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Gradient Estimates for a Nonlinear Diffusion Equation on Complete Manifolds

WU Jiayong*

Department of Mathematics, East China Normal University, Shanghai 200241, China.

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Abstract. Let (M,g) be a complete non-compact Riemannian manifold with the m-dimensional Bakry-Émery Ricci curvature bounded below by a non-positive constant. In this paper, we give a localized Hamilton-type gradient estimate for the positive smooth bounded solutions to the following nonlinear diffusion equation

$$u_t = \Delta u - \nabla\phi\cdot\nabla u - au\log u - bu,$$

where ϕ is a C^2 function, and $a \neq 0$ and b are two real constants. This work generalizes the results of Souplet and Zhang (Bull. London Math. Soc., 38 (2006), pp. 1045-1053) and Wu (Preprint, 2008).

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1 Introduction

Let (M,g) be an n-dimensional non-compact Riemannian manifold with the m-dimensional Bakry-Émery Ricci curvature bounded below. Consider the following diffusion equation:

$$u_t = \Delta u - \nabla\phi\cdot\nabla u - au\log u - bu \tag{1.1}$$

in $B(x_0,R)\times[t_0-T,t_0]\subset M\times(-\infty,\infty)$, where ϕ is a C^2 function, and $a\neq 0$ and b are two real constants. Eq. (1.1) is closely linked with the gradient Ricci solitons, which are the self-similar solutions to the Ricci flow introduced by Hamilton [3]. Ricci solitons have inspired the entropy and Harnack estimates, the space-time formulation of the Ricci flow, and the reduced distance and reduced volume.

Below we recall the definition of Ricci solitons (see also Chapter 4 of [4]).

*Corresponding author. *Email address:* jywu81@yahoo.com (J. Wu)

Definition 1.1. *A Riemannian manifold* (M,g) *is called a* gradient Ricci soliton *if there exists a smooth function* $f: M \to \mathbb{R}$, *sometimes called* potential function, *such that for some constant* $c \in \mathbb{R}$, *it satisfies*

$$Ric(g) + \nabla^g \nabla^g f = cg \tag{1.2}$$

on M, *where* $Ric(g)$ *is the Ricci curvature of manifold* M *and* $\nabla^g \nabla^g f$ *is the Hessian of* f. *A soliton is said to be* shrinking, steady *or* expanding *if the constant* c *is respectively positive, zero or negative.*

Suppose that (M,g) be a gradient Ricci soliton, and c, f are described in Definition A. Letting $u = e^f$, under some curvature assumptions, we can derive from (1.2) that (cf. [5], Eq. (7))

$$\Delta u + 2cu\log u = (A_0 - nc)u, \tag{1.3}$$

for some constant A_0. Eq. (1.3) is a nonlinear elliptic equation and a special case of Eq. (1.1). For this kind of equations, Ma (see Theorem 1 in [5]) obtained the following result.

Theorem A. ([5]) *Let* (M,g) *be a complete non-compact Riemannian manifold of dimension* $n \geq 3$ *with Ricci curvature bounded below by the constant* $-K := -K(2R)$, *where* $R > 0$ *and* $K(2R) \geq 0$, *in the metric ball* $B_{2R}(p)$. *Let* u *be a positive smooth solution to the elliptic equation*

$$\Delta u - au\log u = 0 \tag{1.4}$$

with $a > 0$. *Let* $f = \log u$ *and let* $(f, 2f)$ *be the maximum among* f *and* $2f$. *Then there are two uniform positive constant* c_1 *and* c_2 *such that*

$$\begin{aligned} &|\nabla f|^2 - a(f,2f) \\ &\leq \frac{n\left[(n+2)c_1^2 + (n-1)c_1^2(1+R\sqrt{K}) + c_2\right]}{R^2} + 2n\Big(|a| + K\Big) \end{aligned} \tag{1.5}$$

in $B_R(p)$.

Then Yang (see Theorem 1.1 in [6]) extended the above result and obtained the following local gradient estimate for the nonlinear equation (1.1) with $\phi \equiv c_0$, where c_0 is a fixed constant.

Theorem B. ([6]) *Let* M *be an* n*-dimensional complete non-compact Riemannian Manifold. Suppose the Ricci curvature of* M *is bounded below by* $-K := -K(2R)$, *where* $R > 0$ *and* $K(2R) \geq 0$, *in the metric ball* $B_{2R}(p)$. *If* u *is a positive smooth solution to Eq.* (1.1) *with* $\phi \equiv c_0$ *on* $M \times [0,\infty)$

and $f=\log u$, then for any $\alpha>1$ and $0<\delta<1$,

$$\begin{aligned}&|\nabla f|^2(x,t)-\alpha a f(x,t)-\alpha b-\alpha f_t(x,t)\\ \le&\frac{n\alpha^2}{2\delta t}+\frac{n\alpha^2}{2\delta}\left\{\frac{2\epsilon^2}{R^2}+\frac{\nu}{R^2}+\sigma+\frac{\epsilon^2}{R^2}(n-1)\left(1+R\sqrt{K(2R)}\right)\right.\\ &\left.+\frac{K(2R)}{\alpha-1}+\frac{n\alpha^2\epsilon^2}{8(1-\delta)(\alpha-1)R^2}\right\} \qquad (1.6)\end{aligned}$$

in $B_R(p)\times(0,\infty)$, where $\epsilon>0$ and $\nu>0$ are some constants and where $\sigma=a/2$ if $a>0$; $\sigma=-a$ if $a<0$.

Recently, the author (see Theorem 1.1 in [2]) used Souplet-Zhang's method in [1] and obtained a localized Hamilton-type gradient estimate for the positive smooth bounded solutions of the equation (1.1) with $\phi\equiv c_0$.

Theorem C. ([2]) *Let (M,g) be an n-dimensional non-compact Riemannian manifold with $Ric(M)\ge -K$ for some constant $K\ge 0$. Suppose that $u(x,t)$ is a positive smooth solution to the parabolic equation (1.1) with $\phi\equiv c_0$ in $Q_{R,T}\equiv B(x_0,R)\times[t_0-T,t_0]\subset M\times(-\infty,\infty)$. Let $f:=\log u$. We also assume that there exists non-negative constants α and δ such that $\alpha-f\ge\delta>0$. Then there exist three dimensional constants $\tilde c$, $c(\delta)$ and $c(\alpha,\delta)$ such that*

$$\frac{|\nabla u|}{u}\le\left(\frac{\tilde c}{R}\beta+\frac{c(\alpha,\delta)}{R}+\frac{c(\delta)}{\sqrt{T}}+c(\delta)\left(|a|+K\right)^{1/2}+c(\delta)|a|^{1/2}\beta^{1/2}\right)\left(\alpha-\frac{b}{a}-\log u\right) \qquad (1.7)$$

in $Q_{R/2,T/2}$, where $\beta:=\max\{1,|\alpha/\delta-1|\}$.

The purpose of this paper is to extend Theorem C to the general nonlinear diffusion equation (1.1) via the m-dimensional Bakry-Émery Ricci curvature.

Let us first recall some facts about the m-dimensional Bakry-Émery Ricci curvature (please see [7–10] for more details). Given an n-dimensional Riemannian manifold (M,g) and a C^2 function ϕ, we may define a symmetric diffusion operator $L:=\Delta-\nabla\phi\cdot\nabla$, which is the infinitesimal generator of the Dirichlet form

$$\mathcal{E}(f,g)=\int_M(\nabla f,\nabla g)\mathrm{d}\mu,\ \forall f,g\in C_0^\infty(M),$$

where μ is an invariant measure of L given by $\mathrm{d}\mu=e^{-\phi}\mathrm{d}x$. It is well-known that L is self-adjoint with respect to the weighted measure $\mathrm{d}\mu$.

The ∞-dimensional Bakry-Émery Ricci curvature $Ric(L)$ is defined by

$$Ric(L):=Ric+Hess(\phi),$$

where Ric and $Hess$ denote the Ricci curvature of the metric g and the Hessian respectively. Following the notation used in [10], we also define the m-dimensional Bakry-Émery Ricci curvature of L on an n-dimensional Riemannian manifold as follows

$$Ric_{m,n}(L):=Ric(L)-\frac{\nabla\phi\otimes\nabla\phi}{m-n},$$

where $m := \dim_{BE}(L)$ is called the Bakry-Émery dimension of L. Note that the number m is not necessarily to be an integer and $m \geq n = \dim M$.

The main result of this paper can be stated in the following:

Theorem 1.1. *Let (M,g) be an n-dimensional non-compact Riemannian manifold with $Ric_{m,n}(L) \geq -K$ for some constant $K \geq 0$. Suppose that $u(x,t)$ is a positive smooth solution to the diffusion equation (1.1) in $Q_{R,T} \equiv B(x_0,R) \times [t_0 - T, t_0] \subset M \times (-\infty,\infty)$. Let $f := \log u$. We also assume that there exists non-negative constants α and δ such that $\alpha - f \geq \delta > 0$. Then there exist three dimensional constants $\tilde{c}$, $c(\delta)$ and $c(\alpha,\delta,m)$ such that*

$$\frac{|\nabla u|}{u} \leq \left(\frac{\tilde{c}}{R}\beta + \frac{c(\alpha,\delta,m)}{R} + \frac{c(\delta)}{\sqrt{T}} + c(\delta)\left(|a|+K\right)^{1/2} + c(\delta)|a|^{1/2}\beta^{1/2} \right) \left(\alpha - \frac{b}{a} - \log u \right) \tag{1.8}$$

in $Q_{R/2,T/2}$, where $\beta := \max\{1, |\alpha/\delta - 1|\}$.

We make some remarks on the above theorem below.

Remark 1.1. (i). In Theorem 1.1, it seems that the assumption $\alpha - f \geq \delta > 0$ is reasonable. Because from this assumption, we can get $u \leq e^{\alpha-\delta}$. We say that this upper bound of u can be achieved in some setting. For example, from Corollary 1.2 in [6], we know that positive smooth solutions to the elliptic equation (1.4) with $a < 0$ have $u(x) \leq e^{n/2}$ for all $x \in M$ provided the Ricci curvature of M is non-negative.

(ii). Note that the theorem still holds if m-dimensional Bakry-Émery Ricci curvature is replaced by ∞-dimensional Bakry-Émery Ricci curvature. In fact this result can be obtained by (2.10) in Section 2.

(iii). Theorem 1.1 generalizes the above mentioned Theorem C. When we choose $\phi \equiv c_0$, we return Theorem C. The proof of our main theorem is based on Souplet-Zhang's gradient estimate and the trick used in [2] with some modifications.

In particular, if $u(x,t) \leq 1$ is a positive smooth solution to the diffusion equation (1.1) with $a < 0$, then we have a simple estimate.

Corollary 1.1. *Let (M,g) be an n-dimensional non-compact Riemannian manifold with $Ric_{m,n}(L) \geq -K$ for some constant $K \geq 0$. Suppose that $u(x,t) \leq 1$ is a positive smooth solution to the diffusion equation (1.1) with $a < 0$ in $Q_{R,T} \equiv B(x_0,R) \times [t_0 - T, t_0] \subset M \times (-\infty,\infty)$. Then there exist two dimensional constants c and $c(m)$ such that*

$$\frac{|\nabla u|}{u} \leq \left(\frac{c(m)}{R} + \frac{c}{\sqrt{T}} + c\sqrt{K+|a|} \right) \left(1 - \frac{b}{a} + \log \frac{1}{u} \right) \tag{1.9}$$

in $Q_{R/2,T/2}$.

Remark 1.2. We point out that our localized Hamilton-type gradient estimate can be also regarded as the generalization of the result of Souplet-Zhang [1] for the heat equation on complete manifolds. In fact, the above Corollary 1.1 is similar to the result of Souplet-Zhang (see Theorem 1.1 of [1]). From the inequality (4.4) below, we can conclude that if $\phi \equiv c_0$ and $a = 0$, then our result can be reduced to theirs.

The method of proving Theorem 1.1 is the gradient estimate, which is originated by Yau [11] (see also Cheng-Yau [12]), and developed further by Li-Yau [13], Li [14] and Negrin [15]. Then Hamilton [16] gave an elliptic type gradient estimate for the heat equation. But this type estimate is a global result which requires the heat equation defined on closed manifolds. Recently, a localized Hamilton-type gradient estimate was proved by Souplet and Zhang [1], which can be viewed as a combination of Li-Yau's Harnack inequality [13] and Hamilton's gradient estimate [16]. In this paper, we obtain a localized Hamilton-type gradient estimate for a general diffusion equation (1.1) as Souplet and Zhang in [1] did for the heat equation on complete manifolds. To prove Theorem 1.1, we mainly follow the arguments of Souplet-Zhang in [1], together with some facts about Bakry-Émery Ricci curvature. Note that the diffusion equation (1.1) is nonlinear. So our case is a little more complicated than theirs.

The structure of this paper is as follows. In Section 2, we will give a basic lemma to prepare for proving Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we will prove Corollary 1.1 in the case $0<u\leq 1$ with $a<0$.

2 A basic lemma

In this section, we will prove the following lemma which is essential in the derivation of the gradient estimate of Eq. (1.1). Replacing u by $e^{-b/a}u$, we only need to consider positive smooth solutions of the following diffusion equation:

$$u_t=\Delta u-\nabla\phi\cdot\nabla u-au\log u. \tag{2.1}$$

Suppose that $u(x,t)$ is a positive smooth solution to the diffusion equation (1.1) in $Q_{R,T}\equiv B(x_0,R)\times[t_0-T,t_0]$. Define a smooth function

$$f(x,t):=\log u(x,t)$$

in $Q_{R,T}$. By (2.1), we have

$$\left(L-\frac{\partial}{\partial t}\right)f+|\nabla f|^2-af=0. \tag{2.2}$$

Then we have the following lemma, which is a generalization of the computation carried out in [1,2].

Lemma 2.1. *Let (M,g) be an n-dimensional non-compact Riemannian manifold with $Ric_{m,n}(L)\geq -K$ for some constant $K\geq 0$. Let $f(x,t)$ is a smooth function defined on $Q_{R,T}$ satisfying the diffusion equation* (2.2)*. We also assume that there exist non-negative constants α and δ such that $\alpha-f\geq\delta>0$. Then for all (x,t) in $Q_{R,T}$ the function*

$$\omega:=|\nabla\log(\alpha-f)|^2=\frac{|\nabla f|^2}{(\alpha-f)^2} \tag{2.3}$$

satisfies the following inequality

$$\left(L-\frac{\partial}{\partial t}\right)\omega \geq \frac{2(1-\alpha)+2f}{\alpha-f}\langle\nabla f,\nabla\omega\rangle+2(\alpha-f)\omega^2+2(a-K)\omega+\frac{2af}{\alpha-f}\omega. \tag{2.4}$$

Proof. By (2.3), we have

$$\omega_j=\frac{2f_if_{ij}}{(\alpha-f)^2}+\frac{2f_i^2f_j}{(\alpha-f)^3}, \tag{2.5}$$

$$\Delta\omega=\frac{2|f_{ij}|^2}{(\alpha-f)^2}+\frac{2f_if_{ijj}}{(\alpha-f)^2}+\frac{8f_if_jf_{ij}}{(\alpha-f)^3}+\frac{2f_i^2f_{jj}}{(\alpha-f)^3}+\frac{6f_i^2f_j^2}{(\alpha-f)^4}, \tag{2.6}$$

and

$$\begin{aligned}L\omega&=\Delta\omega-\phi_j\omega_j\\&=\frac{2|f_{ij}|^2}{(\alpha-f)^2}+\frac{2f_if_{ijj}}{(\alpha-f)^2}+\frac{8f_if_jf_{ij}}{(\alpha-f)^3}+\frac{2f_i^2f_{jj}}{(\alpha-f)^3}+\frac{6f_i^4}{(\alpha-f)^4}-\frac{2f_{ij}f_i\phi_j}{(\alpha-f)^2}-\frac{2f_i^2f_j\phi_j}{(\alpha-f)^3}\\&=\frac{2|f_{ij}|^2}{(\alpha-f)^2}+\frac{2f_i(Lf)_i}{(\alpha-f)^2}+\frac{2(R_{ij}+\phi_{ij})f_if_j}{(\alpha-f)^2}+\frac{8f_if_jf_{ij}}{(\alpha-f)^3}+\frac{2f_i^2\cdot Lf}{(\alpha-f)^3}+\frac{6f_i^4}{(\alpha-f)^4},\end{aligned} \tag{2.7}$$

where $f_i:=\nabla_i f$ and $f_{ijj}:=\nabla_j\nabla_j\nabla_i f$, etc. By (2.3) and (2.2), we also have

$$\begin{aligned}\omega_t&=\frac{2\nabla_i f\cdot\nabla_i\left[Lf+|\nabla f|^2-af\right]}{(\alpha-f)^2}+\frac{2|\nabla f|^2\left[Lf+|\nabla f|^2-af\right]}{(\alpha-f)^3}\\&=\frac{2\nabla f\nabla Lf}{(\alpha-f)^2}+\frac{4f_if_jf_{ij}}{(\alpha-f)^2}-\frac{2a|\nabla f|^2}{(\alpha-f)^2}+\frac{2f_i^2Lf}{(\alpha-f)^3}+\frac{2|\nabla f|^4}{(\alpha-f)^3}-\frac{2af|\nabla f|^2}{(\alpha-f)^3}.\end{aligned} \tag{2.8}$$

Combining (2.7) with (2.8), we can get

$$\begin{aligned}\left(L-\frac{\partial}{\partial t}\right)\omega=&\frac{2|f_{ij}|^2}{(\alpha-f)^2}+\frac{2(R_{ij}+\phi_{ij})f_if_j}{(\alpha-f)^2}+\frac{8f_if_jf_{ij}}{(\alpha-f)^3}+\frac{6f_i^4}{(\alpha-f)^4}\\&-\frac{4f_if_jf_{ij}}{(\alpha-f)^2}-\frac{2f_i^4}{(\alpha-f)^3}+\frac{2af_i^2}{(\alpha-f)^2}+\frac{2aff_i^2}{(\alpha-f)^3}.\end{aligned} \tag{2.9}$$

Noting that $Ric_{m,n}(L)\geq -K$ for some constant $K\geq 0$, we have

$$(R_{ij}+\phi_{ij})f_if_j\geq\frac{|\nabla\phi\cdot\nabla f|^2}{m-n}-K|\nabla f|^2\geq -K|\nabla f|^2. \tag{2.10}$$

By (2.5), we have

$$\omega_jf_j=\frac{2f_if_jf_{ij}}{(\alpha-f)^2}+\frac{2f_i^2f_j^2}{(\alpha-f)^3}, \tag{2.11}$$

and consequently,

$$0=-2\omega_j f_j+\frac{4f_i f_j f_{ij}}{(\alpha-f)^2}+\frac{4f_i^4}{(\alpha-f)^3}, \tag{2.12}$$

$$0=\frac{1}{\alpha-f}\left[2\omega_j f_j-\frac{4f_i^4}{(\alpha-f)^3}\right]-\frac{4f_i f_j f_{ij}}{(\alpha-f)^3}. \tag{2.13}$$

Substituting (2.10) into (2.9) and then adding (2.9) with (2.12) and (2.13), we can get

$$\begin{aligned}\left(L-\frac{\partial}{\partial t}\right)\omega\geq&\frac{2|f_{ij}|^2}{(\alpha-f)^2}-\frac{2K|\nabla f|^2}{(\alpha-f)^2}+\frac{4f_i f_j f_{ij}}{(\alpha-f)^3}+\frac{2f_i^4}{(\alpha-f)^4}+\frac{2f_i^4}{(\alpha-f)^3}\\&+\frac{2(1-\alpha)+2f}{\alpha-f}f_i\omega_i+\frac{2af_i^2}{(\alpha-f)^2}+\frac{2aff_i^2}{(\alpha-f)^3}.\end{aligned} \tag{2.14}$$

Note that $\alpha-f\geq\delta>0$ implies

$$\frac{2|f_{ij}|^2}{(\alpha-f)^2}+\frac{4f_i f_j f_{ij}}{(\alpha-f)^3}+\frac{2f_i^4}{(\alpha-f)^4}\geq 0.$$

This, together with (2.14), yields the desired estimate (2.4). □

3 Proof of Theorem 1.1

In this section, we will use Lemma 2.1 and the localization technique of Souplet-Zhang [1] to give the elliptic type gradient estimates on the positive and bounded smooth solutions of the diffusion equation (1.1).

Proof. First we give the well-known cut-off function by Li-Yau [13] (see also [1]) as follows. We caution the reader that the calculation is not the same as that in [13] due to the difference of the first-order term.

Let $\psi=\psi(x,t)$ be a smooth cut-off function supported in $Q_{R,T}$ satisfying the following properties:

(1) $\psi=\psi(d(x,x_0),t)\equiv\psi(r,t)$; $\psi(x,t)=1$ in $Q_{R/2,T/2}$, $0\leq\psi\leq 1$;

(2) ψ is decreasing as a radial function in the spatial variables;

(3) $\dfrac{|\partial_r\psi|}{\psi^\epsilon}\leq\dfrac{C_\epsilon}{R}$, $\dfrac{|\partial_r^2\psi|}{\psi^\epsilon}\leq\dfrac{C_\epsilon}{R^2}$, when $0<\epsilon<1$;

(4) $\dfrac{|\partial_t\psi|}{\psi^{1/2}}\leq\dfrac{C}{T}$.

From Lemma 2.1, by a straight forward calculation, we have

$$\begin{aligned}
&L(\psi\omega)-\frac{2(1-\alpha)+2f}{\alpha-f}\nabla f\cdot\nabla(\psi\omega)-2\frac{\nabla\psi}{\psi}\cdot\nabla(\psi\omega)-(\psi\omega)_t\\
\geq&\, 2\psi(\alpha-f)\omega^2-\left[\frac{2(1-\alpha)+2f}{\alpha-f}\nabla f\cdot\nabla\psi\right]\omega-2\frac{|\nabla\psi|^2}{\psi}\omega\\
&+(L\psi)\omega-\psi_t\omega+2(a-K)\psi\omega+2\frac{af}{\alpha-f}\psi\omega. \qquad (3.1)
\end{aligned}$$

Let (x_1,t_1) be a point where $\psi\omega$ achieves the maximum. By Li-Yau [13], without loss of generality we assume that x_1 is not in the cut-locus of M. Then at this point, we have

$$L(\psi\omega)\leq 0,\quad (\psi\omega)_t\geq 0,\quad \nabla(\psi\omega)=0.$$

Hence at (x_1,t_1), by (3.1), we get

$$\begin{aligned}
2\psi(\alpha-f)\omega^2(x_1,t_1)\leq&\left\{\left[\frac{2(1-\alpha)+2f}{\alpha-f}\nabla f\cdot\nabla\psi\right]\omega+2\frac{|\nabla\psi|^2}{\psi}\omega-(L\psi)\omega\right.\\
&\left.+\psi_t\omega-2(a-K)\psi\omega-2\frac{af}{\alpha-f}\psi\omega\right\}(x_1,t_1). \qquad (3.2)
\end{aligned}$$

In the following, we will introduce the upper bounds for each term of the right-hand side (RHS) of (3.2). Following similar arguments of Souplet-Zhang ([1], p. 1050-1051), we have the estimates of the first term of the RHS of (3.2)

$$\begin{aligned}
&\left[\frac{2f}{\alpha-f}\nabla f\cdot\nabla\psi\right]\omega\\
\leq&\, 2|f|\cdot|\nabla\psi|\cdot\omega^{3/2}=2\left[\psi(\alpha-f)\omega^2\right]^{3/4}\cdot\frac{|f|\cdot|\nabla\psi|}{[\psi(\alpha-f)]^{3/4}}\\
\leq&\,\psi(\alpha-f)\omega^2+\tilde{c}\frac{(f|\nabla\psi|)^4}{[\psi(\alpha-f)]^3}\leq\psi(\alpha-f)\omega^2+\tilde{c}\frac{f^4}{R^4(\alpha-f)^3}; \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
&\left[\frac{2(1-\alpha)}{\alpha-f}\nabla f\cdot\nabla\psi\right]\omega\\
\leq&\, 2|1-\alpha||\nabla\psi|\omega^{3/2}=(\psi\omega^2)^{3/4}\cdot\frac{2|1-\alpha||\nabla\psi|}{\psi^{3/4}}\\
\leq&\,\frac{\delta}{12}\psi\omega^2+c(\alpha,\delta)\left(\frac{|\nabla\psi|}{\psi^{3/4}}\right)^4\leq\frac{\delta}{12}\psi\omega^2+\frac{c(\alpha,\delta)}{R^4}. \qquad (3.4)
\end{aligned}$$

For the second term of the RHS of (3.2), we have

$$2\frac{|\nabla\psi|^2}{\psi}\omega=2\psi^{1/2}\omega\cdot\frac{|\nabla\psi|^2}{\psi^{3/2}}\le\frac{\delta}{12}\psi\omega^2+c(\delta)\left(\frac{|\nabla\psi|^2}{\psi^{3/2}}\right)^2$$
$$\le\frac{\delta}{12}\psi\omega^2+\frac{c(\delta)}{R^4}. \tag{3.5}$$

For the third term of the RHS of (3.2), since $Ric_{m,n}(L)\ge -K$, by the generalized Laplacian comparison theorem (see [9] or [10]),

$$Lr\le(m-1)\sqrt{K}\coth(\sqrt{K}r).$$

Consequently, we have

$$\begin{aligned}-(L\psi)\omega=&-\left[(\partial_r\psi)Lr+(\partial_r^2\psi)\cdot|\nabla r|^2\right]\omega\\
\le&-\left[\partial_r\psi(m-1)\sqrt{K}\coth(\sqrt{K}r)+\partial_r^2\psi\right]\omega\\
\le&-\left[\partial_r\psi(m-1)\left(\frac{1}{r}+\sqrt{K}\right)+\partial_r^2\psi\right]\omega\\
\le&\left[|\partial_r^2\psi|+2(m-1)\frac{|\partial_r\psi|}{R}+(m-1)\sqrt{K}|\partial_r\psi|\right]\omega\\
\le&\psi^{1/2}\omega\frac{|\partial_r^2\psi|}{\psi^{1/2}}+\psi^{1/2}\omega 2(m-1)\frac{|\partial_r\psi|}{R\psi^{1/2}}+\psi^{1/2}\omega(m-1)\frac{\sqrt{K}|\partial_r\psi|}{\psi^{1/2}}\\
\le&\frac{\delta}{12}\psi\omega^2+c(\delta,m)\left[\left(\frac{|\partial_r^2\psi|}{\psi^{1/2}}\right)^2+\left(\frac{|\partial_r\psi|}{R\psi^{1/2}}\right)^2+\left(\frac{\sqrt{K}|\partial_r\psi|}{\psi^{1/2}}\right)^2\right]\\
\le&\frac{\delta}{12}\psi\omega^2+\frac{c(\delta,m)}{R^4}+\frac{c(\delta,m)K}{R^2}.\end{aligned} \tag{3.6}$$

Now we estimate the fourth term:

$$|\psi_t|\omega=\psi^{1/2}\omega\frac{|\psi_t|}{\psi^{1/2}}\le\frac{\delta}{12}\left(\psi^{1/2}\omega\right)^2+c(\delta)\left(\frac{|\psi_t|}{\psi^{1/2}}\right)^2$$
$$\le\frac{\delta}{12}\psi\omega^2+\frac{c(\delta)}{T^2}. \tag{3.7}$$

Notice that we have used Young's inequality below in obtaining (3.3)-(3.7):

$$ab\le\frac{a^p}{p}+\frac{b^q}{q},\quad\forall\, p,q>0\quad\text{with}\ \frac{1}{p}+\frac{1}{q}=1.$$

Finally, we estimate the last two terms:

$$-2(a-K)\psi\omega\le 2(|a|+K)\psi\omega\le\frac{\delta}{12}\psi\omega^2+c(\delta)(|a|+K)^2; \tag{3.8}$$

and

$$-2\frac{af}{\alpha-f}\psi\omega\le 2\frac{|a|\cdot|f|}{\alpha-f}\psi\omega\le\frac{\delta}{12}\psi\omega^2+c(\delta)a^2\frac{f^2}{(\alpha-f)^2}. \tag{3.9}$$

Substituting (3.3)-(3.9) to the RHS of (3.2) at (x_1,t_1), we get

$$\begin{aligned}2\psi(\alpha-f)\omega^2\le\psi(\alpha-f)\omega^2+\tilde{c}\frac{f^4}{R^4(\alpha-f)^3}+\frac{\delta}{2}\psi\omega^2+\frac{c(\alpha,\delta)}{R^4}+\frac{c(\delta)}{R^4}+\frac{c(\delta,m)}{R^4}\\+\frac{c(\delta,m)K}{R^2}+\frac{c(\delta)}{T^2}+c(\delta)(|a|+K)^2+c(\delta)a^2\frac{f^2}{(\alpha-f)^2}.\end{aligned} \tag{3.10}$$

Recall that $\alpha-f\ge\delta>0$, (3.10) implies

$$\begin{aligned}\psi\omega^2(x_1,t_1)\le\tilde{c}\frac{f^4}{R^4(\alpha-f)^4}+\frac{1}{2}\psi\omega^2(x_1,t_1)+\frac{c(\alpha,\delta)}{R^4}+\frac{c(\delta,m)}{R^4}+\frac{c(\delta,m)K}{R^2}\\+\frac{c(\delta)}{T^2}+c(\delta)(|a|+K)^2+c(\delta)a^2\frac{f^2}{(\alpha-f)^2}.\end{aligned} \tag{3.11}$$

Furthermore, we need to estimate the RHS of (3.11). If $f\le 0$ and $\alpha\ge 0$, then we have

$$\frac{f^4}{(\alpha-f)^4}\le 1,\quad \frac{f^2}{(\alpha-f)^2}\le 1; \tag{3.12}$$

if $f>0$, by the assumption $\alpha-f\ge\delta>0$, we know that

$$\frac{f^4}{(\alpha-f)^4}\le\frac{(\alpha-\delta)^4}{\delta^4}=\Big(\frac{\alpha}{\delta}-1\Big)^4,\quad \frac{f^2}{(\alpha-f)^2}\le\Big(\frac{\alpha}{\delta}-1\Big)^2. \tag{3.13}$$

Plugging (3.12) (or (3.13)) into (3.11), we obtain

$$(\psi\omega^2)(x_1,t_1)\le\frac{\tilde{c}\beta^4+c(\alpha,\delta,m)}{R^4}+\frac{c(\delta,m)K}{R^2}+\frac{c(\delta)}{T^2}+c(\delta)(|a|+K)^2+c(\delta)a^2\beta^2, \tag{3.14}$$

where $\beta:=\max\{1,|\alpha/\delta-1|\}$. The above inequality implies, for all (x,t) in $Q_{R,T}$

$$\begin{aligned}(\psi^2\omega^2)(x,t)&\le\psi^2(x_1,t_1)\omega^2(x_1,t_1)\le\psi(x_1,t_1)\omega^2(x_1,t_1)\\&\le\frac{\tilde{c}\beta^4+c(\alpha,\delta,m)}{R^4}+\frac{c(\delta,m)K}{R^2}+\frac{c(\delta)}{T^2}+c(\delta)(|a|+K)^2+c(\delta)a^2\beta^2.\end{aligned} \tag{3.15}$$

Note that $\psi(x,t)=1$ in $Q_{R/2,T/2}$ and $\omega=|\nabla f|^2/(\alpha-f)^2$. Therefore we have

$$\frac{|\nabla f|}{\alpha-f}\le\Big(\frac{\tilde{c}\beta^4+c(\alpha,\delta,m)}{R^4}+\frac{c(\delta,m)K}{R^2}+\frac{c(\delta)}{T^2}+c(\delta)(|a|+K)^2+c(\delta)a^2\beta^2\Big)^{1/4}. \tag{3.16}$$

Since $f=\log u$, we get the following estimate for Eq. (2.1)

$$\frac{|\nabla u|}{u}\le\Big(\frac{\tilde{c}\beta^4+c(\alpha,\delta,m)}{R^4}+\frac{c(\delta)}{T^2}+c(\delta)(|a|+K)^2+c(\delta)a^2\beta^2\Big)^{1/4}\Big(\alpha-\log u\Big). \tag{3.17}$$

Replacing u by $e^{b/a}u$ gives the desired estimate (1.8). This completes the proof of Theorem 1.1. □

4 Proof of Corollary 1.1

Proof. The proof is similar to that of Theorem 1.1. We still use the technique of a cut-off function in a local neighborhood of Riemannian manifolds. For $0<u\leq 1$, we let $f=\log u$. Then $f\leq 0$. Set

$$\omega := |\nabla \log(1-f)|^2 = \frac{|\nabla f|^2}{(1-f)^2}.$$

By Lemma 2.1, we have

$$\left(L-\frac{\partial}{\partial t}\right)\omega \geq \frac{2f}{1-f}\langle \nabla f, \nabla\omega\rangle + 2(1-f)\omega^2 - 2(|a|+K)\omega. \tag{4.1}$$

We define a smooth cut-off function $\psi=\psi(x,t)$ in the same way as Section 3. Follow all steps as in the last section (see also pp. 1050-1051 in [1]), we can easily get the following inequality

$$\begin{aligned} 2(1-f)\psi\omega^2 \leq & (1-f)\psi\omega^2 + \frac{cf^4}{R^4(1-f)^3} + \frac{\psi\omega^2}{2} + \frac{c}{R^4} \\ & + \frac{c(m)}{R^4} + \frac{c(m)K}{R^2} + \frac{c}{T^2} + c(|a|+K)^2, \end{aligned} \tag{4.2}$$

where we used similar estimates (3.3)-(3.9) with the difference that these estimates do not contain the parameter δ. Using the same method as that in proving Theorem 1.1, for all (x,t) in $Q_{R/2,T/2}$ we can get

$$\begin{aligned} \omega^2(x,t) \leq & \frac{c(m)}{R^4} + \frac{c(m)K}{R^2} + \frac{c}{T^2} + c(|a|+K)^2 \\ \leq & \frac{c(m)}{R^4} + \frac{c(m)}{R^2}(|a|+K) + \frac{c}{T^2} + c(|a|+K)^2 \\ \leq & \frac{c(m)}{R^4} + \frac{c}{T^2} + c(|a|+K)^2. \end{aligned} \tag{4.3}$$

Again, using the same argument in the proof of Theorem 1.1 gives

$$\frac{|\nabla f|}{1-f} \leq \frac{c(m)}{R} + \frac{c}{\sqrt{T}} + c\sqrt{K+|a|}, \tag{4.4}$$

where c is a constant depending only on n, $c(m)$ is a constant depending only on n and m.

Since $f=\log u$, we get

$$\frac{|\nabla u|}{u} \leq \left(\frac{c(m)}{R} + \frac{c}{\sqrt{T}} + c\sqrt{K+|a|}\right)\cdot\left(1+\log\frac{1}{u}\right). \tag{4.5}$$

At last, replacing u by $e^{b/a}u$ above yields (1.9). □

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