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Remark on Random Attractors of Stochastic Non-Newtonian Fluid

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Abstract. In this paper, we study the asymptotic behaviors of solution for stochastic non-Newtonian fluid with white noise in two-dimensional domain. In particular, we will prove the existence of random attractors in H.

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1 Introduction

In this paper, we investigate the following stochastic incompressible non-Newtonian fluid in two-dimensional periodic domain D,

$$\mathrm{d}u+\Big(u\cdot\nabla u-\nabla\cdot\tau(e(u))+\nabla\pi\Big)\mathrm{d}t=f(x)\mathrm{d}t+\Phi\mathrm{d}W(t),\quad x\in D,\ t>0, \tag{1.1}$$

$$u(x,0)=u_0(x),\quad x\in D, \tag{1.2}$$

$$\nabla\cdot u(x,t)=0, \tag{1.3}$$

subject to the periodic boundary conditions

$$u(x,t)=u(x+L\chi_j,t),\quad \int_D u(x,t)\mathrm{d}x=0,\quad D=[0,L]^2\ \ (L>0), \tag{1.4}$$

where $\{\chi_j\}_{j=1}^2$ is the natural basis of R^2.

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The unknown vector function u denotes the velocity of the fluid, f is the external force function, and the scalar function π represents the pressure, $\tau_{ij}(e(u))$ is a symmetric stress tensor. There are many fluid materials such as liquid foams, polymeric fluids such as oil in water, blood, etc. whose viscous stress tensors are represented by the form

$$\tau_{ij}(e(u))=2\mu_0\left(\epsilon+|e(u)|^2\right)^{\frac{p-2}{2}}e_{ij}(u)-2\mu_1\Delta e_{ij}(u),\quad i,j=1,2,\ \epsilon>0,\ p>2, \tag{1.5}$$

$$e_{ij}(u)=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big),\quad |e(u)|^2=\sum_{i,j=1}^{2}|e_{ij}(u)|^2.$$

We use

$$W(t)=\sum_i \beta_i(t)h_i \tag{1.6}$$

to describe the cylindrical Wiener process for white noise on Hilbert space H adapted to a filtration $(\mathscr{F}_t)_{t\in R}$ on a fixed probability space $(\Omega,\mathscr{F},\mathbb{P})$, where $\{h_i\}$ is an orthonormal complete basis in Hilbert space H and $\beta_i(t)$ is a family of mutually independent real valued standard Wiener process. Φ is a predictable process in a fixed probability space, which is also a linear mapping.

Next, we set some notations. $L^q(D)$ denotes the Lebesgue space with norm $\|\cdot\|_{L^p}$, particularly, $\|\cdot\|_{L^2}=\|\cdot\|$, and $\|u\|_{L^\infty}=\text{esssup}_{x\in D}|u(x)|$. $H^\sigma(D)$ represents the Sobolev space $\{u\in L^2(D),\ D^k u\in L^2(D),\ k\le\sigma\}$, with $\|\cdot\|_{H^\sigma}=\|\cdot\|_\sigma$. $\mathscr{C}(I,X)$ denotes the space of continuous functions from the interval I to X. $L^q(0,T;X)$ is the space of all measurable functions $u:[0,T]\mapsto X$, with the norm

$$\|u\|^q_{L^q(0,T;X)}=\int_0^T\|u(t)\|^q_X\,\mathrm{d}t,$$

and when $q=\infty$,

$$\|u\|_{L^\infty(0,T;X)}=\text{ess}\sup_{t\in[0,T]}\|u(t)\|_X.$$

Define a space of smooth functions that incorporates the periodicity with respect to x and divergence-free condition

$$\mathscr{V}=\left\{u\in C^\infty_{per}(D):\nabla\cdot u=0,\int_D u\,\mathrm{d}x=0\right\}.$$

We use H to denote the closure of $\mathscr{V}$ in $L^2(D)$ with norm $\|\cdot\|$; $\dot{H}^\sigma(D)$ the closure of $\mathscr{V}$ in $H^\sigma(D)$ with norm $\|\cdot\|_\sigma$ $(\sigma\ge 1)$. Particularly, when $\sigma=2$, $V=\dot{H}^2(D)$. Denote by $(\dot{L}_2^{0,\sigma},\|\cdot\|_{\dot{L}_2^{0,\sigma}})$ the Hilbert space of Hilbert-Schmidt operators from H to $\dot{H}^\sigma(D)$, with the norm

$$\|\Phi\|_{\dot{L}_2^{0,\sigma}}=\left(\sum_i\|\Phi h_i\|^2_{\dot{H}^\sigma}\right)^{\frac{1}{2}}. \tag{1.7}$$

A final restriction on Φ is given: Φ belongs to $\dot{L}_2^{0,5}$.

Obviously, when $p=2$, $\mu_1=0$, and $\Phi=0$, Eq. (1.1) is a deterministic equation and reduces to Navier-Stokes equation. When $\mu_0=\mu_1=0$, it is Euler equation. Both of them are Newtonian fluids. In this paper, we will concentrate our attention on the case $p>2$, $\epsilon>0$, $\mu_0>0$ and $\mu_1>0$. In [1] we have considered the case $1<p<2$.

Many papers have devoted to questions of existence and uniqueness of the solution, and the existence of attractor and manifold for deterministic non-Newtonian equation (see [2–7]). In fact, the deterministic system model usually neglects the impact of many small perturbations, and stochastic equation can conform to physic phenomena better. It is also well known that many authors made efforts to this stochastic field of research, and displayed interesting structures and phenomena in physics. For important equations, such as the stochastic KdV equation, Navier-Stokes equation, Burgers equation, Schrödinger equation etc., there have been more interesting results related to their existence, uniqueness and attractors (see [8–14] for the progresses in these fields). In this paper, we consider stochastic non-Newtonian fluid equation as a simple model for perturbations.

Attractor is an important concept in the study of asymptotic behavior of deterministic dynamical systems. Crauel, Debussche and Flandoli (see [12, 13]) present a general theory to study the random attractors by defining an attracting set as a set that attracts any orbit starting from $-\infty$. The random attractors are compact invariant sets, which depend on chance and move with time. The authors applied the theory to prove the existence of random attractors for two-dimensional stochastic Navier-Stokes equation. In this paper, we will apply the theory to prove the existence of random attractors for two-dimensional stochastic non-Newtonian flow in the case of $p>2$. More recently, the authors in [15–18] considered the existence of random attractors for different stochastic equations. As in the case of the deterministic attractor, the Hausdorff dimension of the random attractor can be estimated. Crauel and Flandoli in [19] developed a method for bounding the Hausdorff dimension when the noise was bounded. Debussche in [20, 21] provided a general way to obtain the Hausdorff dimension of random attractor and proved that the random attractor of reaction-diffusion equation had finite Hausdorff dimension. Langa and Robinson in [22] proved that the fractal dimension enjoyed the same bound as the Hausdorff dimension, and applied the theorem to the two-dimensional stochastic Navier-Stokes equation. Along this line, we can consider the finite Hausdorff and fractal dimension of random attractors obtained here in a forthcoming paper. Of course, the proof is more complicated.

The purpose of this paper is to consider the asymptotic behaviors of stochastic non-Newtonian dynamical system. In the case of $1<p<2$ (see [1]), we only need to assume $\Phi\in\dot{L}_2^{0,2}$, the existence of attractors can be obtained in H. But in the case of $p>2$, we need to assume $\Phi\in\dot{L}_2^{0,5}$, the existence of attractors also can be obtained in H. Obviously, the restriction of Φ is strengthened for the latter. Because the nonlinear term becomes worse as the power p increases, the restriction of Φ must be strengthened in order to obtain the same result. The appearance of the stochastic term has brought great difference for the non-Newtonian fluid, thus the study of it is very important.

In fact, by some careful computations, we find that $2\nabla\cdot(\Delta e(u))$ with the divergence-free condition $\nabla\cdot u=0$ are equivalent to

$$2\nabla\cdot(\Delta e(u))=\Delta^2(u).$$

Eqs. (1.1)-(1.4) can be modified to the following problems in H:

$$\mathrm{d}u+[\mu_1 Au-2\mu_0 A_p u+B(u,u)]\mathrm{d}t=f\mathrm{d}t+\Phi\mathrm{d}W(t),\quad t>s, \tag{1.8}$$

$$u(s)=u_s,\quad s\in R, \tag{1.9}$$

$$u(x,t)=u(x+L\chi_j,t),\quad \int_D u(x,t)\mathrm{d}x=0, \tag{1.10}$$

where $A=P\Delta^2$, $B(u,u)=P(u\cdot\nabla u)$, P is the projection from $L^2(D)$ to H,

$$(A_p u)_i=\frac{\partial}{\partial x_j}\left[\epsilon+|e(u)|^2\right]^{\frac{p-2}{2}}e_{ij}(u).$$

The paper is organized as follows. In Section 2, we recall some definitions and already known results concerning random attractors; In Section 3, we give some properties about Ornstein-Uhlenbeck process $Z(\cdot)$. In Section 4, we prove the existence of random attractors in Hilbert space H with $\Phi\in\dot{L}_2^{0,5}$ and $f\in H$.

For notational simplicity, C is a generic constant and may assume various values from line to line. The summation convention related to repeated indices is used throughout the paper.

2 Preliminaries

We define

$$a(u,v)=\int_D\frac{\partial^2 u}{\partial x_k^2}\frac{\partial^2 v}{\partial x_k^2}\mathrm{d}x,\quad (u,v\in V).$$

Then $a(u,v)$ is the positive definite V-elliptic symmetric bilinear form,

$$(Au,v)=a(u,v)=<f,v>,\quad u\in V,\ f\in V',$$

where V' is the dual space of V, $A:V\to V'$ is a linear operator, and $D(A)=\{u\in V:Au\in H\}$. In fact $A=P\Delta^2$, P is the projection from $L^2(D)$ to H.

According to Rellich Theorem, A^{-1} is compact in H. Then

$$A\phi_n=\lambda_n\phi_n,\quad \phi_n\in D(A), \tag{2.1}$$

where $\{\phi_n\}_{n=1}^{\infty}$ are the eigenfunctions and also are basis in V, $\lambda_n>0,\lambda_n\to\infty$, when $n\to\infty$.

We define a continuous bilinear operator $B(\cdot,\cdot):\dot{H}^1(D)\times\dot{H}^1(D)\to\dot{H}^{-1}(D)$ as follows:

$$(B(u,v),\varpi)=\int_D u_i\frac{\partial v_j}{\partial x_i}\varpi_j\mathrm{d}x,\quad u,v,\varpi\in\dot{H}^1(D),$$

which has the properties:

$$(B(u,v),\varpi)=-(B(u,\varpi),v),\quad (B(u,v),v)=0.$$

For $u\in V$, the operator $A_p(\cdot):V\to V'$ defined by

$$\big(A_p(u),v\big)=-\int_D \gamma(u)e_{ij}(u)e_{ij}(v)\mathrm{d}x,\quad u,v\in V,$$

where

$$\gamma(u)=\big(\epsilon+|e(u)|^2\big)^{\frac{p-2}{2}}.$$

Lemma 2.1 ([4]). *There exist positive constants k_1 and k_2 such that*

$$k_1\|u\|_2^2\le\int_D \frac{\partial e_{ij}(u)}{\partial x_k}\frac{\partial e_{ij}(u)}{\partial x_k}\mathrm{d}x\le k_2\|u\|_2^2.$$

We next recall some definitions and results concerning the random attractors, which can be found in [12, 13]. Let (X,d) be a complete separable metric space and $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space. We will consider a family of mappings $S(t,s;\omega):X\to X,\ -\infty<s\le t<\infty$, parameterized by $\omega\in\Omega$ in the following contexts.

Definition 2.1 (see [12]). *Given $t\in R$ and $\omega\in\Omega$, $K(t,\omega)\subset X$ is an attracting set if for all bounded sets $B\subset X$*

$$d(S(t,s;\omega)B,K(t,\omega))\to 0,\quad s\to-\infty,$$

where $d(A,B)$ is the semidistance defined by

$$d(A,B)=\sup_{x\in A}\inf_{y\in B}d(x,y).$$

Definition 2.2. *A family $A(\omega)$, $\omega\in\Omega$ of the closed subsets of X is measurable if for all $x\in X$ the mapping $\omega\mapsto d(A(\omega),x)$ is measurable.*

Definition 2.3. *Define the random omega limit set of a bounded set $B\subset X$ at time t as*

$$A(B,t,\omega)=\bigcap_{T<t}\overline{\bigcup_{s<T}S(t,s;\omega)B}.$$

Definition 2.4. *Let $S(t,s;\omega)_{t\ge s,\omega\in\Omega}$ be a stochastic dynamical system, and $A(t,\omega)$ be a stochastic set satisfying the following conditions:*

(1) *It is the minimal closed set such that for $t\in R$, $B\subset X$,*

$$d(S(t,s;\omega)B,A(t,\omega))\to 0,\quad s\to-\infty,$$

which implies $A(t,\omega)$ attracts B (B is a deterministic set).

(2) *$A(t,\omega)$ is the largest compact measurable set, which is invariant in the sense that*

$$S(t,s;\omega)A(\theta_s\omega)=A(\theta_t\omega),\quad s\le t.$$

Then $A(t,\omega)$ is said to be the random attractor.

Theorem 2.1 (see [12]). *Let $S(t,s;\omega)_{t\geq s,\omega\in\Omega}$ be a stochastic dynamical system satisfying the following conditions*

(1) *$S(t,r;\omega)S(r,s,;\omega)x=S(t,s;\omega)x$, for all $s\leq r\leq t$ and $x\in X$;*

(2) *$S(t,s;\omega)$ is continuous in X, for all $s\leq t$;*

(3) *for all $s\leq t$ and $x\in X$, the mapping*

$$\omega\mapsto S(t,s;\omega)x$$

is measurable from $(\Omega,\mathscr{F})$ to $(X,\mathscr{B}(X))$;

(4) *for all $t,x\in X$ and P-a.e. ω, the mapping*

$$s\mapsto S(t,s;\omega)x$$

is right continuous at any point.

Assume that there exists a group $\theta_t,t\in R$ of measure preserving mappings such that

$$S(t,s;\omega)x=S(t-s,0;\theta_s\omega)x,\ \ P-\text{a.e.} \tag{2.2}$$

holds and for P-a.e. ω, there exists a compact attracting set $K(\omega)$ at time 0, for P-a.e. $\omega\in\Omega$. We set

$$\Lambda(\omega)=\overline{\bigcup_{B\subset X}A(B,\omega)},$$

where the union is taken over all the bounded subsets of X and $A(B,\omega)$ is given by

$$A(B,\omega)=\bigcap_{T<0}\overline{\bigcup_{s<T}S(0,s;\omega)B}.$$

Then $\Lambda(\omega)$ is the random attractor.

Especially, let $\Omega=\{\omega\in C(R,l^2)|\omega(0)=0\}$, with p being the product measure of two wiener measures on the negative and the positive time parts of Ω. Then

$$(\beta_1(t,\omega),\beta_2(t,\omega),\cdots,\beta_k(t,\omega),\cdots)=\omega(t).$$

In this case, the time shift θ_t is defined as

$$(\theta_t\omega)(s)=\omega(t+s)-\omega(t),\quad s,t\in R. \tag{2.3}$$

As a result, the condition (2.2) is satisfied.

3 Some properties about $Z(\cdot)$

For any $\alpha>0$, we introduce the Ornstein-Uhlenbeck process,

$$Z(t)=\int_{-\infty}^{t}e^{-(\mu_1A+\alpha)(t-s)}\Phi\mathrm{d}W(s), \tag{3.1}$$

where $A=P\Delta^2$ is a positive operator, $\mu_1>0$ is constant, and $Z(t)$ is the solution of the following linear equation,

$$dZ=-(\mu_1 A+\alpha)Zdt+\Phi dW(t), \tag{3.2}$$

$$Z(0)=Z_0=\int_{-\infty}^{0} e^{s(\mu_1 A+\alpha)}\Phi dW(s). \tag{3.3}$$

It is well known and easy to check that $Z(t)$ is a stationary process whose trajectories are $P-a.e.$ continuous. The details are described in the following lemma.

Lemma 3.1. *Assume* $\Phi\in\dot{L}_2^{0,\sigma}$ $(\sigma=0,1,2,3,4,5)$, $s<T\in R$. *Then*

$$Z(\cdot)\in\mathscr{C}(s,T;\dot{H}^\sigma(D)), \tag{3.4}$$

with the estimate

$$\mathbb{E}\Big[\sup_{t\in[s,T]}\|Z(t)\|_\sigma^2\Big]\le C(T-s)\|\Phi\|^2_{\dot{L}_2^{0,\sigma}}.$$

The proof of the lemma is similar to [23], we omit the details here.

Lemma 3.2 (see [11]). *For any* $t\in R$, *let* $\Phi\in\dot{L}_2^{0,5}$. *Then for any* $\delta>0$, *there exists an* $\alpha>0$ *depending on* δ, *such that*

$$\mathbb{E}(\|Z(t)\|_1^2)<\delta. \tag{3.5}$$

Proof. Let $Z\in\dot{H}^1(D)$. By its Fourier series, Z can be expanded as

$$Z=\sum_{m\in\mathcal{Z}^2} c_m e^{2\pi i m\cdot x/L},$$

where

$$\bar{c}_m=c_{-m}\quad\text{and}\quad\sum_{m\in\mathcal{Z}^2}|m|^2|c_m|^2<\infty.$$

Define the operator J_1 as

$$J_1Z:=\sum_{m\in\mathcal{Z}^2} m\, c_m e^{2\pi i m\cdot x/L}.$$

Since $W(t)=\sum_{k=1}^\infty \beta_k(t)h_k$, we can obtain

$$Z(t)=\sum_{k=1}^{\infty}\int_{-\infty}^{t} e^{-(t-s)(\mu_1 A+\alpha)}\Phi h_k d\beta_k(s),$$

and

$$\begin{aligned}\mathbb{E}(\|Z(t)\|_1^2) &= \mathbb{E}(\|J_1 Z(t)\|^2)\\ &= \mathbb{E}\|\sum_{k=1}^{\infty}\int_{-\infty}^{t} e^{-(t-s)(\mu_1 A+\alpha)} J_1 \Phi h_k \mathrm{d}\beta_k(s)\|^2\\ &= \sum_{k=1}^{\infty}\int_{-\infty}^{t} e^{-2(t-s)(\mu_1 A+\alpha)}\|J_1\Phi h_k\|^2 \mathrm{d}s\\ &= \sum_{k=1}^{\infty}\int_{-\infty}^{t} e^{-2(t-s)(\mu_1 A+\alpha)}\|\Phi h_k\|_1^2 \mathrm{d}s\\ &\le \|\Phi\|^2_{\dot{L}_2^{0,1}}\int_{-\infty}^{t} e^{-2(t-s)(\mu_1\lambda_1+\alpha)}\mathrm{d}s\\ &\le \|\Phi\|^2_{\dot{L}_2^{0,5}}\int_{-\infty}^{t} e^{-2(t-s)(\mu_1\lambda_1+\alpha)}\mathrm{d}s = \frac{\|\Phi\|^2_{\dot{L}_2^{0,5}}}{2(\mu_1\lambda_1+\alpha)},\end{aligned}$$

where λ_1 is the first eigenvalue of A. If we let α large enough, then (3.5) holds. □

4 Existence of random attractors in H

In the study of (1.8), it is usual to translate the unknown $v=u-Z$ (Z has the form of (3.1)) to obtain the following equations (see [12,13]),

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\mu_1 Av-2\mu_0 A_p(v+Z)+B(v+Z,v+Z)=\alpha Z+f(x),\quad x\in D, \tag{4.1}$$

$$v(s,\omega)=v_s=u_s-Z(s,\omega),\quad x\in D,\ s\in R, \tag{4.2}$$

$$v(x,t)=v(x+L\chi_j,t),\quad \int_D v(x,t)\mathrm{d}x=0. \tag{4.3}$$

Similarly (see [2,23]), we can prove that the following result holds for $P-a.e.\ \omega\in\Omega$: for $f\in H$, $\Phi\in\dot{L}_2^{0,5}$, $v_s\in H$, $s<T\in R$, there exists a unique weak solution to (4.1)-(4.3) satisfying $v\in\mathscr{C}(s,T;H)\bigcap L^2(s,T;V)$ with $v(s)=v_s$.

We define the stochastic dynamical system $(S(t,s;\omega))_{t\ge s,\omega\in\Omega}$ by

$$\begin{aligned}S(t,s;\omega)u_s &= u(t,\omega;s,u_s)\\ &= v(t,\omega;s,u_s-Z(s,\omega))+Z(t,\omega).\end{aligned}$$

It can be easily checked that the assumptions (1)-(4) are satisfied in Theorem 2.1. In the sequel, we will prove the existence of a compact attracting set $K(\omega)$ at time 0 in H.

Lemma 4.1. *Let $\Phi\in\dot{L}_2^{0,5}$, $f\in H$. There exist random radii $r_0(\omega)$ and $r_1(\omega)$, such that for any given $\rho>0$, there exists $\overline{s}(\omega)\le -1$, such that for all $s\le\overline{s}(\omega)$, and for all $u_s\in H$, with $\|u_s\|\le\rho$,* *the following inequalities*

$$\|v(t,\omega;s,u_s-Z(s,\omega))\|^2\le r_0(\omega),\quad t\in[-1,0],$$
$$\int_{-1}^0\|v(t,\omega;s,u_s-Z(s,\omega))+Z(t)\|_2^2\mathrm{d}t\le r_1(\omega),$$
$$\int_{-1}^0\|e\big(v(t,\omega;s,u_s-Z(s,\omega))+Z(t)\big)\|_{L^p}^p\mathrm{d}t\le r_2(\omega),$$

hold $P-a.e.$, *where* $v(t,\omega;s,u_s-Z(s,\omega))$ *is the solution of* (4.1)-(4.3), *and*

$$\begin{aligned}r_0(\omega)=&2e^{\frac{\mu_1\lambda_1}{4}}\Big(1+\sup_{s\le-1}\exp(s\frac{\mu_1\lambda_1}{8})\|Z(s)\|^2\Big)\\&+2e^{\frac{\mu_1\lambda_1}{4}}\int_{-\infty}^0 g(\sigma)\exp\Big(\sigma(\frac{\mu_1\lambda_1}{4}+\frac{C}{\sigma}\int_\sigma^0\|Z(\tau)\|_1^2\mathrm{d}\tau)\Big)\mathrm{d}\sigma,\\r_1(\omega)=&\frac{64}{\mu_1^2\lambda_1}r_0(\omega)\int_{-1}^0\|Z(\sigma)\|_1^2\mathrm{d}\sigma+\frac{8}{\mu_1}\int_{-1}^0 g(\sigma)\mathrm{d}\sigma\\&+\frac{4}{\mu_1}r_0(\omega)+2\int_{-1}^0\|Z(t)\|_2^2\mathrm{d}t,\\r_2(\omega)=&\frac{16\;r_0(\omega)}{\mu_0\mu_1\lambda_1}\int_{-1}^0\|Z(\sigma)\|_1^2\mathrm{d}\sigma+\frac{1}{\mu_0}\int_{-1}^0 g(\sigma)\mathrm{d}\sigma+\frac{r_0(\omega)}{2\mu_0}.\end{aligned}$$

Proof. Taking the inner product of (4.1) with v in H,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\mu_1\|\Delta v\|^2+2\mu_0\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(v)\mathrm{d}x\\&\qquad+b(v+Z,v+Z,v)=\alpha(Z,v)+(f(x),v),\end{aligned}\tag{4.4}$$

we can obtain the following inequality,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\mu_1\|\Delta v\|^2+2\mu_0\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(v+Z)\mathrm{d}x\\&\le|b(v+Z,v+Z,v)|+\alpha|(Z,v)|+|(f(x),v)|\\&\quad+2\mu_0\Big|\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(Z)\mathrm{d}x\Big|.\end{aligned}\tag{4.5}$$

We will estimate the right terms one by one. Note

$$\alpha|(Z,v)|\le\frac{\mu_1\lambda_1\|v\|_1^2}{8}+\frac{2\alpha^2\|Z\|^2}{\mu_1\lambda_1},$$
$$|(f(x),v)|\le\frac{\mu_1\lambda_1\|v\|_1^2}{8}+\frac{2\|f\|^2}{\mu_1\lambda_1},$$

where λ_1 is the first eigenvalue of operator A. From the divergence-free condition, we know

$$\begin{aligned}&|b(v+Z,v+Z,v)|\\=&|b(v+Z,Z,v+Z)|\le\|v+Z\|_{L^4}\ \|v+Z\|_{L^4}\|Z\|_1\\\le&C\|v+Z\|\ \|v+Z\|_1\|Z\|_1\\\le&\frac{4C}{\mu_1\lambda_1}\|Z\|_1^2\|v+Z\|^2+\frac{\mu_1\lambda_1}{16}\|v+Z\|_1^2\\\le&\frac{8C\|Z\|_1^2}{\mu_1\lambda_1}\|v\|^2+\frac{\mu_1\lambda_1}{8}\|v\|_1^2+\frac{8C}{\mu_1\lambda_1}\|Z\|_1^2\|Z\|^2+\frac{\mu_1\lambda_1}{8}\|Z\|_1^2,\end{aligned}$$

where we have used the Gagliardo-Nirenberg inequality,

$$\|u\|_{L^4}\le C\|u\|^{\frac12}\|\nabla u\|^{\frac12},$$

and ϵ-Young inequality. Note that for $p>2$,

$$\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}\ge|e(v+Z)|^{p-2}.$$

Then

$$\begin{aligned}&2\mu_0\Big|\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(v+Z)\mathrm{d}x\Big|\\\ge&2\mu_0\int_D|e(v+Z)|^p\mathrm{d}x.\end{aligned}$$

The last term in the right hand side of (4.5) is difficult to estimate due to its strong nonlinearity. To estimate it, we mainly apply the Hölder inequality, Sobolev embedding, $H^2(D)\hookrightarrow L^\infty(D)$ and ϵ-Young inequality:

$$\begin{aligned}&2\mu_0\Big|\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(Z)\mathrm{d}x\Big|\\\le&2\mu_0\|e(Z)\|_{L^\infty}\Big|\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)\mathrm{d}x\Big|\\\le&2\mu_0C\|e(Z)\|_{L^\infty}\Big(\int_D\epsilon^{\frac{p-2}{2}}|e_{ij}(v+Z)|\mathrm{d}x+\int_D|e(v+Z)|^{p-1}\mathrm{d}x\Big)\\\le&2\mu_0C\|e(Z)\|_{L^\infty}\Big[\epsilon^{\frac{p-2}{2}}\Big(\int_D|e(v)|\mathrm{d}x+\int_D|e(Z)|\mathrm{d}x\Big)+C\Big(\int_D|e(v+Z)|^p\mathrm{d}x\Big)^{\frac{p-1}{p}}\Big]\\\le&2\mu_0C\|e(Z)\|_{L^\infty}\Big(\|e(v)\|+\|e(Z)\|+\|e(v+Z)\|_{L^p}^{p-1}\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\mu_1}{4}\|v\|_2^2+\frac{4\mu_0^2C\|e(Z)\|_2^2}{\mu_1}+2\mu_0C\|e(Z)\|_2\|e(Z)\| \\
&\quad +\mu_0\|e(v+Z)\|_{L^p}^p+\mu_0(p-1)^{p-1}\Big(\frac{2C\|e(Z)\|_2}{p}\Big)^p \\
&\leq \frac{\mu_1}{4}\|v\|_2^2+\mu_0\|e(v+Z)\|_{L^p}^p+\frac{4\mu_0^2C\|Z\|_3^2}{\mu_1}+2\mu_0C\|Z\|_3\|Z\|_1+C\|Z\|_3^p.
\end{aligned}
\tag{4.6}
$$

Combing the above estimates, we can obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\frac{\mu_1}{4}\|v\|_2^2+\mu_0\|e(v+Z)\|_{L^p}^p+\Big(\frac{\mu_1\lambda_1}{8}-\frac{8C\|Z\|_1^2}{\mu_1\lambda_1}\Big)\|v\|^2 \\
\leq &\frac{2\alpha^2\|Z\|^2}{\mu_1\lambda_1}+\frac{8C}{\mu_1\lambda_1}\|Z\|_1^2\|Z\|^2+\frac{\mu_1\lambda_1}{8}\|Z\|_1^2 \\
&+\frac{4\mu_0^2C\|Z\|_3^2}{\mu_1}+2\mu_0C\|Z\|_3\|Z\|_1+C\|Z\|_3^p+\frac{2\|f\|^2}{\mu_1\lambda_1}.
\end{aligned}
\tag{4.7}
$$

Let

$$
\begin{aligned}
g=&\frac{2\alpha^2\|Z\|^2}{\mu_1\lambda_1}+\frac{8C}{\mu_1\lambda_1}\|Z\|_1^2\|Z\|^2+\frac{\mu_1\lambda_1}{8}\|Z\|_1^2+\frac{4\mu_0^2C\|Z\|_3^2}{\mu_1} \\
&+2\mu_0C\|Z\|_3\|Z\|_1+C\|Z\|_3^p+\frac{2\|f\|^2}{\mu_1\lambda_1}.
\end{aligned}
$$

It follows from (4.7) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\frac{\mu_1}{2}\|v\|_2^2+2\mu_0\|e(v+Z)\|_{L^p}^p+\Big(\frac{\mu_1\lambda_1}{4}-\frac{16C\|Z\|_1^2}{\mu_1\lambda_1}\Big)\|v\|^2\leq 2g. \tag{4.8}
$$

By Gronwall inequality, for $s\leq -1$, and $t\in[-1,0]$, we have

$$
\begin{aligned}
\|v(t)\|^2\leq&\|v(s)\|^2\exp\Big[-\int_s^t\Big(\frac{\mu_1\lambda_1}{4}-\frac{16C\|Z(\sigma)\|_1^2}{\mu_1\lambda_1}\Big)\mathrm{d}\sigma\Big] \\
&+2\int_s^t g(\sigma)\exp\Big[-\int_\sigma^t\Big(\frac{\mu_1\lambda_1}{4}-\frac{16C\|Z(\tau)\|_1^2}{\mu_1\lambda_1}\Big)\mathrm{d}\tau\Big]\mathrm{d}\sigma \\
\leq& e^{\frac{\mu_1\lambda_1}{4}}\|v(s)\|^2\exp\Big[s\Big(\frac{\mu_1\lambda_1}{4}+\frac{C}{s}\int_s^0\|Z(\sigma)\|_1^2\mathrm{d}\sigma\Big)\Big] \\
&+2e^{\frac{\mu_1\lambda_1}{4}}\int_s^0 g(\sigma)\exp\Big[-\int_\sigma^0\Big(\frac{\mu_1\lambda_1}{4}-C\|Z(\tau)\|_1^2\Big)\mathrm{d}\tau\Big]\mathrm{d}\sigma.
\end{aligned}
$$

As the process $\|Z\|_1^2$ is stationary and ergodic, we know from [12] that

$$
-\frac{1}{s}\int_s^0\|Z(\sigma)\|_1^2\mathrm{d}\sigma\to\mathbb{E}(\|Z(0)\|_1^2),\quad s\to-\infty. \tag{4.9}
$$

There exists an $s_0(\omega)$ such that for any $s<s_0(\omega)$,

$$-\frac{1}{s}\int_s^0 \|Z(\sigma)\|_1^2 \mathrm{d}\sigma \le 2\mathbb{E}(\|Z(0)\|_1^2).$$

Applying Lemma 3.2 gives

$$\mathbb{E}(\|Z(0)\|_1^2) \le \frac{\|\Phi\|^2_{L_2^{0,5}}}{2(\mu_1\lambda_1+\alpha)}.$$

We can take α large enough so that

$$\mathbb{E}(\|Z(0)\|_1^2) \le \frac{\mu_1\lambda_1}{16C}, \tag{4.10a}$$

$$\begin{aligned}\exp&\Big[s\Big(\frac{\mu_1\lambda_1}{4}+\frac{C}{s}\int_s^0 \|Z\|_1^2 \mathrm{d}\sigma\Big)\Big]\\ &\le \exp\Big[s\Big(\frac{\mu_1\lambda_1}{4}-C\frac{\mu_1\lambda_1}{8C}\Big)\Big] \le \exp\Big(\frac{s\mu_1\lambda_1}{8}\Big).\end{aligned} \tag{4.10b}$$

Consequently,

$$\begin{aligned}\|v(t)\|^2 \le& e^{\frac{\mu_1\lambda_1}{4}}\|v(s)\|^2 \exp\Big(\frac{s\mu_1\lambda_1}{8}\Big)\\ &+2e^{\frac{\mu_1\lambda_1}{4}}\int_{-\infty}^0 g(\sigma)\exp\Big[\sigma\Big(\frac{\mu_1\lambda_1}{4}+\frac{C_1}{\sigma}\int_\sigma^0 \|Z\|_1^2 \mathrm{d}\tau\Big)\Big]\mathrm{d}\sigma,\end{aligned} \tag{4.11}$$

where $g(\sigma)$ grows at most polynomially, as $\sigma\to-\infty$ $P-a.e.$ (see [13]). Since $g(\sigma)$ is multiplied by a function which decays exponentially, the integral in (4.11) is convergent.

It is now clear that

$$\begin{aligned}\|v(t)\|^2 \le& 2e^{\frac{\mu_1\lambda_1}{4}}\exp\Big(\frac{s\mu_1\lambda_1}{8}\Big)\Big(\|u_s\|^2+\|Z(s)\|^2\Big)\\ &+2e^{\frac{\mu_1\lambda_1}{4}}\int_{-\infty}^0 g(\sigma)\exp\Big[\sigma\Big(\frac{\mu_1\lambda_1}{4}+\frac{C_1}{\sigma}\int_\sigma^0 \|Z\|_1^2 \mathrm{d}\tau\Big)\Big]\mathrm{d}\sigma.\end{aligned} \tag{4.12}$$

Given $\rho>0$, we can choose $\bar{s}(\omega)$, depending only on ω, such that $\exp(\frac{s\mu_1\lambda_1}{8})\rho^2\le 1$, for all $s\le\bar{s}(\omega)$. We can then give the final estimate of $\|v(t)\|$ for $t\in[-1,0]$,

$$\begin{aligned}\|v(t)\|^2 \le r_0(\omega) =& 2e^{\frac{\mu_1\lambda_1}{4}}\Big[1+\sup_{s\le -1}\|Z(s)\|^2\exp\Big(\frac{s\mu_1\lambda_1}{8}\Big)\Big]\\ &+2e^{\frac{\mu_1\lambda_1}{4}}\int_{-\infty}^0 g(\sigma)\exp\Big[\sigma\Big(\frac{\mu_1\lambda_1}{4}+\frac{C_1}{\sigma}\int_\sigma^0 \|Z\|_1^2 \mathrm{d}\tau\Big)\Big]\mathrm{d}\sigma.\end{aligned}$$

Similarly, $Z(s)$ grows at most polynomially, as $s\to-\infty$. Moreover, since $Z(s)$ is multiplied by a function which decays exponentially, the term

$$\sup_{s\le -1}\exp\Big(\frac{s\mu_1\lambda_1}{8}\Big)\|Z(s)\|^2 \text{ is bounded.}$$

Furthermore, we can integrate (4.8) on $[-1,0]$ and deduce

$$
\begin{aligned}
&\int_{-1}^{0}\|v(t)\|_2^2\mathrm{d}t\\
&\quad\le\frac{32}{\mu_1^2\lambda_1}r_0(\omega)\int_{-1}^{0}\|Z(\sigma)\|_1^2\mathrm{d}\sigma+\frac{4}{\mu_1}\int_{-1}^{0}g(\sigma)\mathrm{d}\sigma+\frac{2}{\mu_1}\|v(-1)\|^2,\\
&\int_{-1}^{0}\|v(t)+Z(t)\|_2^2\mathrm{d}t\le r_1(\omega)\\
&\quad=\frac{64}{\mu_1^2\lambda_1}r_0(\omega)\int_{-1}^{0}\|Z(\sigma)\|_1^2\mathrm{d}\sigma+\frac{4}{\mu_1}r_0(\omega)+\frac{8}{\mu_1}\int_{-1}^{0}g(\sigma)\mathrm{d}\sigma+2\int_{-1}^{0}\|Z(t)\|_2^2\mathrm{d}t,\\
&\int_{-1}^{0}\|e\big(v(t)+Z(t)\big)\|_{L^p}^p\mathrm{d}t\le r_2(\omega)\\
&\quad=\frac{16\ r_0(\omega)}{\mu_0\mu_1\lambda_1}\int_{-1}^{0}\|Z(\sigma)\|_1^2\mathrm{d}\sigma+\frac{1}{\mu_0}\int_{-1}^{0}g(\sigma)\mathrm{d}\sigma+\frac{r_0(\omega)}{2\mu_0}.
\end{aligned}
$$

These estimates will be used in the following lemma. □

Lemma 4.2. *Let $\Phi\in\dot{L}_2^{0,5}$, $f\in H$. There exists a random radius $r_3(\omega)$, such that for any given $\rho>0$, there exists $\overline{s}(\omega)\le-1$, such that for all $s\le\overline{s}(\omega)$, and for all $u_s\in H$ with $\|u_s\|\le\rho$, the estimate*

$$\|v(t,\omega;s,u_s-Z(s,\omega))\|_1^2\le r_3(\omega),\quad t\in[-1,0],$$

holds $P-a.e.$. In particular,

$$\|v(0,\omega;s,u_s-Z(s,\omega))\|_1^2\le r_3(\omega).$$

Proof. Taking the inner product of (4.1) with $-\Delta v$ in H,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2+\mu_1\|v\|_3^2-2\mu_0\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(\Delta v)\mathrm{d}x\\
&=\alpha(Z,-\Delta v)+(f(x),-\Delta v)+b(v+Z,v+Z,\Delta v),
\end{aligned}\tag{4.13}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2+\mu_1\|v\|_3^2-2\mu_0\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(\Delta v+\Delta Z)\mathrm{d}x\\
=&\alpha(Z,-\Delta v)+(f(x),-\Delta v)+b(v+Z,v+Z,\Delta v)\\
&-2\mu_0\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(\Delta Z)\mathrm{d}x.
\end{aligned}\tag{4.14}
$$

Let

$$\mathscr{A}=-2\mu_0\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(\Delta v+\Delta Z)\mathrm{d}x.$$

Using the condition $v+Z=u$ gives

$$\begin{aligned}\mathscr{A}=&-2\mu_0\int_D\left(\epsilon+|e(u)|^2\right)^{\frac{p-2}{2}}e_{ij}(u)e_{ij}(\Delta u)\mathrm{d}x\\=&2\mu_0\int_D\left(\epsilon+|e(u)|^2\right)^{\frac{p-2}{2}}\frac{\partial e_{ij}(u)}{\partial x_k}\frac{\partial e_{ij}(u)}{\partial x_k}\mathrm{d}x\\&+(p-2)\int_D\left(\epsilon+|e(u)|^2\right)^{\frac{p-4}{2}}e_{ij}(u)\frac{\partial e_{ij}(u)}{\partial x_k}e_{ij}(u)\frac{\partial e_{ij}(u)}{\partial x_k}.\end{aligned}\tag{4.15}$$

According to $p>2$, then $\mathscr{A}>0$. Consequently, we can drop the term $\mathscr{A}$ in the following computations. We estimate the right terms in (4.14) respectively.

The following estimates can be obtained easily:

$$\begin{aligned}|\alpha(Z,\Delta v)|\le&\frac{\mu_1\lambda_1\|v\|_2^2}{8}+\frac{2\alpha^2\|Z\|^2}{\mu_1\lambda_1},\\|(f(x),\Delta v)|\le&\frac{\mu_1\lambda_1\|v\|_2^2}{8}+\frac{2\|f\|^2}{\mu_1\lambda_1}.\end{aligned}$$

For the third term in the right hand side of (4.14), we use Gagliardo-Nirenberg inequality and ϵ-Young inequality to obtain

$$\begin{aligned}&|b(v+Z,v+Z,\Delta v)|\\\le&\|v+Z\|_{L^\infty}\|\nabla(v+Z)\|\ \|\Delta v\|\\\le&C\|v+Z\|^{\frac12}\|\Delta(v+Z)\|^{\frac12}\|\nabla(v+Z)\|\ \|\Delta v\|\\\le&C\|v+Z\|^{\frac12}\left(\|\Delta v\|^{\frac12}+\|\Delta Z\|^{\frac12}\right)\|\nabla(v+Z)\|\ \|\Delta v\|\\=&C\|v+Z\|^{\frac12}\|\nabla(v+Z)\|\ \|\Delta v\|^{\frac32}+C\|v+Z\|^{\frac12}\|\nabla(v+Z)\|\ \|\Delta Z\|^{\frac12}\|\Delta v\|\\\le&\frac{\mu_1\lambda_1}{8}\|\Delta v\|^2+\frac{54C^4}{\mu_1^3\lambda_1^3}\|v+Z\|^2\|\nabla(v+Z)\|^4+\frac{\mu_1\lambda_1}{8}\|\Delta v\|^2\\&+\frac{2C^2}{\mu_1\lambda_1}\|v+Z\|\ \|\nabla(v+Z)\|^2\|\Delta Z\|,\end{aligned}$$

where the second inequality is due to Gagliardo-Nirenberg inequality

$$\|u\|_{L^\infty}\le\|u\|^{\frac12}\|\Delta u\|^{\frac12},$$

and the last inequality is due to ϵ-Young inequality.

Similarly to (4.6), we mainly apply the Hölder inequality, Sobolev embedding, and

ϵ-Young inequality to estimate the following term,

$$\begin{aligned}
&2\mu_0\Big|\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}e_{ij}(v+Z)e_{ij}(\Delta Z)\mathrm{d}x\Big|\\
\le&2\mu_0\|e(\Delta Z)\|_{L^\infty}\int_D\Big(\epsilon+|e(v+Z)|^2\Big)^{\frac{p-2}{2}}|e_{ij}(v+Z)|\mathrm{d}x\\
\le&2\mu_0\|e(\Delta Z)\|_2\Big(\int_D\epsilon^{\frac{p-2}{2}}|e_{ij}(v+Z)|\mathrm{d}x+\int_D|e(v+Z)|^{p-1}\mathrm{d}x\Big)\\
\le&2\mu_0C\|e(\Delta Z)\|_2\Big(\|e(v)\|+\|e(Z)\|+\|e(v+Z)\|_{L^p}^{p-1}\Big)\\
\le&\frac{\mu_1}{4}\|v\|_3^2+\mu_0\|e(v+Z)\|_{L^p}^p+\frac{4\mu_0^2C\|Z\|_5^2}{\mu_1}+2\mu_0C\|Z\|_5\|Z\|_1+C\|Z\|_5^p. \qquad (4.16)
\end{aligned}$$

From the above estimates, we can obtain the following inequality,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2+\frac{\mu_1}{4}\|v\|_3^2\\
\le&\frac{2\alpha^2\|Z\|^2}{\mu_1\lambda_1}+\mu_0\|e(v+Z)\|_{L^p}^p+\frac{4\mu_0^2C\|Z\|_5^2}{\mu_1}+2\mu_0C\|Z\|_5\|Z\|_1\\
&+C\|Z\|_5^p+\frac{2\|f\|^2}{\mu_1\lambda_1}+\frac{54C^4}{\mu_1^3\lambda_1^3}\|v+Z\|^2\|\nabla(v+Z)\|^4\\
&+\frac{2C^2}{\mu_1\lambda_1}\|v+Z\|\|\nabla(v+Z)\|^2\|\Delta Z\|. \qquad (4.17)
\end{aligned}$$

Let

$$\begin{aligned}
X(t)=&\frac{2\alpha^2\|Z\|^2}{\mu_1\lambda_1}+\mu_0\|e(v+Z)\|_{L^p}^p+\frac{4\mu_0^2C\|Z\|_5^2}{\mu_1}+2\mu_0C\|Z\|_5\|Z\|_1+C\|Z\|_5^p\\
&+\frac{2\|f\|^2}{\mu_1\lambda_1}+\frac{C}{\mu_1^3\lambda_1^3}\|v+Z\|^2\|\nabla Z\|^4+\frac{2C}{\mu_1\lambda_1}\|v+Z\|\ \|\nabla(v+Z)\|^2\|\Delta Z\|,
\end{aligned}$$

and

$$Y(t)=\frac{C}{\mu_1^3\lambda_1^3}\|v+Z\|^2\|\nabla v\|^2.$$

Then the inequality (4.17) can be simplified as the following inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2\le 2X(t)+2Y(t)\|\nabla v\|^2. \qquad (4.18)$$

We deduce that for any $-1\le\theta\le t\le 0$,

$$\begin{aligned}
\|v(t)\|_1^2\le&\|v(\theta)\|_1^2e^{\int_\theta^t 2Y(\sigma)\mathrm{d}\sigma}+\int_\theta^t 2X(\sigma)e^{\int_\sigma^t 2Y(\tau)\mathrm{d}\tau}\mathrm{d}\sigma\\
\le&\Big(\|v(\theta)\|_1^2+\int_{-1}^0 2X(\sigma)\mathrm{d}\sigma\Big)e^{\int_{-1}^0 2Y(\sigma)\mathrm{d}\sigma}. \qquad (4.19)
\end{aligned}$$

Integrating with respect to θ on $[-1,0]$ gives

$$\|v(t)\|_1^2 \leq \Big(\int_{-1}^{0} \|v(\theta)\|_1^2 \mathrm{d}\theta + \int_{-1}^{0} 2X(\sigma)\mathrm{d}\sigma\Big) e^{\int_{-1}^{0} 2Y(\sigma)\mathrm{d}\sigma}. \tag{4.20}$$

Applying Lemma 4.1, obviously, the first term $\int_{-1}^{0} \|v(\theta)\|_1^2 \mathrm{d}\theta$ is bounded.

Combining Lemmas 3.1 and 4.1, we know $\int_{-1}^{0} X(\sigma)\mathrm{d}\sigma$ is also bounded. Then

$$\|v(t)\|_1^2 \leq r_3(\omega), \text{ when } s \leq \bar{s}(\omega).$$

If we let $t=0$, then we have

$$\|v(0)\|_1^2 \leq r_3(\omega), \text{ when } s \leq \bar{s}(\omega).$$

This completes the proof of the lemma. □

Theorem 4.1. *For all $u_s \in H, \Phi \in \dot{L}_2^{0,5}, f \in H$, there exist random attractors for the stochastic non-Newtonian equations* (1.8)-(1.10) *in H.*

Proof. Let $K(\omega)$ be the ball in $\dot{H}^1(D)$ of radius $r_3^{\frac{1}{2}}(\omega) + \|Z(0,\omega)\|_1$. We have proved that for any B bounded in H, there exists $\bar{s}(\omega)$ such that for $s \leq \bar{s}(\omega)$,

$$S(0,s;\omega)B \subset K(\omega) \ P-\text{a.e..}$$

This clearly implies that $K(\omega)$ is an attracting set at time 0, since it is compact in H, and Theorem 2.1 applies. □

Remark 4.1. A random bounded set $\{B(\omega)\}_{\omega \in \Omega}$ of X is called tempered with respect to $(\theta_t)_{t \in R}$, if for $P-a.e.\omega \in \Omega$,

$$\lim_{t \to \infty} e^{-\beta t} \mathrm{d}(B(\theta_{-t}\omega)) = 0 \quad \text{for all } \beta > 0,$$

where θ_t is defined in (2.3), $d(B) = \sup_{x \in B} \|x\|_X$.

In fact, the result can be improved, we can show that the random attractors attract the tempered random subsets of phase space H. Further details refer to [15, 16] and the references therein.

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