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A Multiplicity Result for a Singular and Nonhomogeneous Elliptic Problem in $\mathbb{R}^n$

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Abstract. We establish sufficient conditions under which the quasilinear equation

$$-\mathrm{div}(|\nabla u|^{n-2}\nabla u)+V(x)|u|^{n-2}u=\frac{f(x,u)}{|x|^{\beta}}+\varepsilon h(x)\qquad \text{in } \mathbb{R}^n,$$

has at least two nontrivial weak solutions in $W^{1,n}(\mathbb{R}^n)$ when $\varepsilon>0$ is small enough, $0\le\beta<n$, V is a continuous potential, $f(x,u)$ behaves like $\exp\{\gamma|u|^{n/(n-1)}\}$ as $|u|\to\infty$ for some $\gamma>0$ and $h\not\equiv 0$ belongs to the dual space of $W^{1,n}(\mathbb{R}^n)$.

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1 Introduction and main results

Let $W^{1,n}(\mathbb{R}^n)$ be the usual Sobolev space in $\mathbb{R}^n\,(n\ge 2)$ with the norm

$$\|u\|_{W^{1,n}}=\left(\int_{\mathbb{R}^n}(|\nabla u|^n+|u|^n)\mathrm{d}x\right)^{1/n}.$$

In this paper, we consider the quasilinear differential equation

$$-\Delta_n u+V(x)|u|^{n-2}u=\frac{f(x,u)}{|x|^{\beta}}+\varepsilon h(x)\qquad \text{in } \mathbb{R}^n, \tag{1.1}$$

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where $-\Delta_n u = -\mathrm{div}(|\nabla u|^{n-2}\nabla u)$, V is a continuous potential, $h \not\equiv 0$ belongs to the dual space of $W^{1,n}(\mathbb{R}^n)$, $0 \le \beta < n$ and $f(x,u)$ behaves like $\exp\{\gamma|u|^{n/(n-1)}\}$ as $|u| \to \infty$.

This kind of elliptic problems involving exponential critical growth has been extensively studied by many authors. To get a solution, Moser-Trudinger type inequality and critical point theory are two fundamental tools. For the homogeneous and nonsingular case, that is when $h \equiv 0$ and $\beta = 0$, the existence result in a bounded domain was obtained in [1,2]. When the domain is the whole space, the problem was studied in [3–5]. We can also consider the problem in a Riemannian manifold. For this case one can refer to [6–8] and the references therein. Because of the variational structure of this kind of equations, usually there are both minimum type and mountain-pass type solutions. A nature question is that whether these two types of solutions are different. When $n = 2$ and $\beta = 0$, do Ó, Medeiros and Severo [9] proved that these are two distinct solutions. For general dimensional case, the same authors got the result in [10]. In our paper, the nonlinearity of Eq. (1.1) becomes singular. In [11], do Ó proved that there are two distinct solutions for this singular equation when $n = 2$. Then relevant issues about the general dimensional case should be asked. Our main theorem is to give sufficient conditions under which there are still two solutions to (1.1).

To present our results, we assume the following conditions on the nonlinearity $f(x,s)$:

(H_1) There exist constants $\alpha_0, b_1, b_2 > 0$ such that for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}^+$,

$$|f(x,s)| \le b_1 s^{n-1} + b_2 \Big\{ \exp\{\alpha_0 |s|^{n/(n-1)}\} - B_{n-2}(\alpha_0, s) \Big\},$$

where

$$B_{n-2}(\alpha_0,s) = \sum_{m=0}^{n-2} (\alpha_0^m |s|^{mn/(n-1)})/m!.$$

(H_2) There exist constants $p > n$ and C_p such that

$$f(s) \ge C_p s^{p-1} \qquad \text{for all } s \ge 0,$$

where

$$C_p > \left(\frac{p-n}{p}\right)^{(p-n)/n} \left(\frac{n\alpha_0}{(n-\beta)\alpha_n}\right)^{(n-1)(p-n)/n} S_p^p,$$

$$S_p := \inf_{u \in E \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^n} (|\nabla u|^n + V(x) u^n) \mathrm{d}x\right)^{1/n}}{\left(\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^\beta} \mathrm{d}x\right)^{1/p}}.$$

(H_3) There exists $\mu > n$ such that for all $x \in \mathbb{R}^n$ and $s > 0$,

$$0 < \mu F(x,s) \equiv \mu \int_0^s f(x,t)\mathrm{d}t \le s f(x,s).$$

(H_4) There exist constants $R_0, M_0 > 0$ such that for all $x \in \mathbb{R}^n$ and $s \geq R_0$,

$$F(x,s) \leq M_0 f(x,s).$$

(H_5) $f(x,s) \geq 0$ for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}^+$ and $f(x,s) = 0$ for all $x \in \mathbb{R}^n$ and $s \leq 0$.

Throughout this paper we also assume the following hypotheses on V:

(V_1) $V(x) \geq V_0 > 0$.

(V_2) The function$V^{-1}(x)$ belongs to $L^{1/(n-1)}(\mathbb{R}^n)$.

Define a function space

$$E \triangleq \left\{ u \in W^{1,n}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)|u|^n \mathrm{d}x < \infty \right\}$$

with the norm

$$\|u\| \triangleq \left\{ \int_{\mathbb{R}^n} (|\nabla u|^n + V(x)|u|^n) \mathrm{d}x \right\}^{1/n}.$$

Moreover denote the dual space of E by E^*. Under our assumptions on V, for any $q \geq 1$, the embeddings from E into $W^{1,n}(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ are continuous and compact(cf. [12]).

For any $0 < \beta < n$, we define a singular eigenvalue by

$$\lambda_\beta \triangleq \inf_{u \in E, u \neq 0} \frac{\|u\|^n}{\int_{\mathbb{R}^n} \frac{|u|^n}{|x|^\beta} \mathrm{d}x}. \tag{1.2}$$

Furthermore, we assume

(H_6) $\limsup_{s \to 0^+} \frac{nF(x,s)}{|s|^n} < \lambda_\beta$ uniformly with respect to $x \in \mathbb{R}^n$.

Problem (1.1) is closely related to Moser-Trudinger type inequalities (see, e.g., [13–15]). Here we need the following result of Adimurth and Yang which is a singular Moser-Trudinger type inequality in the whole space $\mathbb{R}^n$.

Theorem A ([16, Theorem 1.1]) *For all* $\alpha \leq (1-\beta/n)\alpha_n$ *and* $0 \leq \beta < n$,

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_{W^{1,n}} \leq 1} \int_{\mathbb{R}^n} \frac{\exp\{\alpha |u|^{n/(n-1)}\} - B_{n-2}(\alpha, u)}{|x|^\beta} \mathrm{d}x < \infty, \tag{1.3}$$

where $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, ω_{n-1} *is the area of the unit sphere* $\mathbb{S}^{n-1}$. *Furthermore this inequality is sharp: when* $\alpha > (1-\beta/n)\alpha_n$, *the integrals in* (1.3) *are still finite, but the supremum is infinity.*

Another key ingredient in our arguments is a singular version of Lion's inequality in $\mathbb{R}^n$.

Theorem 1.1. *Let $\{u_k\}$ be a sequence in E such that $\|u_k\|=1$, $u_k \rightharpoonup u$ in E, $u_k \to u$ in $L^n(\mathbb{R}^n)$, and $\nabla u_k(x) \to \nabla u(x)$ for almost every $x \in \mathbb{R}^n$. Then for any $p<(1-\beta/n)\alpha_n(1-\|u\|^n)^{-1/(n-1)}$,*

$$\sup_k \int_{\mathbb{R}^n} \frac{\exp\{p|u_k|^{n/(n-1)}\}-B_{n-2}(p,u_k)}{|x|^\beta}\mathrm{d}x<+\infty.$$

Combining the above inequalities, we can prove the following multiplicity result.

Theorem 1.2. *Assume* (V_1),(V_2) *and* (H_1)-(H_6). *Then there exists $\epsilon_1>0$ such that for each $0<\epsilon<\epsilon_1$, Eq.* (1.1) *has at least two nontrivial solutions.*

Remark 1.1. After this paper was finished, we know from Yang that he had obtained similar results [17, 18]. Although the methods are both in the framework of variation, but there are differences between the assumptions on the nonlinearity $f(x,.)$ in these two papers. In fact, the assumption (H_5) in [18] and our assumption (H_2) can not cover each other. When $n=2$, for some constant $p>2$, consider the example in [11]:

$$f_0(s)=\begin{cases} 0 & s\in(-\infty,0), \\ C_p s^{p-1}+2s(e^{s^2}-1) & s\in[0,1], \\ C_p s^{p-1}+(e-1)((2s-1)e^{s^2-s}+s) & s\in(1,+\infty). \end{cases}$$

It is obvious that assumptions (H_1), (H_2) and (H_5) are satisfied by f_0. Since

$$F_0(s)=\int_0^s f_0(t)\mathrm{d}t=\begin{cases} 0 & s\in(-\infty,0), \\ \dfrac{C_p}{p}s^p+e^{s^2}-1-s^2 & s\in[0,1], \\ \dfrac{C_p}{p}s^p+(e-1)\left(e^{s^2-s}+(s^2-1)/2\right)-1 & s\in(1,+\infty), \end{cases}$$

we have

$$\lim_{s\to 0^+}\frac{2F_0(s)}{s^2}=0,$$

which implies (H_6). For (H_3), if $0\le s<1$ and $2<\mu_1\le\min\{4,p\}$, then

$$\begin{aligned}\mu_1 F_0(s)&=\frac{\mu_1}{p}C_p s^p+\mu_1(e^{s^2}-1-s^2)\\ &\le C_p s^p+\mu_1\sum_{m=2}^{\infty}\frac{s^{2m}}{m!}\le C_p s^p+\frac{\mu_1}{2}s^2\sum_{m=2}^{\infty}\frac{s^{2(m-1)}}{(m-1)!}\\ &=C_p s^p+\frac{\mu_1}{2}s^2\sum_{m=1}^{\infty}\frac{s^{2m}}{m!}=sf_0(s).\end{aligned}$$

When $1\le s<\infty$, straightforward computations show that there exists some constant $\mu_2>2$ such that

$$\mu_2 F_0(1)\le f_0(1) \qquad \text{and} \qquad 0<\frac{\mathrm{d}(\mu_2 F_0(s))}{\mathrm{d}s}\le\frac{\mathrm{d}(sf_0(s))}{\mathrm{d}s},$$

which implies that

$$\mu_2 F_0(s) \leq s f_0(s) \qquad \text{for } 1 \leq s < \infty.$$

Thus the assumption(H_3) is satisfied for $\mu = \min\{\mu_1, \mu_2\} > 2$. (H_4) can be checked by similar computations and we omit them here. But at the same time we have

$$\lim_{s\to+\infty} s f_0(s) e^{-s^2} = 0.$$

Thus $f_0(s)$ can not satisfy the assumption (H_5) in [18].

2 Preliminaries

For readers' convenience, we first list some lemmas indicating the geometric conditions of the mountain-pass theorem for the functional

$$J_{\epsilon,\beta}(u) = \frac{\|u\|^n}{n} - \int_{\mathbb{R}^n} \frac{F(x,u)}{|x|^\beta} \mathrm{d}x - \epsilon \int_{\mathbb{R}^n} hu \mathrm{d}x. \tag{2.1}$$

For proofs, one can refer to [16] (see earlier work [10] for the case $\beta = 0$).

Lemma 2.1. ([16, Lemma 4.1]) *If* (H_3) *and* (H_4) *are satisfied, then for any nonnegative, compactly supported function* $u \in W^{1,n}(\mathbb{R}^n) \setminus \{0\}$*, there holds* $J_{\epsilon,\beta}(tu) \to -\infty$ *as* $t \to \infty$*.*

Lemma 2.2. ([16, Lemma 4.2]) *If* (H_1) *and* (H_6) *are satisfied, then there exists* $\epsilon_1 > 0$ *such that for* $0 < \epsilon < \epsilon_1$*, there exist* $\rho_\epsilon, r_\epsilon > 0$ *such that* $J_{\epsilon,\beta}(u) \geq \rho_\epsilon$ *for all* u *with* $\|u\| = r_\epsilon$*. Furthermore,* r_ϵ *can be chosen such that* $r_\epsilon \to 0$ *as* $\epsilon \to 0$*.*

Lemma 2.3. ([16, Lemma 4.3]) *Assume* (H_1) *and* $h \not\equiv 0$*. Then there exist* $\tau > 0$ *and* $v \in W^{1,n}$ *with* $\|v\| = 1$ *such that* $J_{\epsilon,\beta}(tv) < 0$ *for all* $0 < t < \tau$*. In particular,*

$$\inf_{\|u\| \leq \tau} J_{\epsilon,\beta}(u) < 0.$$

Next we analyze the compactness of Palais-Smale sequences of $J_{\epsilon,\beta}$.

Lemma 2.4. ([16, Lemma 4.6]) *Assume* (H_1) *and* (H_3)*. If* $\{u_k\}$ *is a Palais-Smale sequence for* $J_{\epsilon,\beta}$ *such that*

$$\liminf_{n\to\infty} \|u_k\|^{n/(n-1)} \leq \left(1 - \frac{\beta}{n}\right) \frac{\alpha_n}{\alpha_0},$$

then $\{u_k\}$ *has a subsequence which converges strongly to a weak solution of* (1.1)*.*

Lemma 2.5. *For* ϵ *small enough, Eq.* (1.1) *has a minimum type solution* u_0 *with* $J_{\epsilon,\beta}(u_0) = c_0 < 0$*, where* c_0 *is defined by*

$$-\infty < c_0 = \inf_{\|u\| \leq r_\epsilon} J_{\epsilon,\beta}(u) < 0. \tag{2.2}$$

Proof. By Lemma 2.2, we can choose ϵ sufficiently small such that

$$(r_\epsilon)^{n/(n-1)} < \left(1-\frac{\beta}{n}\right)\frac{\alpha_n}{\alpha_0}.$$

Since $\bar{B}_{r_\epsilon}$ is a complete, convex and $J_{\epsilon,\beta}$ is of class C^1 and bounded below on $\bar{B}_{r_\epsilon}$, by Ekeland variational principle, there exists a sequence $\{u_k\}$ in $\bar{B}_{r_\epsilon}$ such that

$$J_{\epsilon,\beta}(u_k) \to c_0 = \inf_{\|u\|\le r_\epsilon} J_{\epsilon,\beta}(u_k) \qquad \text{and} \qquad \|J'_{\epsilon,\beta}(u_k)\|_{E^*} \to 0.$$

Observing that

$$\|u_k\|^{n/(n-1)} \le (r_\epsilon)^{n/(n-1)} < \left(1-\frac{\beta}{n}\right)\frac{\alpha_n}{\alpha_0},$$

by Lemma 2.4, there exists a subsequence of $\{u_k\}$ which converges strongly to a solution u_0 of (1.1). It is obviously that $J_{\epsilon,\beta}(u_0) = c_0 < 0$. □

Lemma 2.6. ([16, Lemma 4.4]) *Assume* (H_1) *and* (H_3). *Let* $\{u_k\} \subset E$ *be an arbitrary Palais-Smale sequence. Then there exist a subsequence of* $\{u_k\}$, *still denoted by* $\{u_k\}$, *and* $u \in E$ *such that*

$$\begin{cases} \dfrac{f(x,u_k)}{|x|^\beta} \to \dfrac{f(x,u)}{|x|^\beta} & \text{in } L^1_{loc}(\mathbb{R}^n), \\ \nabla u_k(x) \to \nabla u(x) & \text{for almost every } x \in \mathbb{R}^n, \\ |\nabla u_k|^{n-2}\nabla u_k \rightharpoonup |\nabla u|^{n-2}\nabla u & \text{in } (L^{n/(n-1)}(\mathbb{R}^n))^n. \end{cases}$$

Furthermore, u is a weak solution of (1.1).

Lemma 2.7. *Assume* (H_1)-(H_6). *Then for sufficiently small* ϵ, *the problem* (1.1) *has a mountain-pass type solution* v_0.

Proof. We know from the previous lemmas that $J_{\epsilon,\beta}$ satisfies the mountain-pass conditions except the Palais-Smale condition. Thus, there exists a sequence $\{v_k\}$ in E such that

$$J_{\epsilon,\beta}(v_k) \to c_1 > 0 \qquad \text{and} \qquad \|J'_{\epsilon,\beta}(v_k)\|_{E^*} \to 0,$$

where $c_1 \ge \rho_\epsilon > 0$ is the mountain-pass level. Then by Lemma 2.6, there exists v_0 such that $v_k \rightharpoonup v_0$ in E and v_0 is a weak solution of (1.1). □

To get some more precise information of the minimax level obtained by the mountain-pass theorem, we have

Lemma 2.8. *If* (H_2) *satisfied, then there exist a constant* $\epsilon_0 > 0$ *and a function* $u_p \in E$ *which satisfies* $\|u_p\| = S_p$ *such that for* $M_\epsilon(t): [0,+\infty) \to \mathbb{R}$ *which is given by*

$$M_\epsilon(t) := \frac{t^n}{n}\|u_p\|^n - \int_{\mathbb{R}^n} \frac{F(tu_p)}{|x|^\beta}\mathrm{d}x - t\int_{\mathbb{R}^n} \epsilon h u_p \mathrm{d}x,$$

we have

$$\max_{t\geq 0} M_\epsilon(t) < \frac{(n-\beta)^{n-1}}{n^n}\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1} \qquad \textit{for all } \epsilon<\epsilon_0.$$

Proof. Choose a bounded sequence of functions $\{u_k\}\subset E$ such that

$$\int_{\mathbb{R}^n}\frac{|u_k|^p}{|x|^\beta}\mathrm{d}x=1 \quad \text{and} \quad \|u_k\|\to S_p.$$

Then we can assume that

$$\begin{aligned}
&u_k \rightharpoonup u_p && \text{in } E,\\
&u_k \to u_p && \text{in } L^q(\mathbb{R}^n) \text{ for all } q\in[1,+\infty),\\
&u_k(x)\to u_p(x) && \text{almost everywhere.}
\end{aligned}$$

These imply that

$$\int_{\mathbb{R}^n}\frac{|u_k|^p}{|x|^\beta}\mathrm{d}x\to\int_{\mathbb{R}^n}\frac{|u_p|^p}{|x|^\beta}\mathrm{d}x=1.$$

On the other hand, we have

$$\|u_p\|\leq\liminf_{k\to\infty}\|u_k\|=S_p.$$

Thus we get $\|u_p\|=S_p$. By (H_2) and $\int_{\mathbb{R}^n}|u_p|^p/|x|^\beta\mathrm{d}x=1$, we have

$$\begin{aligned}
M_\epsilon(t)&\leq\frac{t^n}{n}\|u_p\|^n-C_p\frac{t^p}{p}\int_{\mathbb{R}^n}\frac{u_p^p}{|x|^\beta}\mathrm{d}x+\epsilon t\|h\|_{(W^{1,n})^*}\|u_p\|\\
&=\frac{t^n}{n}S_p^n-C_p\frac{t^p}{p}+\epsilon t\|h\|_{(W^{1,n})^*}\|u_p\|\\
&\leq\frac{(p-n)}{np}\frac{S_p^{np/(p-n)}}{C_p^{n/(p-n)}}+\epsilon t_0\|h\|_{(W^{1,n})^*}\|u_p\|,
\end{aligned}$$

where t_0 is a constant which belongs to $[0,+\infty)$ and is independent of the choice of ϵ. By choosing ϵ small enough, we get the desired results from (H_2) immediately. $\square$

Remark 2.1. By Lemmas 2.2 and 2.8, we can conclude that for ϵ sufficiently small

$$0<c_1<c_0+\frac{(n-\beta)^{n-1}}{n^n}\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1}.$$

3 Proof of the main results

For simplicity, we define

$$R_{n-2}(\alpha,s)=\exp\{\alpha s^{n/(n-1)}\}-\sum_{m=0}^{n-2}\alpha_0^m|s|^{mn/(n-1)}/m!.$$

Lemma 3.1. *Let $\alpha>0$ and $r>1$. Then for any $\beta>r$ there exists a positive constant C which only depends on α such that for all $s\in\mathbb{R}^+$*

$$(R_{n-2}(\alpha,s))^r\le CR_{n-2}(\alpha\beta,s).$$

Proof. By using L'Hospital's rule $(n-1)$ times, we get

$$\lim_{s\to 0}\frac{R_{n-2}(\alpha,s)}{R_{n-2}(\alpha\beta,s)}=\lim_{s\to 0}\frac{1}{\beta^{n-1}}\frac{\exp\{\alpha s^{n/(n-1)}\}}{\exp\{\alpha\beta s^{n/(n-1)}\}}=\frac{1}{\beta^{n-1}}.$$

Then we can conclude that

$$\lim_{s\to 0}\frac{(R_{n-2}(\alpha,s))^r}{R_{n-2}(\alpha\beta,s)}=\lim_{s\to 0}(R_{n-2}(\alpha,s))^{r-1}\frac{R_{n-2}(\alpha,s)}{R_{n-2}(\alpha\beta,s)}=0\cdot\frac{1}{\beta^{n-1}}=0.$$

On the other hand, we also have

$$\begin{aligned}&\lim_{s\to\infty}\frac{(R_{n-2}(\alpha,s))^r}{R_{n-2}(\alpha\beta,s)}\\&=\lim_{s\to\infty}\frac{\exp\{r\alpha s^{n/(n-1)}\}}{\exp\{\alpha\beta s^{n/(n-1)}\}}\frac{\left\{1-B_{n-2}(\alpha,s)/\exp\{\alpha s^{n/(n-1)}\}\right\}^r}{1-B_{n-2}(\alpha\beta,s)/\exp\{\alpha\beta s^{n/(n-1)}\}}=0.\end{aligned}$$

Thus we complete the proof of the lemma. □

Proof of Theorem 1.1. Notice that $u_k\to u$ in $L^n(\mathbb{R}^n)$, $\nabla u_k\to\nabla u$ almost everywhere, and $\|u_k\|^n=1$, we obtain from Brezis-Lieb Lemma (cf. [19]) that

$$\|u_k-u\|^n=1-\|u\|^n+o(1). \tag{3.1}$$

Then for k large enough, we have

$$p\|u_k-u\|^{n/(n-1)}<\left(1-\frac{\beta}{n}\right)\alpha_n. \tag{3.2}$$

We claim that for every $x\in\mathbb{R}^n$, we have

$$|u_k(x)|^{n/(n-1)}\le(1+\eta)|u_k(x)-u(x)|^{n/(n-1)}+C|u(x)|^{n/(n-1)} \tag{3.3}$$

for some constant C depending only on n and η, where ϵ is a small positive number to be chosen later. In fact, define a set S by

$$S \triangleq \{x\in\mathbb{R}^n : |u_k(x)-u(x)|>2|u(x)|\}.$$

For $x\notin S$, the inequality in the claim is obvious. For $x\in S$

$$\begin{aligned}
&|u_k(x)|^{n/(n-1)}=|u_k(x)-u(x)+u(x)|^{n/(n-1)}\\
=&|u_k(x)-u(x)|^{n/(n-1)}\left(1+\frac{|u(x)|}{|u_k(x)-u(x)|}\right)^{n/(n-1)}\\
\leq&|u_k(x)-u(x)|^{n/(n-1)}+n/(n-1)(3/2)^{1/(n-1)}|u(x)||u_k(x)-u(x)|^{1/(n-1)}\\
\leq&(1+\eta)|u_k(x)-u(x)|^{n/(n-1)}+(3/2)^{n/(n-1)^2}\eta^{-1/(n-1)}(n-1)^{-1/(n-1)}|u(x)|^{n/(n-1)},
\end{aligned}$$

where we use the mean value theorem at the first inequality and Young's inequality at the second inequality. By a straightforward calculation, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\frac{R_{n-2}(p,u_k)}{|x|^\beta}\mathrm{d}x\\
\leq&\int_{\mathbb{R}^n}\frac{R_{n-2}\left(p,\left((1+\eta)|u_k-u|^{n/(n-1)}+C|u|^{n/(n-1)}\right)^{(n-1)/n}\right)}{|x|^\beta}\mathrm{d}x\\
\leq&\int_{\mathbb{R}^n}\frac{\frac{1}{q}\exp\{pq(1+\eta)|u_k-u|^{n/(n-1)}\}-B_{n-2}((1+\eta)p,(u_k-u))}{|x|^\beta}\mathrm{d}x\\
&+\int_{\mathbb{R}^n}\frac{\frac{1}{r}\exp\{pCr|u|^{n/(n-1)}\}-B_{n-2}(Cp,u)}{|x|^\beta}\mathrm{d}x\\
=&\frac{1}{q}\int_{\mathbb{R}^n}\frac{R_{n-2}\left(pq(1+\eta)\|u_k-u\|^{n/(n-1)},(u_k-u)/\|u_k-u\|\right)}{|x|^\beta}\mathrm{d}x\\
&+\int_{\mathbb{R}^n}\sum_{m=0}^{n-2}\frac{(q^{m-1}-1)p^m(1+\eta)^m|u_k-u|^{mn/(n-1)}}{m!|x|^\beta}\mathrm{d}x\\
&+\int_{\mathbb{R}^n}\frac{\frac{1}{r}\exp\{pCr|u|^{n/(n-1)}\}-B_{n-2}(\alpha,u)}{|x|^\beta}\mathrm{d}x,
\end{aligned}\tag{3.4}$$

where $1/q+1/r=1$. At the first inequality of the above computations, we use (3.3) and the fact that $R_{n-2}(\alpha,s)$ is increasing in $s^{n/(n-1)}$. The second inequality is from Young's inequality and the fact that for any $a,b\geq 0$ and $m\in\mathbb{N}^+$, $(a+b)^m\geq a^m+b^m$. Combining (3.2) and the embedding $E\hookrightarrow L^q$, we conclude that the second term on the right hand side of (3.4) is finite. From (3.2) again, we can choose q, r and η such that

$$pq(1+\eta)\|u_k-u\|^{n/(n-1)}<\left(1-\frac{\beta}{n}\right)\alpha_n.$$

Then Theorem 1.1 follows from (3.4) and Theorem A. □

Proof of Theorem 1.2. Recall that u_0 and v_0 are the two solutions got in Lemma 2.5 and 2.7. From the constructions of u_0 and v_0, we have that there are two sequences $\{u_k\}$ and $\{v_k\}$ in E such that

$$\begin{aligned}
&u_k \to u_0 &&\text{and} && v_k \rightharpoonup v_0,\\
&J_\varepsilon(u_k) \to c_0 < 0 &&\text{and} && J_\varepsilon(v_k) \to c_1 > 0,\\
&J'_\varepsilon(u_k)u_k \to 0 &&\text{and} && J'_\varepsilon(v_k)v_k \to 0.
\end{aligned}$$

We will show a contradiction under the assumption that $u_0 = v_0$.

Here we claim that $F(v_k)/|x|^\beta \to F(u_0)/|x|^\beta$ in $L^1(\mathbb{R}^n)$. To this end, we only need to show that $f(v_k)/|x|^\beta \to f(u_0)/|x|^\beta$ in $L^1(\mathbb{R}^n)$. Then the dominated convergence theorem implies the claim immediately. Since $f(x,s) \geq 0$, it is sufficient to prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^n}\frac{f(x,v_k)}{|x|^\beta}\mathrm{d}x = \int_{\mathbb{R}^n}\frac{f(x,u_0)}{|x|^\beta}\mathrm{d}x.$$

Let M denotes a constant to be determined later. We have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\frac{f(x,v_k)}{|x|^\beta}\mathrm{d}x - \int_{\mathbb{R}^n}\frac{f(x,u_0)}{|x|^\beta}\mathrm{d}x\right|\\
=&\left|\int_{|v_k|<M}\frac{f(x,v_k)}{|x|^\beta}\mathrm{d}x - \int_{|u_0|<M}\frac{f(x,u_0)}{|x|^\beta}\mathrm{d}x\right| + \int_{|v_k|\geq M}\frac{f(x,v_k)}{|x|^\beta}\mathrm{d}x + \int_{|u_0|\geq M}\frac{f(x,u_0)}{|x|^\beta}\mathrm{d}x. \quad (3.5)
\end{aligned}$$

Since $\{v_k\}$ is a Palais-Smale sequence, we have

$$\frac{1}{n}\|v_k\|^n - \int_{\mathbb{R}^n}\frac{F(x,v_k)}{|x|^\beta}\mathrm{d}x - \int_{\mathbb{R}^n}\varepsilon h v_k \mathrm{d}x \to c_1, \quad (3.6)$$

$$\left|\langle J'_\varepsilon(v_k),\varphi\rangle\right| \leq \tau_k\|\varphi\| \qquad \text{for all } \varphi\in E, \quad (3.7)$$

where $\tau_k \to 0$ as $k\to\infty$. Multiplying (3.6) by μ and subtracting (3.7) with $\varphi = v_k$, we obtain from (H_3) that

$$\left(\frac{\mu}{n}-1\right)\|v_k\|^n \leq C(1+\|v_k\|).$$

This implies that $\{v_k\}$ is a bounded sequence and thus

$$\int_{\mathbb{R}^n}\frac{f(x,v_k)v_k}{|x|^\beta}\mathrm{d}x \leq C. \quad (3.8)$$

Since u_0 and $f(x,u_0)/|x|^\beta$ are in $L^1(\mathbb{R}^n)$, for any $\delta > 0$, we can choose some M sufficiently large, for example, $M > C/\delta$, where C is the constant in (3.8), such that

$$\int_{|u_0|\geq M}\frac{f(x,u_0)}{|x|^\beta}\mathrm{d}x < \delta. \quad (3.9)$$

From (3.8), we also have

$$\int_{|v_k|\geq M}\frac{f(x,v_k)}{|x|^\beta}\mathrm{d}x\leq\frac{1}{M}\int_{|v_k|\geq M}\frac{f(x,v_k)v_k}{|x|^\beta}\mathrm{d}x<\delta. \tag{3.10}$$

By (H_1), for $x\in\{x\in\mathbb{R}^n:|v_k|<M\}$, there exists a constant C_1 depending on M such that $|f(x,v_k)|\leq C_1|v_k|^{n-1}$. Since we have $|x|^{-\beta}|v_k|^{n-1}\to|x|^{-\beta}|u_0|^{n-1}$ strongly in $L^1(\mathbb{R}^n)$ and $v_k\to u_0$ almost everywhere in $\mathbb{R}^n$, the Lebesgue's dominated convergence theorem implies that

$$\lim_{k\to\infty}\int_{|v_k|<M}\frac{f(x,v_k)}{|x|^\beta}\mathrm{d}x=\int_{|u_0|<M}\frac{f(x,u_0)}{|x|^\beta}\mathrm{d}x. \tag{3.11}$$

Combining (3.5)-(3.11), we get our claim proved. Let

$$w_k=\frac{v_k}{\|v_k\|}\quad\text{and}\quad w_0=\frac{u_0}{\lim\limits_{k\to\infty}\|v_k\|}.$$

We have $\|w_k\|=1$ and $w_k\rightharpoonup w_0$ in E. In particular $\|w_0\|\leq1$. To proceed, we distinguish two cases:

Case 1. $\|w_0\|=1$. In this case, we have

$$\lim_{k\to\infty}\|v_k\|=\|u_0\|.$$

Therefore, $v_k\to u_0$ in E. Combining the fact that

$$\frac{F(v_k)}{|x|^\beta}\to\frac{F(u_0)}{|x|^\beta}\qquad\text{in } L^1(\mathbb{R}^n),$$

we have

$$J_{\epsilon,\beta}(v_k)\to J_{\epsilon,\beta}(u_0)=c_0,$$

which is a contradiction with our assumption.

Case 2. $\|w_0\|<1$. Since

$$0<c_1<c_0+\frac{(n-\beta)^{n-1}}{n^n}\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1}=J_{\epsilon,\beta}(u_0)+\frac{(n-\beta)^{n-1}}{n^n}\left(\frac{\alpha_n}{\alpha_0}\right)^{n-1},$$

we can choose some $q>1$ sufficiently close to 1 and $\delta>0$ such that

$$q\alpha_0\|v_k\|^{n/(n-1)}\leq(1-\beta/n)\frac{\alpha_n\|v_k\|^{n/(n-1)}}{(n(c_1-J_{\epsilon,\beta}(u_0)))^{1/(n-1)}}-\delta.$$

Notice that

$$\lim_{k\to\infty}\|v_k\|^n(1-\|w_0\|^n)=\lim_{k\to\infty}\|v_k\|^n-\|u_0\|^n=n(c_1-J_{\epsilon,\beta}(u_0)),$$

where we have used $v_k \rightharpoonup u_0$ in E, $v_k \to u_0$ in $L^n(\mathbb{R}^n)$ and $F(x,v_k)/|x|^\beta \to F(x,u_0)/|x|^\beta$ in $L^1(\mathbb{R}^n)$, we get for k sufficiently large,

$$q\alpha_0 \|v_k\|^{n/(n-1)} < \left(1-\frac{\beta}{n}\right)\frac{\alpha_n}{(1-\|w_0\|^n)^{1/(n-1)}}.$$

Then it follows from Lemma 2.6 and Theorem 1.1 that

$$\int_{\mathbb{R}^n} \frac{R_{n-2}(q\alpha_0\|v_k\|^{n/(n-1)},w_k)}{|x|^\beta}\mathrm{d}x \leq C.$$

Therefore by (H_1) and Lemma 3.1 we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \frac{f(x,v_k)(v_k-u_0)}{|x|^\beta}\mathrm{d}x\right| \\
\leq &\left|\int_{\mathbb{R}^n} \frac{b_1 v_k^{n-1}(v_k-u_0)}{|x|^\beta}\mathrm{d}x\right| + \left|\int_{\mathbb{R}^n} \frac{b_2 R_{n-2}(\alpha_0,v_k)(v_k-u_0)}{|x|^\beta}\mathrm{d}x\right| \\
\leq &C\|v_k-u_0\|_{L^p} + C\|v_k-u_0\|_{L^p}\left(\int_M \frac{R_{n-2}(q\alpha_0\|v_k\|^{n/(n-1)},w_k)}{|x|^{q\beta}}\mathrm{d}x\right)^{1/q} \\
\leq &C\|v_k-u_0\|_{L^p} \to 0.
\end{aligned}$$

From this convergence and $J'_\varepsilon(v_k)(v_k-u_0)\to 0$, we get

$$\int_M |\nabla v_k|^{n-2}\nabla v_k(\nabla v_k-\nabla u_0)\mathrm{d}x + \int_M |v_k|^{n-2}v_k(v_k-u_0)\mathrm{d}x \to 0.$$

Moreover, since $v_k \rightharpoonup u_0$, we have

$$\int_M |\nabla u_0|^{n-2}\nabla u_0(\nabla v_k-\nabla u_0)\mathrm{d}x \to 0 \quad \text{and} \quad \int_M |u_0|^{n-2}u_0(v_k-u_0)\mathrm{d}x \to 0.$$

Using the inequality $(|a|^{n-2}a-|b|^{n-2}b)(a-b)\geq 2^{2-n}|a-b|^n$, $\forall a,b\in\mathbb{R}^n$, it follows that

$$\begin{aligned}
&\int_M |\nabla v_k-\nabla u_0|^n\mathrm{d}x + \int_M |v_k-u_0|^n\mathrm{d}x \\
\leq &C\int_M (|\nabla v_k|^{n-2}\nabla v_k - |\nabla u_0|^{n-2}\nabla u_0)(\nabla u_k-\nabla u_0)\mathrm{d}x \\
&+C\int_M (|v_k|^{n-2}v_k-|u_0|^{n-2}u_0)(v_k-u_0)\mathrm{d}x \to 0.
\end{aligned}$$

Therefore we get $v_k \to u_0$ in E. This implies $J_\varepsilon(v_k)\to J_\varepsilon(u_0)=c_0$, which is still a contradiction and the proof is finished. □

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References

[1] Adimurthi, Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **17** (1990), 393-413.

[2] do Ó J. M., Semilinear Dirichlet problems for the N-Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range, *Differential and Integral Equations*, **9** (1996), 967-979.

[3] Cao D. M., Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$, *Comm. Partial Differential Equations*, **17** (1992), 407-435.

[4] do Ó J. M., N-Laplacian equations in $\mathbb{R}^n$ with critical growth, *Abstract and Applied Analysis*, **2** (1997), 301-315.

[5] Figueiredo D. G. de, Miyagaki O. H., and Ruf B., Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, *Calc. Var.*, **3** (1995), 139-153.

[6] do Ó J. M. and Yang Y. Y., A quasi-linear elliptic equation with critical growth on compact Riemannian manifold without boundary, *Ann. Glob. Anal. Geom.*, **38** (2010), 317-334.

[7] Yang Y. Y. and Zhao L., A class of Adams-Fontana type inequalities and related functionals on manifolds, *Nonlinear Diff. Eqn. Appl.*, **17** (2009), 119-135.

[8] Zhao L., Exponential problem on a compact Riemannian manifold without boundary, *Nonlinear Analysis*, **75** (2012), 433-443.

[9] do Ó J. M., Medeiros E., and Severo U., A nonhomogeneous elliptic problem involving critical growth in dimension two, *J. Math. Anal. Appl.*, **345** (2008), 286-304.

[10] do Ó J. M., Medeiros E., and Severo U., On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$, *J. Differential Equations*, **246** (2009), 1363-1386.

[11] do Ó J. M., On a class of singular Trudinger-Moser type inequalities and its applications, unpublished results.

[12] Rabinowitz P. H., On a class of nonlinear Schrödinger equations, *Zeit. Anger. Math. Phys.*, **43** (1992), 270-291.

[13] Moser J., A sharp form of an inequality by N. Trudinger, *Ind. Univ. Math. J.*, **20** (1971), 1077-1091.

[14] Pohozaev S., The Sobolev embedding in the special case $pl=n$, Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964-1965, Mathematics Section (Moskov. Energet. Inst., Moscow) (1965), 158-170.

[15] Trudinger N. S., On embeddings into Orlicz spaces and some applications, *J. Math. Mech.*, **17** (1967), 473-484.

[16] Adimurthi and Yang Y. Y., An interpolation of Hardy inequality and Trudinger-Moser inequality in $\mathbb{R}^N$ and its applications, *International Mathematics Research Notices*, **13** (2010), 2394-2426.

[17] Yang Y. Y., Adams type inequalities and related elliptic partial differential equations in dimension four, *J. Differential Equations*, **252** (2012), 2266-2295.

[18] Yang Y. Y., Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space, *J. Funct. Anal.*, **262** (2012), 1679-1704.

[19] Brézis H. and Lieb E. H., A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.*, **48** (1983), 486-499.