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Nonlinear Hyperbolic-Parabolic System Modeling Some Biological Phenomena

WU Shaohua* and CHEN Hua

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.

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Abstract. In this paper, we study a nonlinear hyperbolic-parabolic system modeling some biological phenomena. By semigroup theory and Leray-Schauder fixed point argument, the local existence and uniqueness of the weak solutions for this system are proved. For the spatial dimension $N=1$, the global existence of the weak solution will be established by the bootstrap argument.

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1 Introduction

The movement behavior of most species is guided by external signals, such as, amoeba moving upwards chemical gradients, insects orienting towards light sources. Let $u(x,t)$ and $v(x,t)$ represent the population of an organism and an external signal at place $x\in\Omega\subset R^N$ and time t respectively. It is well known that the external signal is produced by the individuals, which is described by a nonlinear function $g(v,u)$. If the spatial spread of the external signal is driven by diffusion, the full system for u and v reads (see [1-3])

$$u_t=\nabla(d\nabla u-\chi(v)\nabla v\cdot u), \tag{1.1}$$
$$v_t=d\Delta v+g(v,u). \tag{1.2}$$

Depending on what the type of the external stimulus is, one distinguishes among chemotaxis, haptotaxis, aerotaxis, geotaxis and others. Taking in account of that the external stimulus were based on the light (or the electromagnetic wave), Chen and Wu [4]

*Corresponding author. *Email addresses:* `wush8@sina.com` (S. Wu), `chenhua@whu.edu.cn` (H. Chen)

introduced a hyperbolic-parabolic-type chemotaxis system as follows:

$$u_t=\nabla(d\nabla u-\chi(v)\nabla v\cdot u), \tag{1.3}$$
$$v_{tt}=d\Delta v+g(v,u). \tag{1.4}$$

In [4], Chen and Wu considered the systems (1.3)–(1.4) with

$$g(v,u)=-v+f(u)$$

on a bounded open domain Ω with smooth boundary. For the Neumann boundary problem, they showed the local existence and uniqueness of the solutions, and also achieved the global existence and uniqueness of the solutions of systems (1.3)–(1.4) for $N=1$. In this paper, taking our attention to the case that $g(v,u)$ is nonlinear, on a N-D, compact Riemanian manifold M without boundary, we will obtain some results similar to those given in [4].

Throughout this article, we assume that

$$1<\sigma<2, \tag{1.5}$$
$$N<2\sigma<N+2, \tag{1.6}$$
$$\frac{N}{\sigma-1}<p<\frac{2N}{N-2(\sigma-1)}, \tag{1.7}$$
$$p\geq 4, \tag{1.8}$$

where σ,p are some constants.

It is easy to check that there exist some constants σ and p such that the above four conditions can be satisfied simultaneously for $1\leq N\leq 3$. In fact, we take $\sigma=11/8$ for $N=1$, $\sigma=13/8$ for $N=2$ and $\sigma=15/8$ for $N=3$, which satisfy (1.5) and (1.6), at the same time there exists some constant p such that (1.7) and (1.8) are stratified. The conditions of (1.5)-(1.8) are crucial to our proof of the main results, since the conditions ensure the relevant Sobolev theorems. Set

$$\begin{aligned}
&X_{t_0}=C([0,t_0],H^\sigma(M)), \quad X_\infty=C([0,+\infty),H^\sigma(M)),\\
&Y_{t_0}=C([0,t_0],H^2(M))\cap C^1([0,t_0],H^1(M)),\\
&Y_\infty=C([0,+\infty),H^2(M))\cap C^1([0,+\infty),H^1(M)),\\
&Z_{t_0}=C^1([0,t_0],L^2(M)), \quad Z_\infty=C^1([0,\infty),L^2(M))),\\
&W_{t_0}=C^2([0,t_0],L^2(M)), \quad W_\infty=C^2([0,\infty),L^2(M)),
\end{aligned}$$

where M is a N-D, compact Riemannian manifold without boundary.

2 Main results

Consider

$$\begin{cases} u_t = \nabla(\nabla u - \chi u \nabla v), \\ v_{tt} = \Delta v + g(v,u), \\ u(0,\cdot) = u_0,\ v(0,\cdot) = \varphi,\ v_t(0,\cdot) = \psi, \end{cases} \tag{2.1}$$

where χ is a constant.

Our main results are as follows:

Theorem 2.1. *Under the conditions of* (1.5) *and* (1.6), *for each* $u_0 \in H^\sigma(M)$, $\varphi \in H^2(M)$ *and* $\psi \in H^1(M)$, (2.1) *admits a unique local in time solution* $(u,v) \in X_{t_0} \cap Z_{t_0} \times Y_{t_0} \cap W_{t_0}$ *provided that*

$$g(v,u) = h(v) + f(u),$$

where $h(x), f(x) \in C_0^2(R^1), h(0) = 0$. *If* $N=1$, $\sigma = 5/4$ *and* $u_0 \geq 0$, *then* (2.1) *has a global solution* $(u,v) \in X_\infty \cap Z_\infty \times Y_\infty \cap W_\infty$.

Theorem 2.2. *Suppose* (1.5), (1.7) *and* (1.8) *are satisfied, then for each* $u_0 \in H^\sigma(M)$, $\varphi \in H^2(M)$ *and* $\psi \in H^1(M)$, (2.1) *has a unique local in time solution* $(u,v) \in X_{t_0} \cap Z_{t_0} \times Y_{t_0} \cap W_{t_0}$ *provided that* $g(v,u) = \alpha u v$.

3 The case of $g(v,u) = h(v) + f(u)$

First, we consider two problems below:

$$\begin{cases} u_t = \nabla(\nabla u - \chi u \nabla v), \\ u(0,\cdot) = u_0, \end{cases} \tag{3.1}$$

$$\begin{cases} v_{tt} = \Delta v + h(v) + f(u), \\ v(0,\cdot) = \varphi, \quad v_t(0,\cdot) = \psi. \end{cases} \tag{3.2}$$

Lemma 3.1. *If* $u_0 \in X_{t_0}$ *and* $v \in Y_{t_0}$, σ *and* N *satisfy the conditions* (1.5) *and* (1.6), *then the problem* (3.1) *has a unique solution*

$$u \in X_{t_0} \cap Z_{t_0}.$$

Lemma 3.2. *Under the conditions* (1.5) *and* (1.6), *if* $u \in X_{t_0}$ *is a solution of* (3.1), *then there exists a constant* C *which is independent of* t_0, *such that*

$$\|u\|_{X_{t_0}} \leq C\|u_0\|_{\sigma,2} + Ct_0^{1-\frac{\sigma}{2}} \|v\|_{Y_{t_0}} \cdot \|u\|_{X_{t_0}}, \tag{3.3}$$

where $\|\cdot\|_{k,p}$ *is the norm of the Sobolev space* $W^{k,p}$.

The proof of the above lemmas is identical to the proof of [4, Lemma 3.2–3.3].

Lemma 3.3. *If $\varphi\in H^2(M),\psi\in H^1(M), f(u)\in L^2(0,T;H^1(M))$ and $h(x)\in C_0^2(R^1)$, then there exists a unique local in time solution v for* (3.2) *satisfying*

$$\begin{cases} v\in C([0,t_0];H^2(M)),\\ v_t\in C([0,t_0];H^1(M)),\\ v_{tt}\in L^2(0,t_0;L^2(M)),\end{cases} \tag{3.4}$$

and

$$\begin{aligned}&\|v(t,\cdot)\|^2_{H^2(M)}+\|v_t(t,\cdot)\|^2_{H^1(M)}\\ \le& Ce^{t_0}\Big(\|\varphi\|^2_{H^2(M)}+\|\psi\|^2_{H^1(M)}+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^1(M)}\mathrm{d}\tau\Big),\qquad 0\le t\le t_0.\end{aligned} \tag{3.5}$$

Proof. We verify this lemma by the Leray-Schauder's fixed point argument. Take $\varepsilon\in[0,1]$ and consider

$$\begin{cases} v_{tt}=\Delta v+\varepsilon h(v)+\varepsilon f(u),\\ v(0,\cdot)=\varepsilon\varphi,\quad v_t(0,\cdot)=\varepsilon\psi.\end{cases} \tag{3.6}$$

Choose $X=L^2(0,t_0;H^1(M))$. Then X is a Banach space. For each $w\in X$, consider

$$\begin{cases} v_{tt}=\Delta v+\varepsilon h(w)+\varepsilon f(u),\\ v(0,\cdot)=\varepsilon\varphi,\quad v_t(0,\cdot)=\varepsilon\psi.\end{cases} \tag{3.7}$$

It follows that there is a unique solution v for (3.7) from the hyperbolic regularization of the equation (cf. [4, Lemma 3.1]), and v is subjected to

$$v\in C([0,t_0];H^2(M)),\quad v_t\in C([0,t_0];H^1(M)),\quad v_{tt}\in L^2(0,t_0;L^2(M)),$$

and

$$\begin{aligned}\|v(t,\cdot)\|^2_{H^2(M)}+\|v_t(t,\cdot)\|^2_{H^1(M)}\le& C\varepsilon^2e^{t_0}\Big(\|\varphi\|^2_{H^2(M)}+\|\psi\|^2_{H^1(M)}\\ &+\int_0^{t_0}\|h(w(\tau,\cdot))\|^2_{H^1}\mathrm{d}\tau+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^1(M)}\mathrm{d}\tau\Big),\qquad 0\le t\le t_0.\end{aligned} \tag{3.8}$$

Define

$$T:X\times[0,1]\to X,\quad T(w,\varepsilon)=v(w),$$

where $v(w)$ denotes the unique solution of (3.7). It is obvious that T is compact, and $w\in X$, $T(w,0)=0$.

If $T(w,\varepsilon)=w$, then (3.8) implies

$$\begin{aligned}\|w(t,\cdot)\|^2_{H^2(M)}\le & C\varepsilon^2 e^{t_0}\left(\|\varphi\|^2_{H^2(M)}+\|\psi\|^2_{H^1(M)}+\int_0^{t_0}\|h(w(\tau,\cdot))\|^2_{H^1}\mathrm{d}\tau\right.\\ &\left.+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^1(M)}\mathrm{d}\tau\right)\\ \le & C\varepsilon^2 e^{t_0}\left(\|\varphi\|^2_{H^2}+\|\psi\|^2_{H^1}+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^1}\mathrm{d}\tau\right)\\ &+C\varepsilon^2 e^{t_0}\cdot t_0\cdot\sup_{0\le\tau\le t_0}\|h(w(\tau,\cdot))\|^2_{H^1},\qquad 0\le t\le t_0.\end{aligned}\tag{3.9}$$

Noting that $h(0)=0$, we have

$$\|h(w)\|^2_{H^1}\le\|h\|^2_{c^2}\cdot\|w\|^2_{H^1}\le\|h\|^2_{c^2}\cdot\|w\|^2_{H^2}.$$

Consequently,

$$\begin{aligned}\sup_{0\le t\le t_0}\|w(t,\cdot)\|^2_{H^2(M)}\le & C\varepsilon^2 e^{t_0}\left(\|\varphi\|^2_{H^2(M)}+\|\psi\|^2_{H^1(M)}+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^1(M)}\mathrm{d}\tau\right)\\ &+C\varepsilon^2 e^{t_0}\cdot t_0\cdot\|h\|^2_{C^2}\sup_{0\le\tau\le t_0}\|w(\tau,\cdot)\|^2_{H^1}.\end{aligned}\tag{3.10}$$

Take t_0 small enough, we have

$$\sup_{0\le t\le t_0}\|w(t,\cdot)\|^2_{H^2(M)}\le C\varepsilon^2 e^{t_0}\left(\|\varphi\|^2_{H^2(M)}+\|\psi\|^2_{H^1(M)}+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^1(M)}\mathrm{d}\tau\right),$$

namely

$$\|w\|^2_X\le ce^{t_0}\left(\|\varphi\|^2_{H^2(M)}+\|\psi\|^2_{H^1(M)}+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^1(M)}\mathrm{d}\tau\right),\tag{3.11}$$

for each w ($T(w,\varepsilon)=w$, $\varepsilon\in[0,1]$).

Now Leray-Schauder fixed point theorem indicates that the operator $T(v,1)$ has a fixed point v, which gives our lemma. □

The proof of Theorem 2.1. For $g\in X_{t_0}$, $g(0)=u_0$, let $v=v(g)$ solves

$$\begin{cases}v_{tt}=\Delta v+h(v)+f(g),\\ v(0,\cdot)=\varphi,\quad v_t(0,\cdot)=\psi.\end{cases}\tag{3.12}$$

Then Lemma 3.3 indicates $v\in Y_{t_0}$ and

$$\|v(t,\cdot)\|^2_{H^2(M)}\le Ce^{t_0}\left(\|\varphi\|^2_{H^2(M)}+\|\psi\|^2_{H^1(M)}+\int_0^{t_0}\|f(g(\tau,\cdot))\|^2_{H^1(M)}\mathrm{d}\tau\right),\quad 0\le t\le t_0.\tag{3.13}$$

Furthermore, for above v, i.e., the solution of (3.12) consider

$$\begin{cases} u_t=\nabla(\nabla u-\chi u\nabla v), \\ u(0,\cdot)=u_0=g(0). \end{cases} \tag{3.14}$$

It follows that (3.14) admits a unique solution from Lemma 3.1. Let $u=u(v(g))$ represent the solution of (3.14). Lemma 3.1 also shows that

$$G:\ X_{t_0}\to X_{t_0},\quad Gg=u(v(g)).$$

Set

$$B_M=\Big\{g\in X_{t_0}\Big|g(0)=u_0,\|g(t,\cdot)\|_{\sigma,2}\le M,0\le t\le t_0\Big\},$$

where

$$M=2C\|u_0\|_{\sigma,2}+1,\qquad C\ge 1$$

is given by (3.3). It is easy to check that G maps B_M into B_M.

Assume $g_1,g_2\in X_{t_0}$, and let $v_i(i=1,2)$ denote the corresponding solutions of (3.12). We have

$$\begin{aligned} &Gg_1-Gg_2=u_1-u_2\\ =&-\chi\int_0^t T(t-s)u_1\Delta v_1\mathrm{d}s-\chi\int_0^t T(t-s)\nabla u_1\nabla v_1\mathrm{d}s\\ &+\chi\int_0^t T(t-s)u_2\nabla v_2\mathrm{d}s+\chi\int_0^t T(t-s)\nabla u_2\nabla v_2\mathrm{d}s\\ =&-\chi\int_0^t T(t-s)(u_1\Delta v_1-u_2\Delta v_2)\mathrm{d}s-\chi\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s, \end{aligned}$$

which yields

$$\begin{aligned} &\|Gg_1-Gg_2\|_{X_{t_0}}=\|u_1-u_2\|_{X_{t_0}}\\ \le&2Ct_0^{1-\frac{\sigma}{2}}\|v_1-v_2\|_{Y_{t_0}}+Ct_0^{1-\frac{\sigma}{2}}(\|v_2\|_{Y_{t_0}}+\|v_1\|_{Y_{t_0}})\|Gg_1-Gg_2\|_{X_{t_0}}, \end{aligned} \tag{3.15}$$

and

$$(v_1-v_2)_{tt}=\Delta(v_1-v_2)+h(v_1)-h(v_2)+f(g_1)-f(g_2), \tag{3.16a}$$

$$(v_1-v_2)(0,\cdot)=0,\qquad (v_1-v_2)_t(0,\cdot)=0. \tag{3.16b}$$

It follows from of Lemma 3.3 that

$$\begin{aligned} \|v_1-v_2\|^2_{Y_{t_0}}\le& Ce^{t_0}\int_0^{t_0}\|h(v_1)-h(v_2)+f(g_1)-f(g_2)\|^2_{H^1}\mathrm{d}\tau\\ \le& Ce^{t_0}t_0\sup_{0\le\tau\le t_0}\|h(v_1)-h(v_2)+f(g_1)-f(g_2)\|^2_{H^1} \end{aligned}$$

$$
\begin{aligned}
&\le Ce^{t_0}t_0 \sup_{0\le\tau\le t_0}\Big(\|h(v_1)-h(v_2)\|_{H^1}+\|f(g_1)-f(g_2)\|_{H^1}\Big)^2\\
&\le Ce^{t_0}t_0\Big(\sup_{0\le\tau\le t_0}\|h(v_1)-h(v_2)\|_{H^1}+\sup_{0\le\tau\le t_0}\|f(g_1)-f(g_2)\|_{H^1}\Big)^2\\
&\le Ct_0e^{t_0}\Big((\sup_{0\le\tau\le t_0}\|h(v_1)-h(v_2)\|_{H^1}\Big)^2+\Big(\sup_{0\le\tau\le t_0}\|f(g_1)-f(g_2)\|_{H^1})^2\Big)\\
&\le Ct_0e^{t_0}\Big((\|h\|_{c^2}\|v_1-v_2\|_{Y_{t_0}})^2+(\|f\|_{C^2}\|g_1-g_2\|_{X_{t_0}})^2\Big). \qquad (3.17)
\end{aligned}
$$

If t_0 is sufficiently small, then

$$\|v_1-v_2\|_{Y_{t_0}}\le Ct_0^{\frac{1}{2}}e^{t_0}\|f\|_{C^2}\|g_1-g_2\|_{X_{t_0}}. \qquad (3.18)$$

Since

$$
\begin{aligned}
\|v_1\|_{Y_{t_0}}^2 &\le Ce^{t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+\int_0^{t_0}\|f(g_1(\tau))\|_{H^1}^2\mathrm{d}\tau\right)\\
&\le Ce^{t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+C\int_0^{t_0}(\|f(g_1(\tau))-f(0)\|_{H^1}+\|f(0)\|_{H^1})^2\mathrm{d}\tau\right)\\
&\le Ce^{t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+\int_0^{t_0}(\|f\|_{C^2}\cdot\|g_1(\tau)\|_{H^1}+\|f(0)\|_{H^1})^2\mathrm{d}\tau\right)\\
&\le Ce^{t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+t_0(M+\|f(0)\|_{L^2})^2\right), \qquad (3.19)
\end{aligned}
$$

and

$$\|v_2\|_{Y_{t_0}}^2\le Ce^{t_0}\Big(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+t_0\big(M+\|f(0)\|_{L^2}\big)^2\Big), \qquad (3.20)$$

by virtue of (3.15), (3.18), (3.19) and (3.20), we have that G is a contraction provided that t_0 is small enough. Therefore (2.1) admits a local solution $(u,v)\in X_{t_0}\cap Z_{t_0}\times Y_{t_0}\cap W_{t_0}$, and the uniqueness follows from the contraction of the operator.

Observe that if $s\le 2$, as done in [4, Lemma 5.2], the local solution $(u,v)\in X_{t_0}\cap Z_{t_0}\times Y_{t_0}\cap W_{t_0}$ obeys

$$\|v(t,\cdot)\|_{H^s}^2\le Ce^{t_0}\left(C_0+\int_0^{t_0}\|f(u(\tau,\cdot))\|_{H^{s-1}}^2\mathrm{d}\tau\right),\quad 0\le t\le t_0, \qquad (3.21)$$

where $C_0=\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2$ and C is independent of t_0.

For the unique local solution $(u,v)\in(X_{t_0}\times Y_{t_0})\cap(Z_{t_0}\times W_{t_0})$ of (2.1), if we take $s=1/2$ in (3.21), then

$$\|v(t,\cdot)\|_{H^{\frac{1}{2}}}^2\le Ce^{Cct_0}\left(C_0+\int_0^{t_0}\|f(u(\tau,\cdot))\|_{H^{-\frac{1}{2}}}^2\mathrm{d}\tau\right),\quad 0\le t\le t_0. \qquad (3.22)$$

Since $N=1$, it follows from the Sobolev imbedding theorems, that $W^{0,1}(M)\hookrightarrow H^{-\frac{1}{2}}(M)$. Hence we have

$$\begin{aligned}\|v(t,\cdot)\|^2_{H^{\frac{1}{2}}} &\le Ce^{2Ct_0}\left(C_0+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^{-\frac{1}{2}}}\mathrm{d}\tau\right)\\ &\le Ce^{2Ct_0}\left(C_0+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{L^1}\mathrm{d}\tau\right)\\ &\le Ce^{2ct_0}\left(C_0+\int_0^{t_0}(M_1\|u\|_{L^1}+\|f(0)\|_{L^1})^2\mathrm{d}\tau\right)\\ &=Ce^{2Ct_0}\left(C_0+t_0(M_1\|u_0\|_{L^1}+\|f(0)\|_{L^1})^2\right), \qquad 0\le t\le t_0, \end{aligned}\tag{3.23}$$

where $M_1=\|f\|_{C^2}$. On the other hand, for each $s\le\sigma$ and $0\le\sigma_0<2$,

$$\begin{aligned}\|u(t,\cdot)\|_{H^s} &\le C\|u_0\|_{H^\sigma}+Ct_0^{1-\frac{\sigma_0}{2}}\sup_{0\le t\le t_0}\|\nabla(u\nabla v)\|_{H^{s-\sigma_0}}\\ &\le C\|u_0\|_{H^\sigma}+Ct_0^{1-\frac{\sigma_0}{2}}\sup_{0\le t\le t_0}\|u\nabla v\|_{H^{s-\sigma_0+1}}, \qquad 0\le t\le t_0.\end{aligned}\tag{3.24}$$

In particular for $s=-1/2+1/4$ and $\sigma_0=2-1/8$, we have

$$\|u(t,\cdot)\|_{H^{-\frac{1}{7}2+\frac{1}{4}}}\le C\|u_0\|_{H^\sigma}+Ct_0^{\frac{1}{16}}\sup_{0\le t\le t_0}\|u\nabla v\|_{H^{-1-\frac{1}{8}}}, \qquad 0\le t\le t_0. \tag{3.25}$$

By the Sobolev imbedding theorems and (3.23), we have

$$\begin{aligned}\|u\nabla v\|_{H^{-1-\frac{1}{8}}} &\le C\|u\|_{H^{-1-\frac{1}{8}}}\cdot\|\nabla v\|_{W^{-1-\frac{1}{8},\infty}}\le C\|u\|_{H^{-1}}\cdot\|\nabla v\|_{H^{-\frac{1}{2}}}\le C\|u\|_{L^1}\cdot\|v\|_{H^{\frac{1}{2}}}\\ &\le C\|u_0\|_{L^1}\cdot e^{Ct_0}(C_0^{\frac{1}{2}}+t_0^{\frac{1}{2}}(M_1\|u_0\|_{L^1}+\|f(0)\|_{L^1})), \qquad 0\le t\le t_0.\end{aligned}\tag{3.26}$$

Consequently,

$$\begin{aligned}&\|u(t,\cdot)\|_{H^{-\frac{1}{4}}}\le C\|u_0\|_{H^\sigma}+Ct_0^{\frac{1}{16}}\sup_{0\le t\le t_0}\|u\nabla v\|_{H^{-1-\frac{1}{8}}}\\ \le& C\|u_0\|_{H^\sigma}+Ct_0^{\frac{1}{16}}\|u_0\|_{L^1}\cdot e^{Ct_0}\left(C_0^{\frac{1}{2}}+t_0^{\frac{1}{2}}(M_1\|u_0\|_{L^1}+\|f(0)\|_{L^1})\right), \qquad 0\le t\le t_0.\end{aligned}\tag{3.27}$$

Take $s=1/2+1/4=3/4$ in (3.21), then (3.21) and (3.27) give

$$\begin{aligned}\|v(t,\cdot)\|^2_{H^{\frac{3}{4}}} &\le Ce^{2Ct_0}\left(C_0+\int_0^{t_0}\|f(u(\tau,\cdot))\|^2_{H^{\frac{3}{4}-1}}\mathrm{d}\tau\right)\\ &\le Ce^{2Ct_0}\left(C_0+t_0(M_1\sup_{0\le\tau\le t_0}\|u(\tau,\cdot)\|_{H^{-\frac{1}{4}}}+\|f(0)\|_{H^{-\frac{1}{4}}})^2\right)\\ &\le Ce^{2Ct_0}\Big\{C_0+t_0[M_1[C\|u_0\|_{H^\sigma}+Ct_0^{\frac{1}{16}}\|u_0\|_{L^1}\cdot e^{Ct_0}(C_0^{\frac{1}{2}}\\ &\qquad +t_0^{\frac{1}{2}}(M_1\|u_0\|_{L^1}+\|f(0)\|_{L^1}))]+\|f(0)\|_{H^{-\frac{1}{4}}}]^2\Big\}, \qquad 0\le t\le t_0.\end{aligned}\tag{3.28}$$

Take $s=-1/2+1/4+1/4=0$ and $\sigma_0=2-1/8$ in (3.24), we obtain

$$\begin{aligned}\|u(t,\cdot)\|_{L^2} \le & C\|u_0\|_{H^\sigma}+Ct_0^{1-\frac{\sigma_0}{2}}\sup_{0\le t\le t_0}\|\nabla(u\nabla v)\|_{H^{-\sigma_0}}\\ \le & C\|u_0\|_{H^\sigma}+Ct_0^{\frac{1}{16}}\sup_{0\le t\le t_0}\|u\nabla v\|_{H^{-\sigma_0+1}}\\ \le & C\|u_0\|_{H^\sigma}+Ct_0^{\frac{1}{16}}\sup_{0\le t\le t_0}\|u\nabla v\|_{H^{-1+\frac{1}{8}}}, \qquad 0\le t\le t_0. \end{aligned} \tag{3.29}$$

Since

$$\begin{aligned}\|u\nabla v\|_{H^{-1+\frac{1}{8}}} \le & C\|u\|_{H^{-1+\frac{1}{8}}}\cdot\|\nabla v\|_{W^{-1+\frac{1}{8},\infty}}\\ \le & C\|u\|_{H^{-\frac{1}{4}}}\cdot\|\nabla v\|_{H^{-\frac{1}{2}+\frac{1}{4}}}\\ \le & C\|u\|_{H^{-\frac{1}{4}}}\cdot\|v\|_{H^{\frac{3}{4}}}, \qquad 0\le t\le t_0, \end{aligned} \tag{3.30}$$

we have that

$$\begin{aligned}\|u(t,\cdot)\|_{L^2} \le & C\|u_0\|_{H^\sigma}+Ct_0^{1-\frac{\sigma_0}{2}}\sup_{0\le t\le t_0}\|\nabla(u\nabla v)\|_{H^{-\sigma_0}}\\ \le & C\|u_0\|_{H^\sigma}+Ct_0^{\frac{1}{16}}\sup_{0\le t\le t_0}\|u\nabla v\|_{H^{-1+\frac{1}{8}}}\\ \le & C\|u_0\|_{H^\sigma}+Ct_0^{\frac{1}{16}}\cdot\sup_{0\le t\le t_0}\|u\|_{H^{-\frac{1}{4}}}\cdot\|v\|_{H^{\frac{3}{4}}}, \qquad 0\le t\le t_0. \end{aligned} \tag{3.31}$$

It follows from (3.27) and (3.28) that $\|u(t,\cdot)\|_{L^2}$ grows by a bounded manner in time from (3.31).

If we take $s=1/2+1/4+1/4=1$ in (3.21), then (3.21) and (3.31) imply that $\|v(t,\cdot)\|_{H^1}$ grows also by a bounded manner in time.

Take $s=-1/2+1/4+1/4+1/4=1/4$ and $\sigma_0=2-1/8$ in (3.24) once more. Since $\|v(t,\cdot)\|_{H^1}$ grows by a bounded manner in time, similar to what we have done in (3.29), (3.30) and (3.31), we can deduce that $\|u(t,\cdot)\|_{H^{1/4}}$ grows by a bounded manner in time.

If we repeat the above processes four times, we can prove that $\|u(t,\cdot)\|_{H^{\frac{5}{4}}}$ and $\|v(t,\cdot)\|_{H^2}$ grow by a bounded manner in time, as desired. □

4 The case of $g(v,u)=\alpha uv$

In this section we consider the case of $g(v,u)=\alpha uv$ as follows:

$$\begin{cases} u_t=\nabla(\nabla u-\chi u\nabla v),\\ v_{tt}=\Delta v+\alpha uv,\\ u(0,\cdot)=u_0, v(0,\cdot)=\varphi, v_t(0,\cdot)=\psi. \end{cases} \tag{4.1}$$

We will establish the local in time solution for this case. To obtain Theorem 2.2, we first divide (4.1) into two parts:

$$\begin{cases} u_t=\nabla(\nabla u-\chi u\nabla v), \\ u(0,\cdot)=u_0, \end{cases} \tag{4.2}$$

$$\begin{cases} v_{tt}=\Delta v+\alpha uv, \\ v(0,\cdot)=\varphi, \quad v_t(0,\cdot)=\psi. \end{cases} \tag{4.3}$$

Lemma 4.1. *If $u\in X_{t_0}$, $v\in L^2(0,t_0;W^{1,4}(M))$, then $uv\in L^2(0,t_0;H^1(M))$ and*

$$\|uv\|_{H^1}\leq C\Big(\|u\|_{L^4}\cdot\|v\|_{W^{1,4}}+\|v\|_{L^4}\cdot\|u\|_{W^{1,4}}\Big), \qquad 0\leq t\leq t_0. \tag{4.4}$$

Proof. Since $u\in X_{t_0}$, we know that $u(t,\cdot)\in H^\sigma(M)$, $(0\leq t\leq t_0)$. Using the Sobolev theorem gives that $H^\sigma\subset W^{1,q}$, where $q=2n/[n-2(\sigma-1)]$. Using (1.5), (1.7)–(1.8), we have $W^{1,q}\subset W^{1,4}$. Thus $u\in W^{1,4}$. Now the Cauchy inequality implies

$$\|uv\|_{L^2}\leq\|u\|_{L^4}\cdot\|v\|_{L^4}, \qquad 0\leq t\leq t_0. \tag{4.5}$$

Furthermore $\nabla(uv)=\nabla u\cdot v+u\cdot\nabla v$, so

$$\|\nabla(uv)\|_{L^2}\leq\|\nabla u\cdot v\|_{L^2}+\|u\cdot\nabla v\|_{L^2}, \qquad 0\leq t\leq t_0. \tag{4.6}$$

Again applying the Cauchy inequality gives

$$\|\nabla u\cdot v\|_{L^2}\leq\|\nabla u\|_{L^4}\cdot\|v\|_{L^4}, \qquad 0\leq t\leq t_0. \tag{4.7}$$

$$\|\nabla v\cdot u\|_{L^2}\leq\|\nabla v\|_{L^4}\cdot\|u\|_{L^4}, \qquad 0\leq t\leq t_0, \tag{4.8}$$

Consequently, (4.5)–(4.8) indicate that (4.4) is valid.

Because $u\in C([0,t_0];H^\sigma(M))$ and $H^\sigma\subset W^{1,4}$, $u\in C([0,t_0];W^{1,4}(M))$ and the lemma follows from (4.4) immediately. □

Lemma 4.2. *If $\varphi\in H^2(M)$, $\psi\in H^1(M)$ and $u\in X_{t_0}$, then* (4.3) *has a unique local in time solution $v\in Y_{t_0}$ and*

$$\|v\|^2_{Y_{t_0}}\leq Ce^{|\alpha|t_0}\Big(\|\varphi\|^2_{H^2}+\|\psi\|^2_{H^1}+t_0\|u\|^2_{X_{t_0}}\cdot\|v\|^2_{Y_{t_0}}\Big). \tag{4.9}$$

Proof. We use the Leray-Schauder fixed point argument. Choose $X=L^2(0,t_0;W^{1,4}(M))$, then X is a Banach space.

For each $w\in X$, let $v=v(w)$ solve following initial value problem:

$$\begin{cases} v_{tt}=\Delta v+\varepsilon\alpha wu, \\ v(0,\cdot)=\varepsilon\varphi, \quad v_t(0,\cdot)=\varepsilon\psi, \end{cases} \tag{4.10}$$

where $\varepsilon\in[0,1]$. Thanks to Lemma 4.1, we have $wu\in L^2(0,t_0;H^1(M))$. Thus hyperbolic regularity of the equation asserts that (4.10) admits a solution $v\in Y_{t_0}$ and

$$\|v(t,\cdot)\|_{H^2}^2\le Ce^{|\alpha|t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+\int_0^{t_0}\|wu\|_{H^1}^2\mathrm{d}\tau\right),\qquad 0\le t\le t_0. \tag{4.11}$$

Under the condition of $1\le n\le 3$, the Sobolev theorems show $H^2\hookrightarrow W^{1,4}$. Thus (4.11) asserts $v\in X$. Now we define

$$T:\ X\times[0,1]\to X,\quad T(w,\varepsilon)=v(w).$$

It is obvious that T is compact and $T(w,0)=0$. If $T(w,\varepsilon)=w$, then

$$\|w(t,\cdot)\|_{H^2}^2\le Ce^{|\alpha|t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+\int_0^{t_0}\|wu\|_{H^1}^2\mathrm{d}\tau\right),\qquad 0\le t\le t_0. \tag{4.12}$$

Using Lemma 4.1 once more, we have

$$\begin{aligned}\|uw\|_{H^1}&\le C\left(\|u\|_{L^4}\cdot\|w\|_{W^{1,4}}+\|w\|_{L^4}\cdot\|u\|_{W^{1,4}}\right)\\&\le 2C\|u\|_{W^{1,4}}\cdot\|w\|_{W^{1,4}}\\&\le 2C\|u\|_{H^\sigma}\cdot\|w\|_{H^2},\qquad 0\le t\le t_0.\end{aligned} \tag{4.13}$$

Hence

$$\begin{aligned}\|w(t,\cdot)\|_{H^2}^2&\le Ce^{|\alpha|t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+\int_0^{t_0}\|u\|_{H^\sigma}^2\|w\|_{H^2}^2\mathrm{d}\tau\right)\\&\le Ce^{|\alpha|t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+t_0\|u\|_{X_{t_0}}^2\|w\|_{Y_{t_0}}^2\right),\qquad 0\le t\le t_0.\end{aligned} \tag{4.14}$$

In other words,

$$\|w\|_{Y_{t_0}}^2\le Ce^{|\alpha|t_0}(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2)+Ct_0e^{|\alpha|t_0}\|u\|_{X_{t_0}}^2\cdot\|w\|_{Y_{t_0}}^2. \tag{4.15}$$

As $Y_{t_0}\subset X$, (4.15) implies $\|w\|_X\le M_1$ provided that t_0 is small enough, where M_1 is independent of w and ε.

The Leray-Schauder fixed point theorem indicates that $T(w,1)$ has a fixed point $v=T(v,1)$, which shows that there is a solution of (4.3), and (4.15) asserts that (4.9) is true. The uniqueness follows from the estimation of (4.4). □

Proof of Theorem 2.2. Choose $g\in X_{t_0}$, $g(0)=u_0$ and let $v=v(g)$ be the solution of the following problem:

$$\begin{cases}v_{tt}=\Delta v+\alpha g v,\\ v(0,\cdot)=\varphi,\quad v_t(0,\cdot)=\psi.\end{cases} \tag{4.16}$$

It follows from Lemma 4.2 that (4.16) has a unique solution $v\in Y_{t_0}$ and

$$\|v\|^2_{Y_{t_0}}\le Ce^{|\alpha|t_0}(\|\varphi\|^2_{H^2}+\|\psi\|^2_{H^1})+Ct_0e^{|\alpha|t_0}\|g\|^2_{X_{t_0}}\cdot\|v\|^2_{Y_{t_0}}. \tag{4.17}$$

Let $u=u(v(g))$ solves the following problem:

$$\begin{cases} u_t=\nabla(\nabla u-\chi u\nabla v),\\ u(0,\cdot)=u_0=g(0). \end{cases} \tag{4.18}$$

Note that the conditions of (1.7)–(1.8) imply (1.6), in fact, (1.7) indicates $N>2(\sigma-1)$ and $N/(\sigma-1)<2N/[N-2(\sigma-1)]$, thus $N<4(\sigma-1)$. Using (1.5), we get $N<2\sigma$.

Thus Lemmas 3.1 and 3.2 show that $G:X_{t_0}\to X_{t_0}$, where $Gg=u(v(g))$, and

$$\|Gg\|_{X_{t_0}}\le C\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma}{2}}\cdot\|v\|_{Y_{t_0}}\cdot\|Gg\|_{X_{t_0}}. \tag{4.19}$$

Set

$$B_M=\Big\{g\in X_{t_0}\Big|g(0)=u_0,\|g(t,\cdot)\|_{\sigma,2}\le M,0\le t\le t_0\Big\},$$

where $M=2C\|u_0\|_{\sigma,2}+1$ $(C\ge 1)$ is determined by (4.19). For $g\in B_M$, by (4.17) we have

$$\begin{aligned}\|v\|^2_{Y_{t_0}}&\le Ce^{|\alpha|t_0}\Big(\|\varphi\|^2_{H^2}+\|\psi\|^2_{H^1}\Big)+Ct_0e^{|\alpha|t_0}\|g\|^2_{X_{t_0}}\cdot\|v\|^2_{Y_{t_0}}\\ &\le Ce^{|\alpha|t_0}\Big(\|\varphi\|^2_{H^2}+\|\psi\|^2_{H^1}\Big)+Ct_0e^{|\alpha|t_0}M^2\cdot\|v\|^2_{Y_{t_0}}.\end{aligned} \tag{4.20}$$

If t_0 is small enough, then

$$\|v\|_{Y_{t_0}}\le 2Ce^{|\alpha|t_0}\Big(\|\varphi\|_{H^2}+\|\psi\|_{H^1}\Big). \tag{4.21}$$

Combining (4.19) and (4.21), yields that

$$\|Gg\|_{X_{t_0}}\le C\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma}{2}}\cdot 2Ce^{|\alpha|t_0}\Big(\|\varphi\|_{H^2}+\|\psi\|_{H^1}\Big)\cdot\|Gg\|_{X_{t_0}}.$$

Choose t_0 small enough such that $2C^2t_0\cdot e^{|\alpha|t_0}(\|\varphi\|_{H^2}+\|\psi\|_{H^1})\le 1/2$. Then

$$\|Gg\|_{X_{t_0}}\le 2C\|u_0\|_{H^\sigma}=M,$$

so G maps B_M into B_M.

Finally we prove that the operator G is contract provided that t_0 is small enough. To this end, for $g_1,g_2\in X_{t_0}$, let v_i $(i=1,2)$ be corresponding solutions of (4.16),

$$\begin{aligned}Gg_1-Gg_2&=u_1-u_2\\ &=-\chi\int_0^t T(t-s)u_1\Delta v_1\mathrm{d}s-\chi\int_0^t T(t-s)\nabla u_1\nabla v_1\mathrm{d}s\\ &\quad+\chi\int_0^t T(t-s)u_2\nabla v_2\mathrm{d}s+\chi\int_0^t T(t-s)\nabla u_2\nabla v_2\mathrm{d}s\\ &=-\chi\int_0^t T(t-s)(u_1\Delta v_1-u_2\Delta v_2)\mathrm{d}s-\chi\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s,\end{aligned}$$

we have

$$
\begin{aligned}
&\|Gg_1-Gg_2\|_{X_{t_0}}=\|u_1-u_2\|_{X_{t_0}}\\
\leq&2Ct_0^{1-\frac{\sigma}{2}}\|v_1-v_2\|_{Y_{t_0}}+Ct_0^{1-\frac{\sigma}{2}}(\|v_2\|_{Y_{t_0}}+\|v_1\|_{Y_{t_0}})\cdot\|Gg_1-Gg_2\|_{X_{t_0}}.
\end{aligned}
\tag{4.22}
$$

Otherwise

$$
\begin{aligned}
(v_1-v_2)_{tt}=&\Delta(v_1-v_2)+\alpha(g_1v_1-g_2v_2)\\
=&\Delta(v_1-v_2)+\alpha v_1(g_1-g_2)+\alpha g_2(v_1-v_2),
\end{aligned}
\tag{4.23a}
$$

$$
(v_1-v_2)(0,\cdot)=0,\quad (v_1-v_2)_t(0,\cdot)=0,
\tag{4.23b}
$$

and by using Lemma 3.3 gives

$$
\begin{aligned}
\|v_1-v_2\|_{Y_{t_0}}\leq&Ce^{|\alpha|t_0}\left(\int_0^{t_0}\|\alpha v_1(g_1-g_2)+\alpha g_2(v_1-v_2)\|_{H^1}^2\mathrm{d}\tau\right)^{\frac{1}{2}}\\
\leq&C|\alpha|e^{|\alpha|t_0}\cdot t_0^{\frac{1}{2}}\cdot\sup_{0\leq\tau\leq t_0}\|v_1(g_1-g_2)+g_2(v_1-v_2)\|_{H^1}\\
\leq&C|\alpha|e^{|\alpha|t_0}\cdot t_0^{\frac{1}{2}}\cdot\sup_{0\leq\tau\leq t_0}\|v_1(g_1-g_2)\|_{H^1}\\
&+C|\alpha|e^{|\alpha|t_0}\cdot t_0^{\frac{1}{2}}\cdot\sup_{0\leq\tau\leq t_0}\|g_2(v_1-v_2)\|_{H^1}.
\end{aligned}
\tag{4.24}
$$

Furthermore we declare

$$
\begin{aligned}
\|v_1-v_2\|_{Y_{t_0}}\leq&C|\alpha|e^{|\alpha|t_0}\cdot t_0^{\frac{1}{2}}\cdot\sup_{0\leq\tau\leq t_0}\left(\|v_1\|_{H^2}\cdot\|g_1-g_2\|_{H^\sigma}\right)\\
&+C|\alpha|e^{|\alpha|t_0}\cdot t_0^{\frac{1}{2}}\cdot\sup_{0\leq\tau\leq t_0}\left(\|g_2\|_{H^\sigma}\cdot\|v_1-v_2\|_{H^2}\right)\\
\leq&C|\alpha|e^{|\alpha|t_0}\cdot t_0^{\frac{1}{2}}\cdot\|v_1\|_{Y_{t_0}}\cdot\|g_1-g_2\|_{X_{t_0}}\\
&+C|\alpha|e^{|\alpha|t_0}\cdot t_0^{\frac{1}{2}}\cdot\|g_2\|_{X_{t_0}}\cdot\|v_1-v_2\|_{Y_{t_0}}.
\end{aligned}
\tag{4.25}
$$

For sufficiently small t_0, it is obvious that

$$
\|v_1-v_2\|_{Y_{t_0}}\leq 2C|\alpha|t_0^{\frac{1}{2}}e^{|\alpha|t_0}\cdot\|v_1\|_{Y_{t_0}}\cdot\|g_1-g_2\|_{X_{t_0}},
\tag{4.26a}
$$

$$
\begin{aligned}
\|v_1\|_{Y_{t_0}}^2\leq&Ce^{|\alpha|t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+t_0\|g_1\|_{X_{t_0}}^2\cdot\|v\|_{Y_{t_0}}^2\right)\\
\leq&Ce^{|\alpha|t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+t_0M^2\cdot\|v\|_{Y_{t_0}}^2\right),
\end{aligned}
\tag{4.26b}
$$

$$
\begin{aligned}
\|v_2\|_{Y_{t_0}}^2\leq&Ce^{|\alpha|t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+t_0\|g_2\|_{X_{t_0}}^2\cdot\|v\|_{Y_{t_0}}^2\right)\\
\leq&Ce^{|\alpha|t_0}\left(\|\varphi\|_{H^2}^2+\|\psi\|_{H^1}^2+t_0M^2\cdot\|v\|_{Y_{t_0}}^2\right).
\end{aligned}
\tag{4.26c}
$$

Consequently, for t_0 small enough we have

$$\|v_1\|_{Y_{t_0}} \le C_1 e^{|\alpha| t_0} \cdot \big(\|\varphi\|_{H^2} + \|\psi\|_{H^1} \big), \tag{4.27}$$

$$\|v_2\|_{Y_{t_0}} \le C_1 e^{|\alpha| t_0} \cdot \big(\|\varphi\|_{H^2} + \|\psi\|_{H^1} \big). \tag{4.28}$$

In view of (4.22),(4.26a),(4.27) and (4.28), we have show that G is a contract mapping, as required. □

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