e}^{Kmt},\\
a_1\bar{u}^{\alpha_1}=&a_1\mathrm{e}^{\alpha_1Kt}\Big(M+A^{-1}\mathrm{e}^{-A^2\varphi_0\mathrm{e}^{K(m-1)t/2}}\Big)^{\alpha_1}\leq a_1\mathrm{e}^{mKt}(M+A^{-1})^{\alpha_1},
\end{aligned}$$

provided that $A\geq\frac{m-1}{2\mathrm{e}(M-1)}$. Hence

$$(\bar{u}^m)_t\geq\Delta\bar{u}+a_1\bar{u}^{\alpha_1},\quad (x,t)\in\Omega\times\mathbb{R}^+ \tag{3.1}$$

provided that

$$K\geq\frac{A(\lambda_0+A^2c_1^2)+a_1(M+A^{-1})^{\alpha_1}}{m}.$$

Similarly, we have also

$$\begin{aligned}
(\bar{v}^n)_t=&B^nLn\mathrm{e}^{Lnt}\big(M+\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)/2}}\big)^{n-1}\Big(M+\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)/2}}\\
&-\frac{(\beta_1-1)\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)/2}}{2}\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)/2}}\Big)\geq 0,\\
\Delta\bar{v}=&B\mathrm{e}^{Lt}\Big(\lambda_0\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)/2}+|\nabla\varphi_0|^2(B\mathrm{e}^{Lt})^{\beta_1-1}\Big)\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)/2}}\\
\leq&(\lambda_0+c_1^2)B^{\beta_1}\mathrm{e}^{L\beta_1t},\\
a_2\bar{v}^{\beta_1}=&a_2B^{\beta_1}\mathrm{e}^{L\beta_1t}\Big(M+\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)/2}}\Big)^{\beta_1}\geq a_2B^{\beta_1}M^{\beta_1}\mathrm{e}^{L\beta_1t}.
\end{aligned}$$

Consequently,

$$(\bar{v}^n)_t \geq \Delta\bar{v} - a_2\bar{v}^{\beta_1}, \quad (x,t)\in\Omega\times\mathbb{R}^+ \tag{3.2}$$

provided that

$$M \geq \max\Big(\frac{\beta_1-1}{2\mathrm{e}}, \big(\frac{\lambda_0+c_1^2}{a_2}\big)^{\frac{1}{\beta_1}}\Big).$$

Moreover, on the boundary $\partial\Omega$,

$$\begin{aligned}
&\frac{\partial\bar{u}}{\partial\eta} \geq c_2 A\mathrm{e}^{K(m+1)t/2}, \quad \bar{u}^{\alpha_2}\bar{v}^p \leq B^p(M+1)^{p+\alpha_2}\mathrm{e}^{K\alpha_2 t+Lpt},\\
&\frac{\partial\bar{v}}{\partial\eta} \geq c_2(B\mathrm{e}^{Lt})^{(\beta_1+1)/2}, \quad \bar{u}^q\bar{v}^{\beta_2} \leq B^{\beta_2}(M+1)^{q+\beta_2}\mathrm{e}^{Kqt+L\beta_2 t}.
\end{aligned}$$

Observe that

$$pq \leq \Big(\frac{1+m}{2}-\alpha_2\Big)\Big(\frac{1+\beta_1}{2}-\beta_2\Big).$$

Letting

$$L = K\Big(\frac{m+1}{2}-\alpha_2\Big)/p, \quad B^{\frac{\beta_1+1}{2}-\beta_2} \geq (M+1)^{q+\beta_2}/c_2, \quad A \geq B^p(M+1)^{p+\alpha_2}/c_2,$$

we can have

$$\frac{\partial\bar{u}}{\partial\eta} \geq \bar{u}^{\alpha_2}\bar{v}^p, \quad \frac{\partial\bar{v}}{\partial\eta} \geq \bar{u}^q\bar{v}^{\beta_2}, \quad (x,t)\in\partial\Omega\times\mathbb{R}^+. \tag{3.3}$$

For $M \geq \max(\|u_0\|_\infty, \|v_0\|_\infty)$, we have in addition that $\bar{u}(x,0) \geq u_0(x)$ and $\bar{v}(x,0) \geq v_0(x)$. Thus, $(\bar{u},\bar{v})$ is a global supersolution of (1.1)-(1.3). □

Lemma 3.2. *Assume* $0<m$, $n<1$ *with* $\beta_1>n$, *and the condition* (e) *holds. Then the solutions of* (1.1)-(1.3) *are global.*

Proof. We know

$$\mu = m, \quad \gamma = n + \Big(\beta_1 - n\Big)_+ - \Big(\frac{\beta_1-1}{2}\Big)_+$$

with $0<m,n<1$ by (1.15). Thus, $\gamma=(1+\beta_1)/2$ if $\beta_1>1>n$, and $\gamma=\beta_1$ if $n<\beta_1<1$. There are two subcases for the lemma: (i) $\beta_1>1$ with $pq \leq (m-\alpha_2)(\frac{1+\beta_1}{2}-\beta_2) < (1-\alpha_2)(\frac{1+\beta_1}{2}-\beta_2)$; (ii) $n<\beta_1\leq 1$ with $pq \leq (m-\alpha_2)(\beta_1-\beta_2) < (1-\alpha_2)(\beta_1-\beta_2)$. Construct

$$\begin{aligned}
&\bar{u}(x,t) = A\mathrm{e}^{Kt}\log\big[h_0(x)\mathrm{e}^{K(m-1)t}+M\big],\\
&\bar{v}(x,t) = B\mathrm{e}^{Lt}\big(M+\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)/2}}\big) \quad \text{for (i)},\\
&\bar{v}(x,t) = B\mathrm{e}^{Lt}\big(M+\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)}}\big) \quad \text{for (ii)},
\end{aligned}$$

with positive constants K, L, A, B, M to be determined. In the subcase (i), we have

$$\begin{aligned}(\bar{u}^m)_t=&KmA^m\mathrm{e}^{Kmt}\Big(\log[h_0(x)\mathrm{e}^{K(m-1)t}+M]\Big)^m+mA^m\mathrm{e}^{Kmt}\Big(\log[h_0(x)\mathrm{e}^{K(m-1)t}+M]\Big)^{m-1}\\&\times\frac{K(m-1)h_0(x)\mathrm{e}^{K(m-1)t}}{h_0(x)\mathrm{e}^{K(m-1)t}+M}\geq\frac{1}{2}KmA^m\mathrm{e}^{Kmt}(\log M)^m,\\ \Delta\bar{u}=&\frac{A\mathrm{e}^{Kmt}\Delta h_0}{h_0(x)\mathrm{e}^{K(m-1)t}+M}-\frac{A\mathrm{e}^{Kt}(\nabla[h_0(x)\mathrm{e}^{K(m-1)t}+M])^2}{[h_0(x)\mathrm{e}^{K(m-1)t}+M]^2}\leq\frac{A\lambda_h\mathrm{e}^{Kmt}}{M},\\ a_1\bar{u}^{\alpha_1}=&a_1A^{\alpha_1}\mathrm{e}^{\alpha_1Kt}\Big(\log[h_0(x)\mathrm{e}^{K(m-1)t}+M]\Big)^{\alpha_1}\leq a_1A^m\mathrm{e}^{mKt}\Big(\log[c_5+M]\Big)^{\alpha_1},\end{aligned}$$

provided that $M\log M\geq 2c_5(1-m)$, and hence

$$(\bar{u}^m)_t\geq\Delta\bar{u}+a_1\bar{u}^{\alpha_1},\quad (x,t)\in\Omega\times\mathbb{R}^+ \tag{3.4}$$

with

$$K\geq\frac{2A^{1-m}\lambda_h+2a_1M(\log[c_5+M])^{\alpha_1}}{mM(\log M)^m}.$$

Similarly to the proof of Lemma 3.1,

$$(\bar{v}^n)_t\geq\Delta\bar{v}-a_2\bar{v}^{\beta_1},\quad (x,t)\in\Omega\times\mathbb{R}^+ \tag{3.5}$$

provided that

$$M\geq\max\Big(\frac{\beta_1-1}{2\mathrm{e}},\Big(\frac{\lambda_0+c_1^2}{a_2}\Big)^{\frac{1}{\beta_1}}\Big).$$

Moreover, for $(x,t)\in\partial\Omega\times\mathbb{R}^+$, we have

$$\begin{aligned}&\frac{\partial\bar{u}}{\partial\eta}\geq\frac{A\mathrm{e}^{Kmt}}{M+c_5},\quad\frac{\partial\bar{v}}{\partial\eta}\geq c_2B^{\frac{\beta_1+1}{2}}\mathrm{e}^{\frac{L(\beta_1+1)t}{2}},\\&\bar{u}^{\alpha_2}\bar{v}^p\leq A^{\alpha_2}B^p(\log[c_5+M])^{\alpha_2}(M+1)^p\mathrm{e}^{(K\alpha_2+Lp)t},\\&\bar{u}^q\bar{v}^{\beta_2}\leq A^qB^{\beta_2}(\log[c_5+M])^q(M+1)^{\beta_2}\mathrm{e}^{(Kq+L\beta_2)t}.\end{aligned}$$

Since

$$pq\leq(m-\alpha_2)\Big(\frac{1+\beta_1}{2}-\beta_2\Big)<(1-\alpha_2)\Big(\frac{1+\beta_1}{2}-\beta_2\Big),$$

by taking $L=K(m-\alpha_2)/p$ and

$$\begin{aligned}&\frac{A^{1-\alpha_2}}{M+c_5}>B^p(\log[c_5+M])^{\alpha_2}(M+1)^p,\\&c_2B^{\frac{\beta_1+1}{2}-\beta_2}>A^q(\log[c_5+M])^q(M+1)^{\beta_2},\end{aligned}$$

we can get

$$\frac{\partial \bar{u}}{\partial \eta} \geq \bar{u}^{\alpha_2} \bar{v}^p, \quad \frac{\partial \bar{v}}{\partial \eta} \geq \bar{u}^q \bar{v}^{\beta_2}, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+. \tag{3.6}$$

If A, B are sufficiently large then $\bar{u}(x,0) \geq u_0(x)$, $\bar{v}(x,0) \geq v_0(x)$. Hence $(\bar{u},\bar{v})$ is just a supersolution of (1.1)-(1.3). In the subcase (ii), by virtue of the calculation of subcase (i), we have

$$(\bar{u}^m)_t \geq \Delta\bar{u} - a_1\bar{u}^{\alpha_1}, \quad (x,t) \in \Omega \times \mathbb{R}^+ \tag{3.7}$$

provided $M\log M \geq 2c_5(1-m)$ and

$$K \geq \frac{2A^{1-m}\lambda_h + 2a_1 M(\log[c_5+M])^{\alpha_1}}{mM(\log M)^m}.$$

In addition,

$$\begin{aligned}
(\bar{v}^n)_t =& B^n Ln\mathrm{e}^{Lnt}\big(M+\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)}}\big)^{n-1}\Big(M+\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)}} \\
&-(\beta_1-1)\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)}\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)}}\Big) \geq 0, \\
\Delta\bar{v} =& B^{\beta_1}\mathrm{e}^{L\beta_1 t}\big(\lambda_0\varphi_0 + |\nabla\varphi_0|^2(B\mathrm{e}^{Lt})^{\beta_1-1}\big)\mathrm{e}^{-\varphi_0(B\mathrm{e}^{Lt})^{(\beta_1-1)}} \leq (\lambda_0+c_1^2)B^{\beta_1}\mathrm{e}^{L\beta_1 t}, \\
a_2\bar{v}^{\beta_1} \geq& a_2 B^{\beta_1}M^{\beta_1}\mathrm{e}^{L\beta_1 t},
\end{aligned}$$

and hence

$$(\bar{v}^n)_t \geq \Delta\bar{v} - a_2\bar{v}^{\beta_1}, \quad (x,t) \in \Omega \times \mathbb{R}^+ \tag{3.8}$$

provided $M \geq (\lambda_0+c_1^2)^{\frac{1}{\beta_1}} a_2^{-\frac{1}{\beta_1}}$. On the boundary $\partial\Omega \times \mathbb{R}^+$,

$$\begin{aligned}
&\frac{\partial \bar{u}}{\partial \eta} \geq \frac{A\mathrm{e}^{Kmt}}{M+c_5}, \quad \frac{\partial \bar{v}}{\partial \eta} \geq c_2(B\mathrm{e}^{Lt})^{\beta_1}, \\
&\bar{u}^{\alpha_2}\bar{v}^p \leq A^{\alpha_2}B^p(\log[c_5+M])^{\alpha_2}(M+1)^p\mathrm{e}^{(K\alpha_2+Lp)t}, \\
&\bar{u}^q\bar{v}^{\beta_2} \leq A^q B^{\beta_2}(\log[c_5+M])^q(M+1)^{\beta_2}\mathrm{e}^{(Kq+L\beta_2)t}.
\end{aligned}$$

Since $pq \leq (m-\alpha_2)(\beta_1-\beta_2) < (1-\alpha_2)(\beta_1-\beta_2)$, there exist sufficiently large A, B, K, and $L = K(m-\alpha_2)/p$ such that

$$\frac{\partial \bar{u}}{\partial \eta} \geq \bar{u}^{\alpha_2} \bar{v}^p, \quad \frac{\partial \bar{v}}{\partial \eta} \geq \bar{u}^q \bar{v}^{\beta_2}, \quad (x,t) \in \partial\Omega \times \mathbb{R}^+. \tag{3.9}$$

For A, B large that $\bar{u}(x,0) \geq u_0(x)$ and $\bar{v}(x,0) \geq v_0(x)$, $(\bar{u},\bar{v})$ is a global supersolution of (1.1)-(1.3). □

Lemma 3.3. *Assume* $0<m<1$, $n\geq 1$ *with* $\beta_1>n$, *and the condition* (e) *holds. Then the solutions of* (1.1)-(1.3) *are global.*

Proof. Construct

$$\bar{u}(x,t)=Ae^{Kt}\log\left[h_0(x)e^{K(m-1)t}+M\right],\quad \bar{v}(x,t)=Be^{Lt}\left(M+e^{-\varphi_0(Be^{Lt})^{(\beta_1-1)/2}}\right)$$

with K, L, A, B, $M>0$ to be determined. Similarly to the proof of Lemmas 3.1 and 3.2, it is easy to prove that $(\bar{u},\bar{v})$ can be a supersolution of (1.1)-(1.3). □

Lemma 3.4. *Assume* $0<n<1$, $m\geq 1$ *with* $\beta_1>n$, *and the condition* (e) *holds. Then the solutions of* (1.1)-(1.3) *are global.*

Proof. We know

$$\mu=\frac{1+m}{2},\quad \gamma=n+\Big(\beta_1-n\Big)_+-\Big(\frac{\beta_1-1}{2}\Big)_+$$

with $m\geq 1$, $0<n<1$ by (1.17). There are two subcases for the lemma: (i) $\beta_1>1$ with $pq\leq(\frac{m+1}{2}-\alpha_2)(\frac{1+\beta_1}{2}-\beta_2)$; (ii) $n<\beta_1\leq 1$ with $pq\leq(\frac{m+1}{2}-\alpha_2)(\beta_1-\beta_2)$. Construct

$$\begin{aligned}&\bar{u}(x,t)=e^{Kt}\big(M+A^{-1}e^{-A^2\varphi_0 e^{K(m-1)t/2}}\big),\\ &\bar{v}(x,t)=Be^{Lt}\big(M+e^{-\varphi_0(Be^{Lt})^{(\beta_1-1)/2}}\big)\quad\text{for (i)},\\ &\bar{v}(x,t)=Be^{Lt}\big(M+e^{-\varphi_0(Be^{Lt})^{(\beta_1-1)}}\big)\quad\text{for (ii)}.\end{aligned}$$

By a similar argument as that in the proof of Lemmas 3.1 and 3.2, we can take K, L, A, B, M sufficiently large such that $(\bar{u},\bar{v})$ is a global supersolution of (1.1)-(1.3). □

Lemma 3.5. *Assume* m, $n\geq 1$ *with* $\beta_1\leq n$, *and the condition* (e) *holds. Then the solutions of* (1.1)-(1.3) *are global.*

Proof. We know

$$\mu=\frac{1+m}{2},\quad \gamma=\frac{1+n}{2}+\Big(\frac{\beta_1-n}{2}\Big)_+=\frac{1+n}{2}$$

with m, $n\geq 1$ and $\beta_1\leq n$ by (1.14). Thus $pq\leq(\frac{1+m}{2}-\alpha_2)(\frac{1+n}{2}-\beta_2)$. Construct

$$\bar{u}(x,t)=e^{Kt}\big(M+A^{-1}e^{-A^2\varphi_0 e^{K(m-1)t/2}}\big),\ \bar{v}(x,t)=e^{Lt}\Big(M+B^{-1}e^{-B^2\varphi_0 e^{L(n-1)t/2}}\Big).$$

Similar to the proof of Lemma 3.1, by letting

$$A\geq\frac{m-1}{2e(M-1)},\quad K\geq\frac{A(\lambda_0+A^2c_1^2)+a_1(M+A^{-1})^{\alpha_1}}{m},$$

we can get

$$(\bar{u}^m)_t\geq\Delta\bar{u}+a_1\bar{u}^{\alpha_1},\quad (x,t)\in\Omega\times\mathbb{R}^+. \tag{3.10}$$

Moreover, by taking $B \geq \frac{n-1}{2\mathrm{e}(M-1)}$, we have

$$(\bar{v}^n)_t \geq Ln\mathrm{e}^{Lnt}, \;\; \Delta\bar{v} \leq B(\lambda_0 + B^2 c_1^2)\mathrm{e}^{Lnt},$$

and hence with $L \geq \frac{B(\lambda_0+B^2c_1^2)}{n}$ we have

$$(\bar{v}^n)_t \geq \Delta\bar{v} - a_2\bar{v}^{\beta_1}, \quad (x,t) \in \Omega\times\mathbb{R}^+. \tag{3.11}$$

Furthermore, on the boundary $\partial\Omega$

$$\begin{aligned}
&\frac{\partial\bar{u}}{\partial\eta} \geq c_2 A\mathrm{e}^{K(m+1)t/2}, \;\; \bar{u}^{\alpha_2}\bar{v}^p \leq (M+1)^{p+\alpha_2}\mathrm{e}^{K\alpha_2 t+Lpt},\\
&\frac{\partial\bar{v}}{\partial\eta} \geq c_2 B\mathrm{e}^{L(n+1)t/2}, \;\; \bar{u}^q\bar{v}^{\beta_2} \leq (M+1)^{q+\beta_2}\mathrm{e}^{Kqt+L\beta_2 t}.
\end{aligned}$$

Since $pq \leq (\frac{1+m}{2}-\alpha_2)(\frac{1+n}{2}-\beta_2)$, we can choose

$$L = \frac{K(\frac{m+1}{2}-\alpha_2)}{p}, \quad B \geq \frac{(M+1)^{q+\beta_2}}{c_2}, \quad A \geq \frac{(M+1)^{p+\alpha_2}}{c_2}$$

to get

$$\frac{\partial\bar{u}}{\partial\eta} \geq \bar{u}^{\alpha_2}\bar{v}^p, \;\; \frac{\partial\bar{v}}{\partial\eta} \geq \bar{u}^q\bar{v}^{\beta_2}, \quad (x,t)\in\partial\Omega\times\mathbb{R}^+. \tag{3.12}$$

It is easy to see that $\bar{u}(x,0) \geq u_0(x)$, $\bar{v}(x,0) \geq v_0(x)$ with $M \geq \max(\|u_0\|_\infty, \|v_0\|_\infty)$. Thus, $(\bar{u},\bar{v})$ is a global supersolution of (1.1)-(1.3). ☐

Lemma 3.6. *Assume* $0<m$, $n<1$ *with* $\beta_1 \leq n$, *and the condition* (e) *holds. Then the solutions of* (1.1)-(1.3) *are global.*

Proof. We have

$$\mu = m, \quad \gamma = n + \Big(\beta_1 - n\Big)_+ - \Big(\frac{\beta_1-1}{2}\Big)_+ = n$$

with $0<m$, $n<1$ and $\beta_1\leq n$ by (1.15). Thus $pq \leq (m-\alpha_2)(n-\beta_2) < (1-\alpha_2)(1-\beta_2)$ for the lemma. Construct

$$\bar{u}(x,t) = A\mathrm{e}^{Kt}\log\big[h_0(x)\mathrm{e}^{K(m-1)t}+M\big], \; \bar{v}(x,t) = B\mathrm{e}^{Lt}\log\big[h_0(x)\mathrm{e}^{L(n-1)t}+M\big],$$

where

$$\begin{aligned}
&M\log M = \max\{2c_5(1-m), 2c_5(1-n)\},\\
&K \geq \frac{2A^{1-m}\lambda_h + 2a_1 M(\log[c_5+M])^{\alpha_1}}{mM(\log M)^m}, \;\; L \geq \frac{2B^{1-n}\lambda_h}{nM(\log M)^n}.
\end{aligned}$$

By a simple computation, we can get

$$(\bar{u}^m)_t \geq \Delta\bar{u} + a_1\bar{u}^{\alpha_1}, \quad (\bar{v}^n)_t \geq \Delta\bar{v} - a_2\bar{v}^{\beta_1}, \quad (x,t)\in\Omega\times\mathbb{R}^+. \tag{3.13}$$

Moreover, for $(x,t)\in\partial\Omega\times\mathbb{R}^+$,

$$\begin{aligned}
&\frac{\partial\bar{u}}{\partial\eta} \geq \frac{A\mathrm{e}^{Kmt}}{M+c_5}, \quad \frac{\partial\bar{v}}{\partial\eta} \geq \frac{B\mathrm{e}^{Lnt}}{M+c_5},\\
&\bar{u}^{\alpha_2}\bar{v}^p \leq A^{\alpha_2}B^p(\log[c_5+M])^{\alpha_2+p}\mathrm{e}^{(K\alpha_2+Lp)t},\\
&\bar{u}^q\bar{v}^{\beta_2} \leq A^qB^{\beta_2}(\log[c_5+M])^{q+\beta_2}\mathrm{e}^{(Kq+L\beta_2)t}.
\end{aligned}$$

Since $pq\leq(m-\alpha_2)(n-\beta_2)<(1-\alpha_2)(1-\beta_2)$, there exist sufficiently large A, B, K and $L=K(m-\alpha_2)/p$ such that

$$\Big((M+c_5)(\log[c_5+M])^{\alpha_2+p}\Big)^{\frac{1}{1-\alpha_2}}B^{\frac{p}{1-\alpha_2}} < A < \Big((c_5+M)(\log[c_5+M])^{q+\beta_2}\Big)^{-\frac{1}{q}}B^{\frac{1-\beta_2}{q}},$$

and $Ln\geq Kq+\beta_2L$. Therefore

$$\frac{\partial\bar{u}}{\partial\eta} \geq \bar{u}^{\alpha_2}\bar{v}^p, \quad \frac{\partial\bar{v}}{\partial\eta} \geq \bar{u}^q\bar{v}^{\beta_2}, \quad (x,t)\in\partial\Omega\times\mathbb{R}^+. \tag{3.14}$$

With A, B large that $\bar{u}(x,0)\geq u_0(x)$, $\bar{v}(x,0)\geq v_0(x)$, $(\bar{u},\bar{v})$ is a global supersolution of (1.1)-(1.3). □

Lemma 3.7. *Assume* $0<m<1$, $n\geq1$ *with* $\beta_1\leq n$, *and the condition* (e) *holds. Then the solutions of* (1.1)-(1.3) *are global.*

Proof. By (1.16), we have $pq\leq(m-\alpha_2)(\frac{n+1}{2}-\beta_2)<(1-\alpha_2)(\frac{n+1}{2}-\beta_2)$ here. Construct

$$\bar{u}(x,t)=A\mathrm{e}^{Kt}\log\big[h_0(x)\mathrm{e}^{K(m-1)t}+M\big],\ \bar{v}(x,t)=\mathrm{e}^{Lt}\Big(M+B^{-1}\mathrm{e}^{-B^2\varphi_0\mathrm{e}^{L(n-1)t/2}}\Big).$$

By using the arguments for Lemmas 3.2 and 3.5, we can obtain with M, K, L large that $(\bar{u},\bar{v})$ is a global supersolution of (1.1)-(1.3). □

Lemma 3.8. *Assume* $0<n<1$, $m\geq1$ *with* $\beta_1\leq n$, *and the condition* (e) *holds. Then the solutions of* (1.1)-(1.3) *are global.*

Proof. By (1.17), we know $pq\leq(\frac{m+1}{2}-\alpha_2)(n-\beta_2)<(\frac{m+1}{2}-\alpha_2)(1-\beta_2)$. Construct

$$\bar{u}(x,t)=\mathrm{e}^{Kt}\big(M+A^{-1}\mathrm{e}^{-A^2\varphi_0\mathrm{e}^{K(m-1)t/2}}\big),\ \bar{v}(x,t)=B\mathrm{e}^{Lt}\log\big[h_0(x)\mathrm{e}^{L(n-1)t}+M\big].$$

Similar to the arguments used for the proofs of Lemmas 3.5 and 3.6, we can take K, L, A, B, M sufficiently large such that $(\bar{u},\bar{v})$ is a global supersolution of (1.1)-(1.3). □

Combining Lemmas 3.1-3.8 yields the conclusion of Theorem 1.3.

4 Critical situation

Finally, let us treat the critical case of $(1/\rho_1, 1/\rho_2) = (0,0)$ in Theorem 1.4. In fact, the conclusions are included in Theorems 1.2 and 1.3 already:

Proof of Theorem 1.4. Notice that $(1/\rho_1, 1/\rho_2) = (0,0)$ is equivalent to $pq = (\mu - \alpha_2)(\gamma - \beta_2)$, which with $pq > 0$ implies either (i) $\mu - \alpha_2$, $\gamma - \beta_2 < 0$, or (ii) $\mu - \alpha_2$, $\gamma - \beta_2 > 0$. It is easy to find that the case of $\alpha_2 > \mu$, $\beta_2 > \gamma$ is included in the case (a) of the proof for Theorem 2 (treated by Lemma 2.1), while the case of $\alpha_2 < \mu$, $\beta_2 < \gamma$ with $pq = (\mu - \alpha_2)(\gamma - \beta_2)$ is covered by (e) in Section 3, which is assumed in Lemmas 3.1-3.8 for proving Theorem 1.3. □

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