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Synchronization of Stochastic Two-Layer Geophysical Flows

HAN Yongqian*

Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, China.

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Abstract. In this paper, the two-layer quasigeostrophic flow model under stochastic wind forcing is considered. It is shown that when the layer depth or density difference across the layers tends to zero, the dynamics on both layers synchronizes to an averaged geophysical flow model.

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1 Introduction

We consider the two-layer quasigeostrophic flow model ([1], p. 423; [2], p. 87):

$$\frac{\partial q_1}{\partial t}+J(\psi_1,q_1+\beta y)=\nu\Delta^2\psi_1+f(x,y)+\dot{W}_1(t,x,y), \tag{1.1a}$$

$$\frac{\partial q_2}{\partial t}+J(\psi_2,q_2+\beta y)=\nu\Delta^2\psi_2-r\Delta\psi_2+\dot{W}_2(t,x,y), \tag{1.1b}$$

where potential vorticities $q_1(x,y,t)$, $q_2(x,y,t)$ for the top layer and the bottom layer are defined via stream functions $\psi_1(x,y,t)$, $\psi_2(x,y,t)$, respectively,

$$\begin{aligned} q_1&=\Delta\psi_1-F\cdot(\psi_1-\psi_2),\\ q_2&=\Delta\psi_2-F\cdot(\psi_2-\psi_1). \end{aligned} \tag{1.2}$$

*Corresponding author. *Email address:* han_yongqian@iapcm.ac.cn (Y. Han)

Here $(x,y)\in O:=(0,L)\times(0,L)\in\mathbb{R}^2$, L is the characteristic scale for horizontal length of the flows; F is positive defined by (see also [2], p.87)

$$F=\frac{f_0^2}{gh}\frac{\rho_0}{\rho_2-\rho_1}, \tag{1.3}$$

g is the gravitational acceleration, h is the depth of layers with the assumption that the depth of top and bottom layers is equal, ρ_1 and ρ_2 are the densities ($\rho_2>\rho_1$) of top and bottom layers, respectively; ρ_0 is the characteristic scale for density of the flows, $f_0+\beta y$ (with f_0,β constants) is the Coriolis parameter and β is the meridional gradient of the Coriolis parameter; $\nu>0$ is the viscosity. Note that $r=f_0\delta_E/(4h)$ is the Ekman constant ([3], p.29). Here $\delta_E=\sqrt{2\nu/f_0}$ is the Ekman layer thickness ([1], p.188). Moreover, $J(h,g)=h_xg_y-h_yg_x$ is the Jacobi operator and $\Delta=\partial_{xx}+\partial_{yy}$ is the Laplace operator in $\mathbb{R}^2$. Finally, $f(x,y)$ is the mean (deterministic) wind forcing, two-sided Wiener processes $W_1(t)$ and $W_2(t)$, which describe the fluctuating part of the external wind forcing in the fluid, either are mutual independent or $W_1(t)=W_2(t)$. In this paper, we consider the case when the covariance operators Q_1 and Q_2 of the Wiener processes $W_1(t)$ and $W_2(t)$ have a finite trace, respectively.

The two-layer quasigeostrophic flow model has been used as a theoretical and numerical model to understand basic mechanisms in large scale geophysical flows, such as baroclinic effects [1], wind-driven circulation [4, 5], the Gulf Stream [6], fluid stability [7] and subtropical gyres [3, 8]. Recently Salmon [9] introduced a generalized two-layer ocean flow model.

We assume Dirichlet boundary conditions for $\psi=(\psi_1,\psi_2)$:

$$\psi|_{\partial O}=\Delta\psi|_{\partial O}=0. \tag{1.4}$$

We also assume an appropriate initial condition $\psi(x,y,0)=\psi_0(x,y)$.

The stochastically forced quasigeostrophic model has been used to investigate various phenomena in geophysical flows [10–15]. This stochastic model has also be investigated in the context of stochastic dynamical systems [16–20].

For this stochastic two-layer model, following [16], we can establish the well-posedness and the existence of pullback attractors. The purpose of this paper is to establish the synchronization for the stochastic two-layer model.

In this paper, we first recall some basic facts about random dynamical systems in Section 2. In Section 3, we establish the well-posedness of the stochastic two-layer quasigeostrophic model by transforming it into a coupled system of random partial differential equations. In Section 4, we show the existence of pullback attractors for the random two-layer model. In Section 5, we establish the synchronization for the random two-layer model. In Section 6, the main results concerning stochastic two-layer quasigeostrophic model are established.

2 Random dynamical systems

In order to investigate the long time dynamics of the two-layer fluid system (1.1), we need some appropriate concepts and tools from the theory of *random dynamical systems*. For detailed presentation of random dynamical systems we refer to the monograph by Arnold [21] and references [16,24].

A random dynamical system (RDS) consists of two components. The first component is a *metric dynamical system* $(\Omega,\mathcal{F},\mathbb{P},\theta)$ as a model for a noise, where $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space and θ is a $(\mathcal{F}\otimes\mathcal{B}(\mathbb{R}),\mathcal{F})$-measurable flow which satisfies

$$\theta_0=\mathrm{id},\qquad \theta_{t+\tau}=\theta_t\circ\theta_\tau=:\theta_t\theta_\tau,$$

for $t,\tau\in\mathbb{R}$. The measure $\mathbb{P}$ is supposed to be ergodic with respect to θ, i.e., $\theta_t\mathbb{P}=\mathbb{P}$ for all $t\in\mathbb{R}$. The second component of a random dynamical system is a $(\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}\otimes\mathcal{B}(H),\mathcal{B}(H))$-measurable mapping φ satisfying the *cocycle* property

$$\varphi(t+\tau,\omega,x)=\varphi(t,\theta_\tau\omega,\varphi(\tau,\omega,x)),\qquad \varphi(0,\omega,x)=x,\ \ \forall x\in H,$$

where the phase space H is a separable metric space. We will denote this random dynamical system by symbol φ.

We can associate a metric dynamical system θ with the Wiener process W as follows. Let U be a separable Hilbert space. We consider the probability space

$$\Big(C_0(\mathbb{R},U),\mathcal{B}\big(C_0(\mathbb{R},U)\big),\mathbb{P}\Big),$$

where $C_0(\mathbb{R},U)$ is the Fréchet space of continuous functions on $\mathbb{R}$ which are zero at zero and $\mathcal{B}(C_0(\mathbb{R},U))$ is the corresponding Borel σ-algebra. The Wiener process W can be interpreted in the canonical sense $W(\cdot,\omega)=\omega(\cdot)$. The flow θ_t is given by

$$\theta_t\omega=\omega(\cdot+t)-\omega(t),\qquad \text{for}\ \ \omega\in C_0(\mathbb{R},U). \tag{2.1}$$

The flow θ_t is called the *Wiener shift*. The measure $\mathbb{P}$ which is ergodic with respect to θ_t is called the *Wiener measure*.

A closed set $B(\omega)$, depending on ω, in a separable Hilbert space H is called random if the distance mapping $\omega\to\sup_{x\in B(\omega)}\|x-y\|_H$ is a random variable for any $y\in H$. In addition, we will assume that $B(\omega)$ is forward invariant:

$$\varphi(t,\omega,B(\omega))\subset B(\theta_t\omega),\qquad t>0.$$

In the following we also need the concept of *tempered random variables*. A random variable x is called tempered if $t\to|x(\theta_t\omega)|$ is subexponentially growing:

$$\limsup_{t\to\pm\infty}\frac{\log^+|x(\theta_t\omega)|}{|t|}=0\quad \text{a.s.}$$

(see Arnold [21]). A random set $\{B(\omega)\}$ is said to be tempered if there exists a $v_0 \in H$ such that $B(\omega) \subset \{v \in H: \mathrm{dist}(v,v_0) \le r(\omega)\}$ for all $\omega \in \Omega$, where the random variable $r(\omega) > 0$ is tempered. We denote by $\mathcal{B}$ the collection of all tempered random set $\{B(\omega)\}$.

Below we need the concept of a random (global) pullback attractor for RDSs (see, e.g., [21,22]), which extends the corresponding definition of a global universal attractor in autonomous systems [23].

Definition 2.1. *Let (θ,φ) be a RDS with the phase space H. A random closed set $\{A(\omega)\}$ from $\mathcal{B}$ is said to be a random pullback attractor for (θ,φ) in $\mathcal{B}$ if*

(i) *$A(\omega)$ is a forward invariant set,*

(ii) *$\{A(\omega)\}$ is pullback attracting in $\mathcal{B}$, i.e.,*

$$\lim_{t\to+\infty} dist_H\Big\{\varphi\big(t,\theta_{-t}\omega,B(\theta_{-t}\omega)\big),A(\omega)\Big\}=0, \qquad \omega\in\Omega,$$

for all $\{B(\omega)\}\in\mathcal{B}$, where $dist_H(A,B)=\sup_{a\in A} dist_H(a,B)$.

The following result [22,24] ensures the existence of a random attractor for a RDS.

Theorem 2.1. *Let (θ,φ) be a RDS on $\Omega\times H$ such that $\varphi(t,\omega,\cdot): H\to H$ is a compact operator for each fixed $t>0$ and P-a.e. $\omega\in\Omega$. If there exists a tempered random set $\{D(\omega)\}$ and $T=T\big(\{B(\omega)\},\omega\big)\ge 0$ such that*

$$\varphi\big(t,\theta_{-t}\omega,B(\theta_{-t}\omega)\big)\subset D(\omega), \qquad \forall t\ge T, \ \text{P-a.e.}\ \omega\in\Omega,$$

for every tempered random set $\{B(\omega)\}$, then the RDS (θ,φ) has a random pullback attractor $\{A(\omega)\}$ with the component subsets defined for P-a.e. $\omega\in\Omega$ by

$$A(\omega)=\cap_{s>0}\overline{\cup_{t\ge s}\varphi\big(t,\theta_{-t}\omega,D(\theta_{-t}\omega)\big)}.$$

The family $\{D(\omega)\}$ ia called a pullback absorbing random set for the RDS.

3 Well-posedness of the random two-layer fluid system

Following [16], the well-posedness of the random two-layer fluid system can be established. Here we introduce the outline.

Let $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ denote the standard scalar product and norm in L^2, respectively. We also denote $\mathbf{L}^2=L^2\times L^2$ and $\mathbf{H}^s=H^s\times H^s$. We work on the phase space $\mathbf{H}^{-1}$ with the scalar product

$$(q,\overline{q})_*=(\nabla\psi_1,\nabla\overline{\psi}_1)_0+(\nabla\psi_2,\nabla\overline{\psi}_2)_0+F(\psi_1-\psi_2,\overline{\psi}_1-\overline{\psi}_2)_0,$$

where $q=(q_1,q_2)$, $\overline{q}=(\overline{q}_1,\overline{q}_2)$ and $\psi=(\psi_1,\psi_2)\in\mathbf{H}_0^1$, $\overline{\psi}=(\overline{\psi}_1,\overline{\psi}_2)$. The relation between q (resp. $\overline{q}$) and ψ (resp. $\overline{\psi}$) is defined by (1.2). The norm induced by this scalar product

$$\|q\|_*^2=(q,q)_*=\|\nabla\psi_1\|_0^2+\|\nabla\psi_2\|_0^2+F\|\psi_1-\psi_2\|_0^2 \tag{3.1}$$

is equivalent to the usual norm on $\mathbf{H}^{-1}$.

To treat the nonlinearity in the two-layer fluid model we need the following lemma.

Lemma 3.1. **([16])** *The Jacobian operator has the following properties:*

$$J(u,v)=-J(v,u),\qquad (J(u,v),v)_0=0, \tag{3.2}$$

$$(J(u,v),w)_0=(J(v,w),u)_0, \tag{3.3}$$

for u,v,w in H_0^1. Moreover, the following estimates hold:

$$|(J(u,v),\Delta u)_0|\le c_0\|\Delta v\|_0\cdot\|\nabla u\|_0\cdot\|\Delta u\|_0,\qquad u,v\in H^2\cap H_0^1; \tag{3.4a}$$

$$|(J(u,v),w)_0|\le c_1\|\Delta u\|_0\cdot\|\Delta v\|_0\cdot\|w\|_0,\qquad u,v\in H^2\cap H_0^1, w\in L^2; \tag{3.4b}$$

$$|(J(u,v),w)_0|\le c_1\|\nabla u\|_0\cdot\|\Delta v\|_0\cdot\|\nabla w\|_0,\qquad u,w\in H_0^1, v\in H^2\cap H_0^1, \tag{3.5}$$

where $c_1=c_0\lambda_1^{-1/2}$, c_0 is a constant.

We introduce two Ornstein-Uhlenbeck processes $\eta_1(x,y,t,\omega)$ and $\eta_2(x,y,t,\omega)$ in L^2 which are defined by the solutions of the following linear stochastic partial differential equations

$$\frac{\partial\eta_1}{\partial t}=\nu(k+1)\Delta\eta_1+\dot{W}_1, \tag{3.6a}$$

$$\frac{\partial\eta_2}{\partial t}=\nu(k+1)\Delta\eta_2+\dot{W}_2, \tag{3.6b}$$

with Dirichlet boundary condition $\eta_1|_{\partial O}=\eta_2|_{\partial O}=0$ and with some initial condition η_{10}, $\eta_{20}\in H^1$, where $k>0$ is a free *control* parameter and W_j $(j=1,2)$ are Wiener processes in L^2. We suppose that the covariance operators Q_j $(j=1,2)$ of these Wiener processes have finite traces. These processes η_1 and η_2 can be written in the form

$$\eta_1(t,\omega):=\Big(\int_{-\infty}^t S(t-s)\mathrm{d}W_1(s)\Big)(\omega), \tag{3.7a}$$

$$\eta_2(t,\omega):=\Big(\int_{-\infty}^t S(t-s)\mathrm{d}W_2(s)\Big)(\omega), \tag{3.7b}$$

where $S(r)=e^{r\nu(k+1)\Delta}$. As mentioned in Section 2 such a Wiener process generates a metric dynamical systems denoted by $(\Omega,\mathcal{F},\mathbb{P},\theta)$ where θ is the Wiener shift introduced in (2.1). It is well known that these equations have two stationary solutions which are generated by two *Gaussian* random variables η_1 and η_2 in H_0^1. In particular, the mapping

$$t\to\eta_j(\theta_t\omega)\in L^2_{loc}(-\infty,\infty;H_0^1),\qquad j=1,2$$

solves Eq. (3.6). Moreover, we can assume that the random variables η_j and the processes $(t,\omega)\to\eta_j(\theta_t\omega)$ $(j=1,2)$ are defined for *all* $\omega\in\Omega$ what follows by a perfection argument for Ornstein–Uhlenbeck processes in Hilbert spaces, see Chueshov and Scheutzov [25] Proposition 3.1. For moments of η_j $(j=1,2)$ we obtain due to [26]:

$$\begin{aligned}\mathbb{E}\|\eta_1(t)\|_1^2&=\int_{-\infty}^t \mathrm{tr}_0\{(-\Delta)S(t-r)Q_1S^*(t-r)\}\mathrm{d}r\\&=\sum_{j=1}^{\infty}\int_{-\infty}^t \lambda_j e^{-2\nu(k+1)\lambda_j(t-r)}\mathrm{d}r\Big(Q_1\varphi_j,\varphi_j\Big)=\frac{\mathrm{tr}_0Q_1}{2\nu(k+1)},\end{aligned}\tag{3.8a}$$

$$\mathbb{E}\|\eta_1(t)\|_1^{2n}\le C_n\left(\frac{\mathrm{tr}_0Q_1}{\nu(k+1)}\right)^n,\tag{3.8b}$$

$$\mathbb{E}\|\eta_2(t)\|_1^2=\frac{\mathrm{tr}_0Q_2}{2\nu(k+1)},\quad \mathbb{E}\|\eta_2(t)\|_1^{2n}\le C_n\left(\frac{\mathrm{tr}_0Q_2}{\nu(k+1)}\right)^n,\qquad n\in\mathbb{N},\ C_n>0,\ \forall t\in\mathbb{R}.\tag{3.8c}$$

We introduce new variables [16]

$$\widetilde{q}_1:=q_1-\eta_1,\quad \widetilde{q}_2:=q_2-\eta_2,\quad \widetilde{\psi}_1:=\psi_1+\xi_1,\quad \widetilde{\psi}_2:=\psi_2+\xi_2,\tag{3.9}$$

where the stationary processes η_1 and η_2 solve the problem (3.6) and ξ_1 and ξ_2 are defined such that the elliptic equations (1.2) keep the same form

$$\begin{aligned}\widetilde{q}_1&=\Delta\widetilde{\psi}_1-F\cdot(\widetilde{\psi}_1-\widetilde{\psi}_2),\\ \widetilde{q}_2&=\Delta\widetilde{\psi}_2-F\cdot(\widetilde{\psi}_2-\widetilde{\psi}_1).\end{aligned}$$

The processes ξ_1 and ξ_2 are solutions of the linear elliptic equations

$$\Delta\xi_1-F\cdot(\xi_1-\xi_2)=-\eta_1,\tag{3.10a}$$
$$\Delta\xi_2-F\cdot(\xi_2-\xi_1)=-\eta_2,\tag{3.10b}$$
$$\xi_1|_{\partial O}=\xi_2|_{\partial O}=0.\tag{3.10c}$$

By simple calculations we have the estimates

$$\|\xi_1+\xi_2\|_{s+2}\le\|\eta_1+\eta_2\|_s,\quad 0\le s\le 1,\tag{3.11a}$$
$$\|\nabla\xi_1-\nabla\xi_2\|_0^2+F\|\xi_1-\xi_2\|_0^2\le\frac{1}{4F}\|\eta_1-\eta_2\|_0^2,\tag{3.11b}$$
$$\|\Delta\xi_1-\Delta\xi_2\|_0^2+F\|\nabla\xi_1-\nabla\xi_2\|_0^2\le\frac{1}{4F}\|\nabla(\eta_1-\eta_2)\|_0^2,\tag{3.11c}$$
$$\|\Delta\xi_1\|_0+\|\Delta\xi_2\|_0\le 2\big(\|\eta_1\|_0+\|\eta_2\|_0\big),\tag{3.11d}$$
$$\|\nabla\Delta\xi_1\|_0+\|\nabla\Delta\xi_2\|_0\le 2\big(\|\nabla\eta_1\|_0+\|\nabla\eta_2\|_0\big).\tag{3.11e}$$

Using (3.10) we have

$$\begin{aligned}-\nu\Delta^2\xi_1-\nu(k+1)\Delta\eta_1&=-\nu F(\Delta\xi_1-\Delta\xi_2)-\nu k\Delta\eta_1,\\ -\nu\Delta^2\xi_2-\nu(k+1)\Delta\eta_2&=-\nu F(\Delta\xi_2-\Delta\xi_1)-\nu k\Delta\eta_2.\end{aligned}$$

Thus we finally get the coupled system of random partial differential equations (for convenience, we drop the tilde)

$$\frac{\partial q_1}{\partial t}+J(\psi_1-\xi_1,q_1+\eta_1+\beta y)=\nu\Delta^2\psi_1+f-\nu F(\Delta\xi_1-\Delta\xi_2)-\nu k\Delta\eta_1, \tag{3.12a}$$

$$\frac{\partial q_2}{\partial t}+J(\psi_2-\xi_2,q_2+\eta_2+\beta y)=\nu\Delta^2\psi_2-r\Delta\psi_2+r\Delta\xi_2-\nu F(\Delta\xi_2-\Delta\xi_1)-\nu k\Delta\eta_2, \tag{3.12b}$$

with the Dirichlet boundary condition (1.4) and the coupling condition

$$q_1=\Delta\psi_1-F\cdot(\psi_1-\psi_2), \tag{3.13a}$$

$$q_2=\Delta\psi_2-F\cdot(\psi_2-\psi_1), \tag{3.13b}$$

and with initial data

$$q(x,y,0)=\big(q_{01}(x,y)-\eta_1(x,y,0,\omega),q_{02}-\eta_2(x,y,0,\omega)\big)\in\mathbf{H}^{-1}, \tag{3.14a}$$

$$\psi(x,y,0)=\big(\psi_{01}(x,y)+\xi_1(x,y,0,\omega),\psi_{02}+\xi_2(x,y,0,\omega)\big)\in\mathbf{H}^{1}, \tag{3.14b}$$

where η_1 and η_2 are the stationary solutions to (3.6) and ξ_1 and ξ_2 solve (3.10) in $H^2\cap H^1_0$.

For the rest of this paper, we work on this coupled system (3.12). As in [16,28], using the Galerkin method and the compactness argument we can prove the following assertion on the well-posedness of problems (3.12) and (3.13).

Theorem 3.1. (Well-posedness) *Let $q_0\in\mathbf{H}^{-1}$ and $f\in L^2$. Then for P-a.e. $\omega\in\Omega$ and for all $\tau>0$, the systems* (3.12) *and* (3.13) *have an unique solution $\{q(t),\psi(t)\}$ such that*

$$q\in C([0,\infty);\mathbf{H}^{-1})\cap L^2_{loc}(0,\infty;\mathbf{L}^2)\cap L^2_{loc}(\tau,\infty;\mathbf{H}^1_0).$$

The function ψ associated to q by (3.13) *satisfies*

$$\psi\in C\big([0,\infty);\mathbf{H}^1_0\big)\cap L^2_{loc}\big(0,\infty;\mathbf{H}^2\cap\mathbf{H}^1_0\big)\cap L^2_{loc}\Big(\tau,\infty;\mathbf{H}^3\cap\big\{\psi|_{\partial O}=\Delta\psi|_{\partial O}=0\big\}\Big).$$

The solution depends continuously on the initial condition $q_0\in\mathbf{H}^{-1}$.

By the uniqueness assertion of the last theorem the solution of (3.12) $t\to q(t)$ generates a random dynamical system (θ,φ) on $\mathbf{\Omega}\times\mathbf{H}^{-1}$, where $\mathbf{\Omega}=C_0(\mathbb{R},\mathbf{H}^{-1})$. Moreover, the mapping $\varphi:\mathbf{H}^{-1}\to\mathbf{H}^{-1}$ which maps q_0 to $\varphi(t,\omega,q_0)=q(t)$ is continuous.

Let $\psi_{01}=\psi_{02}$, $\overline{\psi}_0=(\psi_{01},\psi_{02})$, and $\psi(t)=(\psi_1(t),\psi_2(t))$ be the solution of system (3.12) with initial data $\psi_0=\overline{\psi}_0$. By the Theorem 3.1 (3.12) also generates two RDSs (θ,φ_{1F}) and (θ,φ_{2F}) on $\Omega\times H^1_0$ as follows: mapping $\varphi_{1F}:H^1_0\to H^1_0$ which maps ψ_{01} to

$$\varphi_{1F}(t,\omega,\psi_{01})=\psi_1(t),$$

and mapping $\varphi_{2F}:H^1_0\to H^1_0$ which maps ψ_{02} to

$$\varphi_{2F}(t,\omega,\psi_{02})=\psi_2(t).$$

Here the mappings φ_{1F} and φ_{2F} are continuous, $\Omega=C_0(\mathbb{R},H_0^1)$. We investigate the asymptotic behavior of the above RDSs (θ,φ_{1F}) and (θ,φ_{2F}) as $F\to\infty$ at the level of pullback attractors.

Now using inverse transformation we define the cocycles $\overline{\varphi}$, $\overline{\varphi}_{1F}$ and $\overline{\varphi}_{2F}$ for problem (1.1) by the formulas

$$\begin{aligned}&\overline{\varphi}(t,\omega,\cdot)=R_m^{-1}(\theta_t\omega)\circ\varphi\big(t,\omega,R_m(\omega)\cdot\big),\\&\overline{\varphi}_{1F}(t,\omega,\cdot)=R_{m1}^{-1}(\theta_t\omega)\circ\varphi_{1F}\big(t,\omega,R_{m1}(\omega)\cdot\big),\\&\overline{\varphi}_{2F}(t,\omega,\cdot)=R_{m2}^{-1}(\theta_t\omega)\circ\varphi_{2F}\big(t,\omega,R_{m2}(\omega)\cdot\big),\end{aligned}$$

where $R_m(\omega):\mathbf{H}^{-1}\to\mathbf{H}^{-1}$, $R_{m1}(\omega):H_0^1\to H_0^1$ and $R_{m2}(\omega):H_0^1\to H_0^1$ are random mappings of the form

$$\begin{aligned}&R_m(\omega)U=U-(\eta_1,\eta_2), && U=(U_1,U_2)\in\mathbf{H}^{-1},\\&R_{m1}(\omega)V=V+\xi_1, && V\in H_0^1,\\&R_{m2}(\omega)V=V+\xi_2, && V\in H_0^1.\end{aligned}$$

It is clear that $R_m(\omega)$ maps tempered random sets in $\mathbf{H}^{-1}$ into tempered sets in $\mathbf{H}^{-1}$, and $R_{m1}(\omega)$ and $R_{m2}(\omega)$ map tempered random sets in H_0^1 into tempered sets in H_0^1. Therefore all statements concerning the RDSs $(\theta,\overline{\varphi})$, $(\theta,\overline{\varphi}_{1F})$ and $(\theta,\overline{\varphi}_{2F})$ can be easily reformulated as statements concerning the RDSs (θ,φ), (θ,φ_{1F}) and (θ,φ_{2F}). In our further considerations we deal with the RDSs (θ,φ), (θ,φ_{1F}) and (θ,φ_{2F}).

4 Pullback attractors of the random two-layer fluid system

In this section we prove the existence of random pullback attractors for the RDSs (θ,φ), (θ,φ_{1F}) and (θ,φ_{2F}) by continuing the line of research introduced in [16].

We first construct an absorbing forward invariant set for the random dynamical system generated by (3.12).

Theorem 4.1. *For P-a.e. $\omega\in\Omega$, there exist a compact random set $\{B(\omega)\}$ in the space $\mathbf{H}^{-1}$, and compact random sets $\{B_{1F}(\omega)\}$ and $\{B_{2F}(\omega)\}$ in H_0^1 such that*

$$\begin{aligned}&\varphi(t,\omega,B(\omega))\subset B(\theta_t\omega), && \text{for } t\geq 0, && (4.1a)\\&\varphi(t,\omega,q(\omega))\in B(\theta_t\omega), && \text{for } t\geq t_0(\omega,q), && (4.1b)\\&\varphi_{1F}(t,\omega,B_{1F}(\omega))\subset B_{1F}(\theta_t\omega), && \text{for } t\geq 0, && (4.2a)\\&\varphi_{1F}(t,\omega,\psi_1(\omega))\in B_{1F}(\theta_t\omega), && \text{for } t\geq t_1(\omega,\psi_1), && (4.2b)\\&\varphi_{2F}(t,\omega,B_{2F}(\omega))\subset B_{2F}(\theta_t\omega), && \text{for } t\geq 0, && (4.3a)\\&\varphi_{2F}(t,\omega,\psi_2(\omega))\in B_{2F}(\theta_t\omega), && \text{for } t\geq t_2(\omega,q_2), && (4.3b)\end{aligned}$$

where the RDSs (θ,φ), (θ,φ_{1F}) and (θ,φ_{1F}) are generated by (3.12)*, q is a random variable with values in $\mathbf{H}^{-1}$, ψ_1 and ψ_2 are random variables with values in H_0^1.*

By using Theorem 2.1 and 4.1, we obtain the following theorem which is the main result of this section.

Theorem 4.2. (I) *In the space* $\mathbf{H}^{-1}$ *the RDS* (θ,φ) *generated by* (3.12) *has a compact pullback attractor* $\{A(\omega)\}$ *for P-a.e.* $\omega\in\Omega$. *Moreover, there exists a tempered random variable* $R(\omega)$, *which doesn't depend on F, such that*

$$A(\omega)\subset\Big\{q\in\mathbf{H}^{-1}:\|q\|_*^2\le R(\omega)\Big\}. \tag{4.4}$$

(II) *In the space* H_0^1 *the RDSs* (θ,φ_{1F}) *and* (θ,φ_{1F}) *generated by* (3.12) *possess compact pullback attractors* $\{A_{1F}(\omega)\}$ *and* $\{A_{2F}(\omega)\}$ *for P-a.e.* $\omega\in\Omega$, *respectively. Moreover,*

$$A_{1F}(\omega)\subset\{\psi_1\in H_0^1:\|\nabla\psi_1\|_0^2\le R(\omega)\}, \tag{4.5}$$

$$A_{2F}(\omega)\subset\{\psi_2\in H_0^1:\|\nabla\psi_2\|_0^2\le R(\omega)\}. \tag{4.6}$$

We now divide the proof of Theorem 4.1 into some lemmas. We start with the following lemma.

Lemma 4.1. *Let* $q(t)$ *be the solution of* (3.12). *Then* $q(t)$ *satisfies the following inequality*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|q(t)\|_*^2+\nu\Big(\|\Delta\psi_1(t)\|_0^2+\|\Delta\psi_2(t)\|_0^2\Big)\\ &\le d_0\Big(\|\eta_1(\theta_t\omega)\|_1^2+\|\eta_2(\theta_t\omega)\|_1^2\Big)\cdot\Big(\|\nabla\psi_1\|_0^2+\|\nabla\psi_2\|_0^2\Big)+m(\theta_t\omega),\end{aligned}$$

where

$$m(\omega)=d_1(\|\eta_1(\omega)\|_1^4+\|\eta_2(\omega)\|_1^4)+d_2(\|\eta_1(\omega)\|_0^2+\|\eta_2(\omega)\|_0^2)+d_3$$

and constants d_0, d_1, d_2 *and* d_3 *are independent of F.*

Proof. Taking the scalar product (3.12) with (ψ_1,ψ_2) and integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\nabla\psi_1(t)\|_0^2+\|\nabla\psi_2(t)\|_0^2+F\|\psi_1(t)-\psi_2(t)\|_0^2\Big\}\\ &\qquad+\nu\Big\{\|\Delta\psi_1(t)\|_0^2)+\|\Delta\psi_2(t)\|_0^2\Big\}+r\|\nabla\psi_2(t)\|_0^2\\ &=-\int_O\Big\{J(\xi_1,q_1+\eta_1+\beta y)\psi_1+J(\xi_2,q_2+\eta_2+\beta y)\psi_2\Big\}\mathrm{d}x\mathrm{d}y-r(\Delta\xi_2,\psi_2)_0-(f,\psi_1)_0.\\ &\qquad+\nu F(\Delta\xi_1-\Delta\xi_2,\psi_1-\psi_2)_0+\nu k(\Delta\eta_1,\psi_1)_0+\nu k(\Delta\eta_2,\psi_2)_0.\end{aligned} \tag{4.7}$$

We know that

$$\begin{aligned}&\nu F(\Delta\xi_1-\Delta\xi_2,\psi_1-\psi_2)_0+\nu k(\Delta\eta_1,\psi_1)_0+\nu k(\Delta\eta_2,\psi_2)_0-r(\Delta\xi_2,\psi_2)_0\\ &=\nu F(\xi_1-\xi_2,\Delta\psi_1-\Delta\psi_2)_0+\nu k(\eta_1,\Delta\psi_1)_0+\nu k(\eta_2,\Delta\psi_2)_0-r(\xi_2,\Delta\psi_2)_0\\ &\le\frac{\nu}{8}\Big\{\|\Delta\psi_1\|_0^2+\|\Delta\psi_2\|_0^2\Big\}+12\Big\{\nu F^2\|\xi_1-\xi_2\|_0^2+\nu k^2\|\eta_1\|_0^2+\nu k^2\|\eta_2\|_0^2+\frac{r^2}{\nu}\|\xi_2\|_0^2\Big\},\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&\int_O \Big\{ J(\xi_1,\eta_1+\beta y)\psi_1 + J(\xi_2,\eta_2+\beta y)\psi_2 \Big\} \mathrm{d}x\mathrm{d}y - (f,\psi_1)_0 \\
\le& \frac{\nu\lambda_1^2}{8}\Big\{ \|\psi_1\|_0^2 + \|\psi_2\|_0^2 \Big\} + \frac{12}{\nu\lambda_1^2}\Big\{ \beta^2\|\xi_1\|_1^2 + \beta^2\|\xi_2\|_1^2 + \|\xi_1\|_3^2\|\eta_1\|_1^2 + \|\xi_2\|_3^2\|\eta_2\|_1^2 + \|f\|_0^2 \Big\} \\
\le& \frac{\nu}{8}\Big\{ \|\Delta\psi_1\|_0^2 + \|\Delta\psi_2\|_0^2 \Big\} + \frac{12}{\nu\lambda_1^2}\Big\{ \beta^2\|\xi_1\|_1^2 + \beta^2\|\xi_2\|_1^2 + \|\xi_1\|_3^2\|\eta_1\|_1^2 + \|\xi_2\|_3^2\|\eta_2\|_1^2 + \|f\|_0^2 \Big\},
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
&\int_O \{ J(\xi_1,\Delta\psi_1)\psi_1 + J(\xi_2,\Delta\psi_2)\psi_2 \} \mathrm{d}x\mathrm{d}y \\
=& \int_O \Big\{ J(\psi_1,\xi_1)\Delta\psi_1 + J(\psi_2,\xi_2)\Delta\psi_2 \Big\} \mathrm{d}x\mathrm{d}y \\
\le& \frac{\nu}{8}\Big\{ \|\Delta\psi_1\|_0^2 + \|\Delta\psi_2\|_0^2 \Big\} + \frac{C}{\nu}\Big\{ \|\xi_1\|_3^2\|\nabla\psi_1\|_0^2 + \|\xi_2\|_3^2\|\nabla\psi_2\|_0^2 \Big\},
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
&F\int_O \Big\{ J(\xi_1,\psi_2-\psi_1)\psi_1 + J(\xi_2,\psi_1-\psi_2)\psi_2 \Big\} \mathrm{d}x\mathrm{d}y \\
=& F\int_O J(\xi_1-\xi_2,\psi_2)(\psi_1-\psi_2)\mathrm{d}x\mathrm{d}y \le F\|\xi_1-\xi_2\|_1\|\nabla\psi_2\|\|\psi_1-\psi_2\|_{L^\infty} \\
\le& \frac{\nu}{8}\Big\{ \|\Delta\psi_1\|_0^2 + \|\Delta\psi_2\|_0^2 \Big\} + \frac{C}{\nu}F^2\|\xi_1-\xi_2\|_1^2\|\nabla\psi_2\|_0^2.
\end{aligned} \tag{4.11}
$$

Putting (4.7)–(4.11) and (3.11) together, we complete the proof of the lemma. □

If k is chosen large enough then particular moments of η_1 and η_2 are small. Especially we can formulate:

Lemma 4.2. *Let W_1 and W_2 be Wiener processes in L^2 with finite trace of the covariances Q_1 and Q_2, respectively. Then under assumptions*

$$
\frac{2d_0a_0(\mathrm{tr}_0Q_1+\mathrm{tr}_0Q_2)}{\lambda_1\nu^2(k+1)} < 1, \qquad \frac{8d_0(\mathrm{tr}_0Q_1+\mathrm{tr}_0Q_2)}{\lambda_1\nu^2(k+1)^2} \le 1, \tag{4.12}
$$

the random variable

$$
R_0(\omega) := \int_{-\infty}^0 \exp\Big\{ \frac{\nu\lambda_1}{a_0}\tau + d_0\int_\tau^0 (\|\eta_1(\theta_{\tau'}\omega)\|_1^2 + \|\eta_2(\theta_{\tau'}\omega)\|_1^2)\mathrm{d}\tau' \Big\} m(\theta_\tau\omega)\mathrm{d}\tau
$$

is finite and tempered, where $a_0 = 1+2F/\lambda_1$. Moreover

$$
\big(\mathbb{E}R_0^2\big)^{\frac{1}{2}} \le e\, d_4 \Big(\frac{a_0}{\nu\lambda_1} \Big)^{\frac{1}{2}} \Big(\frac{\lambda_1\nu}{a_0} - \frac{2d_0(\mathrm{tr}_0Q_1+\mathrm{tr}_0Q_2)}{\nu(k+1)} \Big)^{-\frac{1}{2}},
$$

where

$$
d_4 = C_8^{\frac{1}{2}} \frac{d_1(\mathrm{tr}_0Q_1+\mathrm{tr}_0Q_2)^2}{\nu^2(k+1)^2} + C_4^{\frac{1}{2}} \frac{d_2(\mathrm{tr}_0Q_1+\mathrm{tr}_0Q_2)}{\nu(k+1)} + d_3
$$

is an estimate for $(\mathbb{E}m^2)^{1/2}$ (the constants C_8, C_4 are defined in (3.8)*).*

The proof of this lemma can be found in Chueshov *et. al* [16] for an Ornstein-Uhlenbeck process in another Hilbert space. However the argument given there is of a general nature.

We now construct some sets satisfying (4.1a), (4.2a) and (4.3a), respectively.

Lemma 4.3. *Let* $R(\omega):=aR_0(\omega)$ *for some* $a>1$ *and* R_0 *as in Lemma* 4.2. *Provided conditions* (4.12) *hold, then the closed* $\mathbf{H}^{-1}$*–ball* $B(0,R(\omega)^{1/2})$ *fulfills* (4.1a), *and the closed* H_0^1*–balls* $B(0,R(\omega)^{1/2})$ *fulfill* (4.2a) *and* (4.3a).

Proof. Using Lemma 4.1 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q(t)\|_*^2\le\left(-\frac{\nu\lambda_1}{a_0}+d_0\cdot(\|\eta_1(\theta_t\omega)\|_1^2+\|\eta_2(\theta_t\omega)\|_1^2)\right)\cdot\|q(t)\|_*^2+m(\theta_t\omega).$$

Let $q_0=q(0)$ and $\rho(t,\omega,\|q_0\|_*^2)$ be the solution of the one dimensional random affine equation

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t}+\frac{\nu\lambda_1}{a_0}\rho=d_0\Big(\|\eta_1(\theta_t\omega)\|_1^2+\|\eta_2(\theta_t\omega)\|_1^2\Big)\rho+m(\theta_t\omega), \tag{4.13a}$$

$$\rho(0,\omega,\|q_0\|_*^2)=\|q_0\|_*^2. \tag{4.13b}$$

A comparison argument gives that

$$\|\varphi(t,\omega,q_0)\|_*^2\equiv\|q(t)\|_*^2\le\rho(t,\omega,\|q_0\|_*^2).$$

Here φ is the dynamical system introduced in Section 3: $\varphi(t,\omega,q_0)=q(t)$, where $q(t)$ is the solution to (3.12) with the initial data q_0. Eq. (4.13) has the stationary solution given by $\rho(t,\omega,R_0(\omega))=R_0(\theta_t\omega)$. This can be checked by the variation of constants formula. It follows from (3.8) that for a sufficient small $\varepsilon>0$,

$$\mathbb{E}\int_0^t d_0(\|\eta_1(\theta_\tau\omega)\|_1^2+\|\eta_2(\theta_\tau\omega)\|_1^2)\mathrm{d}\tau<\frac{\nu\lambda_1-\varepsilon}{a_0}t,\qquad \frac{\nu\lambda_1-\varepsilon}{a_0}>0,$$

for large $t>0$. Thus this solution is exponentially attracting:

$$\begin{aligned}&|R_0(\theta_t\omega)-\rho(t,\omega,\|q_0\|_*^2)|=|\rho(t,\omega,R_0(\omega))-\rho(t,\omega,\|q_0\|_*^2)|\\ &\le\exp\left\{\int_0^t\left(d_0(\|\eta_1(\theta_\tau\omega)\|_1^2+\|\eta_2(\theta_\tau\omega)\|_1^2)-\frac{\nu\lambda_1}{a_0}\right)\mathrm{d}\tau\right\}\left(R(\omega)+\|q_0\|_*^2\right)\to 0.\end{aligned}$$

This completes the proof of lemma. □

It remains to prove the existence of compact sets $B(\omega)$ satisfying (4.1a), $B_{1F}(\omega)$ satisfying (4.2a) and $B_{2F}(\omega)$ satisfying (4.3a).

Lemma 4.4. *Let the random variable $R(\omega)$ be defined in Lemma* 4.3.

(I) *The set*

$$B(\omega):=\overline{\varphi(1,\theta_{-1}\omega,B(0,R(\theta_{-1}\omega)^{\frac{1}{2}}))}$$

is a compact absorbing forward invariant random set in $\mathbf{H}^{-1}$ for P-a.e. $\omega\in\Omega$. Moreover,

$$\omega\mapsto\sup\Big\{\|\Delta\psi_1\|_0^2+\|\Delta\psi_2\|_0^2:(q_1,q_2)\in B(\omega)\Big\}, \tag{4.14}$$

is a tempered random variable (ψ_1 and ψ_2 are defined by (3.13)).

(II) *The sets*

$$B_{1F}(\omega):=\overline{\varphi_{1F}(1,\theta_{-1}\omega,B(0,R(\theta_{-1}\omega)^{\frac{1}{2}}))},$$

and

$$B_{2F}(\omega):=\overline{\varphi_{2F}(1,\theta_{-1}\omega,B(0,R(\theta_{-1}\omega)^{\frac{1}{2}}))}$$

are compact absorbing forward invariant random sets in H_0^1 for P-a.e. $\omega\in\Omega$.

Proof. The regularity assertion of Theorem 3.1 and some standard techniques (see Bernier [28]) imply that the sets $B(\omega)\subset\mathbf{H}^{-1}$ are compact. Since R is a random variable, the ball $B(0,R^{1/2})$ is a random set. The continuity of $\varphi(t,\omega,\cdot)$ allows us to conclude that B is a random set. The construction of B ensures that the set is absorbing and forward invariant. The temperedness of (4.14) can be proved in the same way as in [16].

Similarly, we can prove that $B_{1F}(\omega)$ and $B_{1F}(\omega)$ are compact random sets in H_0^1. □

Putting the above four lemmas together gives Theorem 4.1.

5 Synchronization

We show that dynamics (θ,φ_{1F}) and (θ,φ_{2F}) on both layers synchronize to a synchronized model. To this end, we need a few estimates.

Lemma 5.1. *Suppose that there exists a constant C_0 such that $\mathbb{E}\|\psi_{01}-\psi_{02}\|_0^2\le C_0/F$. For any time interval $[0,T]$, we have that*

$$\lim_{F\to\infty}\sup_{0\le t\le T}\mathbb{E}\|\psi_1(\cdot,t)-\psi_2(\cdot,t)\|_0^2=0, \tag{5.1}$$

$$\sup_{0\le t\le T}\mathbb{E}(\|\nabla\psi_1(\cdot,t)\|_0^2+\|\nabla\psi_2(\cdot,t)\|_0^2)+\mathbb{E}\int_0^T\Big(\|\Delta\psi_1(s)\|_0^2+\|\Delta\psi_2(s)\|_0^2\Big)\mathrm{d}s\le K_1,\ \ \forall T\in[0,\infty), \tag{5.2}$$

where constant K_1 is independent of F defined in (1.3).

Proof. Using Lemma 4.1, we have

$$\begin{aligned}
&\mathbb{E}\|\nabla\psi_1(\cdot,t)\|_0^2+\mathbb{E}\|\nabla\psi_2(\cdot,t)\|_0^2+F\mathbb{E}\|\psi_1(\cdot,t)-\psi_2(\cdot,t)\|_0^2\\
&\quad+\nu\mathbb{E}\int_0^t\Big(\|\Delta\psi_1(s)\|_0^2+\|\Delta\psi_2(s)\|_0^2\Big)\mathrm{d}s\\
\le& C_2d_0\int_0^t\mathbb{E}\Big(\|\eta_1(\theta_t\omega)\|_1^2+\|\eta_2(\theta_t\omega)\|_1^2\Big)\cdot\mathbb{E}\Big(\|\nabla\psi_1\|_0^2+\|\nabla\psi_2\|_0^2\Big)\mathrm{d}s\\
&\quad+\mathbb{E}\int_0^t m(\theta_t\omega)\mathrm{d}s+2\mathbb{E}\Big(\|\nabla\psi_{01}\|_0^2+\|\nabla\psi_{02}\|_0^2\Big)+\widetilde{C}_0,
\end{aligned}\tag{5.3}$$

where

$$\widetilde{C}_0=\frac{16}{\lambda_1}\mathbb{E}(\|\eta_1(0)\|_0^2+\|\eta_2(0)\|_0^2)+\frac{1}{2F}\mathbb{E}\|\eta_1(0)-\eta_2(0)\|_0^2+2C_0.$$

By the Gronwall inequality, we have

$$\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}\|\nabla\psi_1(\cdot,t)\|_0^2+\mathbb{E}\|\nabla\psi_2(\cdot,t)\|_0^2\\
\le&\Big(2\mathbb{E}(\|\nabla\psi_{01}\|_0^2+\|\nabla\psi_{02}\|_0^2)+\widetilde{C}_0+\mathbb{E}\int_0^T m(\theta_t\omega)\mathrm{d}s\Big)\\
&\qquad\times\exp\Big\{C_2d_0\mathbb{E}\int_0^T(\|\eta_1(\theta_t\omega)\|_1^2+\|\eta_2(\theta_t\omega)\|_1^2)\mathrm{d}t\Big\}.
\end{aligned}\tag{5.4}$$

Estimates (5.3) and (5.4) imply that

$$\begin{aligned}
&\sup_{0\le t\le T}F\mathbb{E}\|\psi_1(\cdot,t)-\psi_2(\cdot,t)\|_0^2+\nu\mathbb{E}\int_0^T(\|\Delta\psi_1(s)\|_0^2+\|\Delta\psi_2(s)\|_0^2)\mathrm{d}s\\
\le&\Big(2\mathbb{E}(\|\nabla\psi_{01}\|_0^2+\|\nabla\psi_{02}\|_0^2)+\widetilde{C}_0+\mathbb{E}\int_0^T m(\theta_t\omega)\mathrm{d}s\Big)\\
&\quad\times\Big\{C_2d_0\mathbb{E}\int_0^T(\|\eta_1(\theta_t\omega)\|_1^2+\|\eta_2(\theta_t\omega)\|_1^2)\mathrm{d}t\\
&\quad\times\exp\Big(C_2d_0\mathbb{E}\int_0^T(\|\eta_1(\theta_t\omega)\|_1^2+\|\eta_2(\theta_t\omega)\|_1^2)\mathrm{d}t\Big)+1\Big\}.
\end{aligned}\tag{5.5}$$

Consequently, estimates (5.4) and (5.5) imply (5.1) and (5.2) . □

We claim that the following SPDE is the synchronized model for the two-layer QG dynamics (1.1): the synchronized vorticity q and the corresponding streamfunction u satisfy the following system

$$\frac{\partial}{\partial t}q+J(u,q+\beta y)=\nu\Delta^2u-\frac{r}{2}\Delta u+\frac{1}{2}f(x,y,t)+\frac{\dot{W}_1+\dot{W}_2}{2},\qquad q=\Delta u,\tag{5.6}$$

in the spatial domain O with the Dirichlet boundary condition and initial condition:

$$u\big|_{\partial O}=\Delta u\big|_{\partial O}=0,\qquad u(x,y,0)=u_0(x,y).\tag{5.7}$$

Let $\eta(x,y,t,\omega)$ be the Ornstein-Uhlenbeck process which is defined by the solution of the following linear stochastic partial differential equation

$$\frac{\partial \eta}{\partial t}=\nu(k+1)\Delta\eta+\frac{\dot{W}_1+\dot{W}_2}{2}, \qquad \eta\big|_{\partial O}=0, \tag{5.8}$$

with some initial condition $\eta_0\in L^2$, where $k>0$ is a free *control* parameter. Let the process $\xi(x,y,t,\omega)$ be the solution of the linear elliptic equation

$$\Delta\xi=-\eta\,, \qquad \xi|_{\partial O}=0\,. \tag{5.9}$$

Then $\eta=\frac{1}{2}(\eta_1+\eta_2)$. Using the same change of unknown variable as in Section 3

$$\widetilde{q}:=q-\eta, \qquad \widetilde{u}:=u+\xi,$$

Eq. (5.6) is transformed into the following random PDE

$$\begin{aligned}&\frac{\partial}{\partial t}\Delta u+J(u-\xi,\Delta u+\eta+\beta y)\\ =&\nu\Delta^2u-\frac{r}{2}\Delta(u-\xi)+\frac{1}{2}f(x,y,t)-\nu k\Delta\eta\,,\end{aligned} \tag{5.10}$$

in the spatial domain O with the Dirichlet boundary condition and an appropriate initial condition:

$$u\big|_{\partial O}=\Delta u\big|_{\partial O}=0, \qquad u(x,y,0)=u_0(x,y)+\xi(x,y,0). \tag{5.11}$$

The same argument as in Section 3 allows us to prove that problem (5.10) and (5.11) is well-posed and generates a RDS (θ,φ^∞) in H_0^1. The same argument as in Section 4 allows us to prove that RDS (θ,φ^∞) possesses a compact pullback attractor $\{A^\infty(\omega)\}$. Thus we obtain the following theorem.

Theorem 5.1. (Well-posedness, RDS, pullback attractor, limiting model)
(I) *Let $u_0\in H_0^1$ and $f\in L^2$. Then for all $\omega\in\Omega$ and for all $\tau>0$, the systems* (5.10) and (5.11) *have an unique solution $u(t)$ such that*

$$u\in C\big([0,\infty);H_0^1\big)\cap L^2_{loc}\big(0,\infty;H^2\cap H_0^1\big)\cap L^2_{loc}\big(\tau,\infty;H^3\cap\{u\big|_{\partial O}=\Delta u\big|_{\partial O}=0\}\big).$$

The solution depends continuously on the initial condition $u_0\in H_0^1$.
(II) *In the space H_0^1, problem* (5.10) and (5.11) *generates a RDS (θ,φ^∞) possessing a compact pullback attractor $\{A^\infty(\omega)\}$. Here $\varphi^\infty(t,\omega,u_0)=u(t)$.*

We note that the synchronized QG model (5.6) has the following property.

Lemma 5.2. (I) *Let $u(t)$ be the solution of* (5.10)*. Then $u(t)$ satisfies the following inequality*

$$\sup_{0\le t\le T}\mathbb{E}\|\nabla u(\cdot,t)\|_0^2+\mathbb{E}\int_0^T\|\Delta u(s)\|_0^2\mathrm{d}s\le C_1, \qquad \forall T\in[0,\infty), \tag{5.12}$$

where C_1 is a constant.

(II) *Let $\overline{u}(t)$ and $\widetilde{u}(t)$ be the solution of* (5.10) *with initial data $\overline{u}_0$ and $\widetilde{u}_0$, respectively. Then the following inequality holds:*

$$\sup_{0\le t\le T}\mathbb{E}\|\nabla\{\overline{u}-\widetilde{u}\}(\cdot,t)\|_0^2\le e^{C_2T}\mathbb{E}\|\nabla\{\overline{u}_0-\widetilde{u}_0\}\|_0^2,\qquad \forall T\in[0,\infty), \tag{5.13}$$

where C_2 is a constant.

Proof. The estimate (5.12) can be proved by the same argument as in the proof of estimate (5.2). Let $v=\overline{u}-\widetilde{u}$. Then v satisfies the following equation

$$\frac{\partial}{\partial t}\Delta v+J(v,\Delta\overline{u}+\eta+\beta y)+J(\widetilde{u}-\xi,\Delta v)=\nu\Delta^2 v-\frac{r}{2}\Delta v, \tag{5.14}$$

in the spatial domain O with the Dirichlet boundary condition and an appropriate initial condition:

$$v\big|_{\partial O}=\Delta v\big|_{\partial O}=0,\quad v(x,y,0)=\overline{u}_0(x,y)-\widetilde{u}_0(x,y,0). \tag{5.15}$$

Taking the scalar product (5.14) with v and integration by parts we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v(t)\|_0^2+\nu\|\Delta v(t)\|_0^2+\frac{r}{2}\|\nabla v(t)\|_0^2\\ =&-\int_O J(\widetilde{u}-\xi,v)\Delta v\,\mathrm{d}x\mathrm{d}y\le\|\nabla(\widetilde{u}-\xi)\|_{L^4}\|\nabla v\|_{L^4}\|\Delta v\|_0\\ \le& C\|\nabla(\widetilde{u}-\xi)\|_{L^4}\|\nabla v\|_0^{\frac{1}{2}}\|\Delta v\|_0^{\frac{3}{2}}\\ \le&\frac{\nu}{2}\|\Delta v\|_0^2+C\|\nabla(\widetilde{u}-\xi)\|_{L^4}^4\|\nabla v\|_0^2.\end{aligned} \tag{5.16}$$

By using estimates (5.12), (5.16) and Gronwall inequality, we obtain estimate (5.13). □

Lemma 5.3. *Let ξ_1 and ξ_2 be the solution of* (3.10)*, ξ be the solution of* (5.9)*. Then we have the following estimate*

$$\begin{aligned}&\mathbb{E}\|\nabla(\xi_1-\xi)(\cdot,t)\|_0^2+\mathbb{E}\|\nabla(\xi_2-\xi)(\cdot,t)\|_0^2\\ \le&\frac{1}{16F}\mathbb{E}\|(\eta_1-\eta_2)(\cdot,t)\|_0^2,\qquad\forall t\in[0,\infty).\end{aligned} \tag{5.17}$$

Proof. From (3.10) and (5.9), we get the following system of random partial differential equations

$$\begin{aligned}&\Delta(\xi_1-\xi)-F(\xi_1-\xi_2)=\frac{1}{2}(\eta_2-\eta_1),\\ &\Delta(\xi_2-\xi)-F(\xi_2-\xi_1)=\frac{1}{2}(\eta_1-\eta_2).\end{aligned} \tag{5.18}$$

Taking the scalar product (5.18) with $(\xi_1-\xi,\xi_2-\xi)$, and integration by parts we have

$$\begin{aligned}&\|\nabla(\xi_1-\xi)\|_0^2+\|\nabla(\xi_2-\xi)\|_0^2+F\|\xi_1-\xi_2\|_0^2\\=&\frac{1}{2}(\eta_1-\eta_2,\xi_1-\xi_2)\le F\|\xi_1-\xi_2\|_0^2+\frac{1}{16F}\|\eta_1-\eta_2\|_0^2.\end{aligned}\tag{5.19}$$

This estimate (5.19) implies (5.17). □

Let $u_1=\psi_1-u$ and $u_2=\psi_2-u$ be the differences in the streamfunctions, between the top layer of (3.12) and the synchronized model (5.10), and between the bottom layer and the synchronized model, respectively. By (3.12) and (5.10), we get the following coupled system of random partial differential equations

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\Delta u_1-F(u_1-u_2)\big)+J\big(u_1-(\xi_1-\xi),q_1+\eta_1+\beta y\big)+J\big(u-\xi,\Delta u_1-F(u_1-u_2)+\eta_1-\eta\big)\\=&\nu\Delta^2u_1+\frac{r}{2}\Delta(u-\xi)+\frac{f}{2}-\nu F(\Delta\xi_1-\Delta\xi_2)-\nu k\Delta(\eta_1-\eta),\end{aligned}\tag{5.20a}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\Delta u_2-F(u_2-u_1)\big)+J\big(u_2-(\xi_2-\xi),q_2+\eta_2+\beta y\big)+J\big(u-\xi,\Delta u_2-F(u_2-u_1)+\eta_2-\eta\big)\\=&\nu\Delta^2u_2-r\Delta(u_2-\xi_2+\xi)-\frac{r}{2}\Delta(u-\xi)-\frac{f}{2}-\nu F(\Delta\xi_2-\Delta\xi_1)-\nu k\Delta(\eta_2-\eta),\end{aligned}\tag{5.20b}$$

with the Dirichlet boundary condition and initial data

$$u_1\big|_{\partial O}=\Delta u_1\big|_{\partial O}=0,\qquad u_2\big|_{\partial O}=\Delta u_2\big|_{\partial O}=0,\tag{5.21a}$$

$$u_1(x,y,0)=\psi_{01}(x,y)-u_0(x,y)+\xi_1(x,y,0,\omega)-\xi(x,y,0,\omega),\tag{5.21b}$$

$$u_2(x,y,0)=\psi_{02}(x,y)-u_0(x,y)+\xi_2(x,y,0,\omega)-\xi(x,y,0,\omega),\tag{5.21c}$$

where η_1 and η_2 are the stationary solutions to (3.6), η is the stationary solutions to (5.8), ξ_1 and ξ_2 solve (3.10) in $H^2\cap H_0^1$, and ξ solves (5.9) in $H^2\cap H_0^1$.

Theorem 5.2. (Limit transition on finite time intervals) *Assume that both models start at the same initial data $\psi_{01}=\psi_{02}=u_0$. Then we have the following convergence results on any finite time interval $[0,T]$:*

$$\lim_{F\to\infty}\sup_{0\le t\le T}\mathbb{E}\Big(\|\nabla u_1(\cdot,t)\|_0^2+\|\nabla u_2(\cdot,t)\|_0^2\Big)=0,\tag{5.22}$$

$$\lim_{F\to\infty}\mathbb{E}\Big(\|\nabla u_1(\cdot,T)\|_0^2+\|\nabla u_2(\cdot,T)\|_0^2\Big)=0,\qquad\forall T\in[0,\infty),\tag{5.23}$$

$$\lim_{F\to\infty}\mathbb{E}\int_0^T\Big(\|\Delta u_1(s)\|_0^2+\|\Delta u_2(s)\|_0^2\Big)\mathrm{d}s=0,\qquad\forall T\in[0,\infty),\tag{5.24}$$

where F is defined in (1.3). *That is, as the layer depth $h\to0$ or as density gradient across the two layers $\rho_2-\rho_1\to0$, the two-layer model* (3.12) *synchronizes to the averaged model* (5.10) *on finite time intervals.*

Proof. Taking the scalar product (5.20) with (u_1,u_2) and integration by parts we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\nabla u_1(t)\|_0^2+\|\nabla u_2(t)\|_0^2+F\|u_1(t)-u_2(t)\|_0^2\Big\}\\
&\quad+\nu\Big\{\|\Delta u_1(t)\|_0^2+\|\Delta u_2(t)\|_0^2\Big\}+r\|\nabla u_2(t)\|_0^2\\
=&-\int_O\Big\{J(\xi_1-\xi,q_1+\eta_1+\beta y)u_1+J(\xi_2-\xi,q_2+\eta_2+\beta y)u_2\\
&\quad+J(u-\xi,u_1)\Delta u_1+J(u-\xi,u_2)\Delta u_2+\frac{1}{2}J(u-\xi,u_1-u_2)(\eta_1-\eta_2)\Big\}\mathrm{d}x\mathrm{d}y\\
&\quad+\left(\nu F\Delta(\xi_1-\xi_2)-\frac{f}{2},u_1-u_2\right)_0+\frac{1}{2}\nu k(\Delta(\eta_1-\eta_2),u_1-u_2)_0\\
&\quad-\frac{r}{2}(\Delta(u-\xi),u_1-u_2)_0-r(\Delta(\xi_2-\xi),u_2)_0. \qquad (5.25)
\end{aligned}
$$

Employing estimates (3.11), (5.1), (5.2), (5.12) and (5.17), we know that

$$
\begin{aligned}
&\mathbb{E}\left(\nu F\Delta(\xi_1-\xi_2)-\frac{f}{2},u_1-u_2\right)_0-r\mathbb{E}(\Delta(\xi_2-\xi),u_2)_0\\
=&\mathbb{E}\left(\nu F\Delta(\xi_1-\xi_2)-\frac{f}{2},\psi_1-\psi_2\right)_0+r\mathbb{E}(\nabla(\xi_2-\xi),\nabla u_2)_0\\
&\longrightarrow 0,\quad \text{as}\quad F\to\infty, \qquad (5.26)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\int_0^t\left\{\frac{1}{2}\nu k(\Delta(\eta_1-\eta_2),u_1-u_2)_0-\frac{r}{2}(\Delta(u-\xi),u_1-u_2)_0\right\}\mathrm{d}s\\
=&\mathbb{E}\int_0^t\left\{-\frac{1}{2}\nu k\big(\nabla(\eta_1-\eta_2),\nabla(\psi_1-\psi_2)\big)_0-\frac{r}{2}(\Delta(u-\xi),\psi_1-\psi_2)_0\right\}\mathrm{d}s\\
&\longrightarrow 0,\quad \text{as}\quad F\to\infty,\quad \forall\, t\in[0,T], \qquad (5.27)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\int_0^t\int_O\{J(\xi_1-\xi,q_1+\eta_1+\beta y)u_1+J(\xi_2-\xi,q_2+\eta_2+\beta y)u_2\}\\
=&\mathbb{E}\int_0^t\int_O\{J(\xi_1-\xi,\Delta\psi_1+\eta_1+\beta y)u_1+J(\xi_2-\xi,\Delta\psi_2+\eta_2+\beta y)u_2+FJ(\xi_1-\xi_2,u_2)(\psi_1-\psi_2)\}\\
\le&\mathbb{E}\int_0^t\{\|\nabla(\xi_1-\xi)\|_{L^4}\|\nabla u_1\|_{L^4}\|\Delta\psi_1+\eta_1+\beta y\|_0\\
&\quad+\|\nabla(\xi_2-\xi)\|_{L^4}\|\nabla u_2\|_{L^4}\|\Delta\psi_2+\eta_2+\beta y\|_0+F\|\xi_1-\xi_2\|_0\|\nabla u_2\|_{L^4}\|\nabla(\psi_1-\psi_2)\|_{L^4}\}\mathrm{d}s\\
\le&\mathbb{E}\int_0^t\Big\{K_2\|\nabla(\xi_1-\xi)\|_0^{\frac{1}{2}}\|\Delta(\xi_1-\xi)\|_0^{\frac{1}{2}}\|\nabla u_1\|_{L^4}\|\Delta\psi_1+\eta_1+\beta y\|_0\\
&\quad+K_2\|\nabla(\xi_2-\xi)\|_0^{\frac{1}{2}}\|\Delta(\xi_2-\xi)\|_0^{\frac{1}{2}}\|\nabla u_2\|_{L^4}\|\Delta\psi_2+\eta_2+\beta y\|_0\\
&\quad+K_3F\|\xi_1-\xi_2\|_0\|\nabla u_2\|_{L^4}\|\Delta(\psi_1-\psi_2)\|_0^{\frac{3}{4}}\|\psi_1-\psi_2\|_0^{\frac{1}{4}}\Big\}\mathrm{d}s\\
&\longrightarrow 0,\quad \text{as}\quad F\to\infty,\quad \forall\, t\in[0,T], \qquad (5.28)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\int_0^t\int_O \frac{1}{2}J(u-\xi,u_1-u_2)(\eta_1-\eta_2)\mathrm{d}x\mathrm{d}y\mathrm{d}s\\
=&\mathbb{E}\int_0^t\int_O \frac{1}{2}J(u-\xi,\psi_1-\psi_2)(\eta_1-\eta_2)\mathrm{d}x\mathrm{d}y\mathrm{d}s\\
\le&\frac{1}{2}\mathbb{E}\int_0^t \|\eta_1-\eta_2\|_0\|\nabla(u-\xi)\|_{L^4}\|\nabla(\psi_1-\psi_2)\|_{L^4}\mathrm{d}s\\
\le&\frac{K_3}{2}\mathbb{E}\int_0^t \|\eta_1-\eta_2\|_0\|\nabla(u-\xi)\|_{L^4}\|\Delta(\psi_1-\psi_2)\|_0^{\frac{3}{4}}\|\psi_1-\psi_2\|_0^{\frac{1}{4}}\mathrm{d}s\\
&\to 0,\quad \text{as}\quad F\to\infty,\quad \forall\, t\in[0,T],
\end{aligned}
\tag{5.29}
$$

$$
\begin{aligned}
&\int_O \{J(u-\xi,u_1)\Delta u_1+J(u-\xi,u_2)\Delta u_2\}\mathrm{d}x\mathrm{d}y\\
\le&\|\nabla(u-\xi)\|_{L^4}\{\|\nabla u_1\|_{L^4}\|\Delta u_1\|_0+\|\nabla u_2\|_{L^4}\|\Delta u_2\|_0\}\\
\le&K_2\|\nabla(u-\xi)\|_{L^4}\left\{\|\nabla u_1\|_0^{\frac{1}{2}}\|\Delta u_1\|_0^{\frac{3}{2}}+\|\nabla u_2\|_0^{\frac{1}{2}}\|\Delta u_2\|_0^{\frac{3}{2}}\right\}\\
\le&\frac{\nu}{2}\{\|\Delta u_1\|_0^2+\|\Delta u_2\|_0^2\}+K_4\|\nabla(u-\xi)\|_{L^4}^4\{\|\nabla u_1\|_0^2+\|\nabla u_2\|_0^2\},
\end{aligned}
\tag{5.30}
$$

where constants K_2, K_3 and K_4 are independent of F. Then, $\forall\epsilon>0$, $\exists F_0\gg 1$, such that $\forall F>F_0$, we have that

$$
\begin{aligned}
&\mathbb{E}\|\nabla u_1(t)\|_0^2+\|\nabla u_2(t)\|_0^2+\mathbb{E}\int_0^t\left\{\|\Delta u_1(s)\|_0^2+\|\Delta u_2(s)\|_0^2\right\}\mathrm{d}s\\
\le&\epsilon+K_5\mathbb{E}\int_0^t\left\{\|\nabla u_1\|_0^2+\|\nabla u_2\|_0^2\right\}\mathrm{d}s,\qquad \forall t\in[0,T],
\end{aligned}
$$

where constant K_5 is independent of F. By Gronwall inequality, we obtain that

$$
\begin{aligned}
&\sup_{0\le t\le T}\mathbb{E}\|\nabla u_1(t)\|_0^2+\|\nabla u_2(t)\|_0^2+\mathbb{E}\int_0^T\left\{\|\Delta u_1(s)\|_0^2+\|\Delta u_2(s)\|_0^2\right\}\mathrm{d}s\\
\le&\epsilon e^{K_5T},\qquad \forall\epsilon>0,\quad F>F_0.
\end{aligned}
\tag{5.31}
$$

Estimate (5.31) implies (5.22), (5.23) and (5.24). The proof of this Theorem is therefore complete. □

We are now ready to present our main conclusion on synchronization.

Theorem 5.3. (Synchronization) *Assume that* $W_1=W_2$. *Let* $\{A_{1F}(\omega)\}$ *and* $\{A_{2F}(\omega)\}$ *be the global random pullback attractor for the RDSs* (θ,φ_{1F}) *and* (θ,φ_{2F}) *generated by* (3.12), *respectively. Then*

$$
\lim_{F\to\infty}\sup\left\{\mathrm{dist}_{H_0^1}\big(\psi_1,A^\infty(\omega)\big):\psi_1\in A_{1F}(\omega)\right\}=0,
\tag{5.32}
$$

$$
\lim_{F\to\infty}\sup\left\{\mathrm{dist}_{H_0^1}\big(\psi_2,A^\infty(\omega)\big):\psi_2\in A_{2F}(\omega)\right\}=0.
\tag{5.33}
$$

Here $\{A^\infty(\omega)\}$ *is the random pullback attractor for the RDS* (θ,φ^∞) *generated by* (5.10).

Proof. We continue the line of research introduced in [24]. $W_1 = W_2$ implies that metric dynamical systems θ are same in RDSs (θ,φ_{1F}), (θ,φ_{2F}) and (θ,φ^∞).

Assume that (5.32) does not hold for some $\omega \in \Omega$. Then there exists a sequence $\{F_n\}$ with $F_n \to \infty$ and a sequence $\psi_{1n} \in A_{1F_n}(\omega)$ such that

$$\mathrm{dist}_{H_0^1}\big(\psi_{1n}, A^\infty(\omega)\big) \geq \delta > 0, \qquad \forall n = 1,2,\cdots. \tag{5.34}$$

By the invariance property of the attractor $A_{1F_n}(\omega)$, for every $t>0$ there exists $v_n^t \in A_{1F_n}(\theta_{-t}\omega)$ such that $\psi_{1n} = \varphi_{1F_n}(t,\theta_{-t}\omega,v_n^t)$. Since $A_{1F_n}(\omega)$ is compact and estimate (4.5) holds, we can assume that there exist ψ_{1*} and v_*^t in H_0^1 such that

$$\lim_{n\to\infty} \|\nabla(\psi_{1n} - \psi_{1*})\|_0 = 0, \qquad \lim_{n\to\infty} \|\nabla(v_n^t - v_*^t)\|_0 = 0. \tag{5.35}$$

Therefore, if we show that $\psi_{1*} \in A^\infty(\omega)$, then we obtain a contradiction to (5.34).

From Lemma 5.2, Theorem 5.2 and (5.35), it follows that

$$\begin{aligned}
&\mathbb{E}\|\nabla\big(\psi_{1n} - \varphi^\infty(t,\theta_{-t}\omega,v_*^t)\big)\|_0 \\
\leq &\mathbb{E}\|\nabla\big(\varphi_{1F_n}(t,\theta_{-t}\omega,v_n^t) - \varphi^\infty(t,\theta_{-t}\omega,v_n^t)\big)\|_0 \\
&\quad + \mathbb{E}\|\nabla(\varphi^\infty\big(t,\theta_{-t}\omega,v_n^t) - \varphi^\infty(t,\theta_{-t}\omega,v_*^t)\big)\|_0 \qquad \longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Then $\psi_* = \varphi^\infty(t,\theta_{-t}\omega,v_*^t)$. However, it follows from (4.5) and (5.35) that $v_*^t \in B_0(\omega)$, where

$$B_0(\omega) = \Big\{\psi_1 \in H_0^1 : \|\nabla\psi_1\|_0^2 \leq R(\omega)\Big\}.$$

Thus we have that

$$\psi_* \in \varphi^\infty\big(t,\theta_{-t}\omega,B_0(\omega)\big) \text{ for every } t>0.$$

Since $\varphi^\infty\big(t,\theta_{-t}\omega,B_0(\omega)\big) \to A^\infty(\omega)$ as $t\to\infty$, this implies that $\psi_* \in A^\infty(\omega)$. This contradiction to (5.34) implies (5.32). Moreover, (5.33) can be proved by the same argument. □

6 Conclusions

Now we conclude this paper by stating our main results concerning stochastic two-layer quasi-geostrophic model (1.1). By using inverse transformation we define the cocycle $\overline{\varphi}^\infty$ for (5.6) by the formula

$$\overline{\varphi}^\infty(t,\omega,\cdot) = \big(R_m^\infty\big)^{-1}(\theta_t\omega) \circ \varphi^\infty\big(t,\omega,R_m^\infty(\omega)\cdot\big),$$

where the RDS (θ,φ^∞) is generated by (5.10), $R_m^\infty(\omega): H_0^1 \to H_0^1$ is a random mapping of the form

$$R_m^\infty(\omega)U = U + \xi, \qquad U \in H_0^1.$$

All statements concerning RDSs $(\theta,\overline{\varphi})$, $(\theta,\overline{\varphi}_{1F})$, $(\theta,\overline{\varphi}_{2F})$ and $(\theta,\overline{\varphi}^\infty)$ can be easily reformulated as statements concerning the RDSs (θ,φ), (θ,φ_{1F}), (θ,φ_{2F}) and (θ,φ^∞), respectively. From Theorems 3.1, 4.2, 5.1, 5.2 and 5.3, we obtain the following theorem:

Theorem 6.1. (Well-posedness, pullback attractor and synchronization)

1. *Let* $q_0\in\mathbf{H}^{-1}$ *and* $f\in L^2$*. Then for P-a.e.* $\omega\in\Omega$ *and for all* $\tau>0$*, the system* (1.1)– (1.4) *has a unique solution* $\{q(t),\psi(t)\}$ *such that*

$$q\in C\big([0,\infty);\mathbf{H}^{-1}\big)\cap L^2_{loc}\big(0,\infty;\mathbf{L}^2\big)\cap L^2_{loc}\big(\tau,\infty;\mathbf{H}^1_0\big).$$

The function ψ *associated to* q *by* (1.2) *satisfies*

$$\psi\in C\big([0,\infty);\mathbf{H}^1_0\big)\cap L^2_{loc}\big(0,\infty;\mathbf{H}^2\cap\mathbf{H}^1_0\big)\cap L^2_{loc}\big(\tau,\infty;\mathbf{H}^3\cap\{\psi|_{\partial O}=\Delta\psi|_{\partial O}=0\}\big).$$

The solution depends continuously on the initial condition $q_0\in\mathbf{H}^{-1}$*.*

2. *Problem* (1.1)–(1.4) *generates a RDS* $(\theta,\overline{\varphi})$ *in space* $\mathbf{H}^{-1}$ *with the metric dynamical system* θ *generated by the Wiener process* $W=(W_1,W_2)$ *in*

$$\Big(C_0(\mathbb{R},\mathbf{H}^{-1}),\mathcal{B}(C_0(\mathbb{R},\mathbf{H}^{-1})),\mathbb{P}\Big),$$

and the cocycle $\overline{\varphi}$ *defined by the formula* $\overline{\varphi}(t,\omega,q_0)=q(t)$*.*

3. *Let* $\psi(t)=\big(\psi_1(t),\psi_2(t)\big)$ *be the unique solution of problem* (1.1)–(1.4) *with initial data* $\psi_0=(\psi_{01},\psi_{01})$*. Then problem* (1.1)– (1.4) *generates a RDS* $(\theta,\overline{\varphi}_{1F})$ *in space* H^1_0 *with the metric dynamical system* θ *generated by the Wiener process* W_1 *in* $(C_0(\mathbb{R},H^1_0),\mathcal{B}(C_0(\mathbb{R},H^1_0)),\mathbb{P})$ *and the cocycle* $\overline{\varphi}_{1F}$ *defined by the formula* $\overline{\varphi}_{1F}(t,\omega,\psi_{01})=\psi_1(t)$*.*

4. *Let* $\psi(t)=\big(\psi_1(t),\psi_2(t)\big)$ *be the unique solution of problem* (1.1)–(1.4) *with initial data* $\psi_0=(\psi_{02},\psi_{02})$*. Then problem* (1.1)– (1.4) *generates a RDS* $(\theta,\overline{\varphi}_{2F})$ *in space* H^1_0 *with the metric dynamical system* θ *generated by the Wiener process* W_2 *in* $(C_0(\mathbb{R},H^1_0),\mathcal{B}(C_0(\mathbb{R},H^1_0)),\mathbb{P})$ *and the cocycle* $\overline{\varphi}_{2F}$ *defined by the formula* $\overline{\varphi}_{2F}(t,\omega,\psi_{02})=\psi_2(t)$*.*

5. *Let* $u_0\in H^1_0$ *and* $f\in L^2$*. Then for P-a.e.* $\omega\in\Omega$ *and for all* $\tau>0$*, the system* (5.6) and (5.7) *has a unique solution* $u(t)$ *such that*

$$u\in C([0,\infty);H^1_0)\cap L^2_{loc}(0,\infty;H^2\cap H^1_0)\cap L^2_{loc}(\tau,\infty;H^3\cap\{u|_{\partial O}=\Delta u|_{\partial O}=0\}).$$

The solution depends continuously on the initial condition $u_0\in H^1_0$*.*

6. *Problem* (5.6) and (5.7) *generates a RDS* $(\theta,\overline{\varphi}^\infty)$ *in space* H^1_0 *with the metric dynamical system* θ *generated by the Wiener process* $(W_1+W_2)/2$ *in* $(C_0(\mathbb{R},H^1_0),\mathcal{B}(C_0(\mathbb{R},H^1_0)),\mathbb{P})$ *and the cocycle* $\overline{\varphi}^\infty$ *defined by the formula* $\overline{\varphi}^\infty(t,\omega,u_0)=u(t)$*.*

7. *Assume that both models* (1.1) *and* (5.6) *start at the same initial data* $\psi_{01}=\psi_{02}=u_0$*. Then we have the following convergence results on any finite time interval* $[0,T]$*:*

$$\lim_{F\to\infty}\sup_{0\le t\le T}\mathbb{E}(\|\nabla\{\psi_1-u\}(\cdot,t)\|_0^2+\|\nabla\{\psi_2-u\}(\cdot,t)\|_0^2)=0.$$

8. *The RDS* $(\theta,\overline{\varphi})$ *has a compact pullback attractor* $\{\overline{A}(\omega)\}$ *in phase space* $\mathbf{H}^{-1}$ *for P-a.e.* $\omega\in\Omega$*. The RDSs* $(\theta,\overline{\varphi}_{1F})$*,* $(\theta,\overline{\varphi}_{2F})$ *and* $(\theta,\overline{\varphi}^\infty)$ *have compact pullback attractors* $\{\overline{A}_{1F}(\omega)\}$*,* $\{\overline{A}_{2F}(\omega)\}$ *and* $\{\overline{A}^\infty(\omega)\}$ *in phase space* H^1_0 *for P-a.e.* $\omega\in\Omega$*, respectively.*

9. *Assume that* $W_1 = W_2$. *Then attractors* $\{\overline{A}_{1F}(\omega)\}$ *and* $\{\overline{A}_{2F}(\omega)\}$ *are upper semicontinuous as* $F \to \infty$ *in the sense that*

$$\lim_{F\to\infty} \sup_{\psi_1 \in \overline{A}_{1F}(\omega)} \inf_{u\in \overline{A}^{\infty}(\omega)} \|\nabla(\psi_1 - u)\|_0 = 0, \qquad \forall \omega \in \Omega,$$
$$\lim_{F\to\infty} \sup_{\psi_2 \in \overline{A}_{2F}(\omega)} \inf_{u\in \overline{A}^{\infty}(\omega)} \|\nabla(\psi_2 - u)\|_0 = 0, \qquad \forall \omega \in \Omega.$$

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