JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **25**, No. 1, pp. 66-78

doi: 10.4208/jpde.v25.n1.5
February 2012

# Boundary Layers Associated with a Coupled Navier-Stokes/Allem-Cahn System: The Non-Characteristic Boundary Case

XIE Xiaoqiang*

*School of Science, Shanghai Second Polytechnic University, Shanghai 201209, China.*

Received 6 July 2011; Accepted 24 November 2011

---

**Abstract.** The goal of this article is to study the boundary layer of Navier-Stokes/Allen-Cahn system in a channel at small viscosity. We prove that there exists a boundary layer at the outlet (down-wind) of thickness $\nu$, where $\nu$ is the kinematic viscosity. The convergence in $L^2$ of the solutions of the Navier-Stokes/Allen-Cahn equations to that of the Euler/Allen-Cahn equations at the vanishing viscosity was established. In two dimensional case we are able to derive the physically relevant uniform in space and time estimates, which is derived by the idea of better control on the tangential derivative and the use of an anisotropic Sobolve imbedding.

**AMS Subject Classifications**: 35Q35, 35K55, 76D05

**Chinese Library Classifications**: O175.29

**Key Words**: Boundary layers; Navier-Stokes; Euler equations; Allen-Cahn; vanishing viscosity limit.

---

## 1 Introduction

Consider the incompressible Navier-Stokes/Allen-Cahn system of the form

$$\begin{cases} \frac{\partial}{\partial t}\overrightarrow{u}^\nu+(\overrightarrow{u}^\nu\cdot\nabla)\overrightarrow{u}^\nu+\nabla p^\nu=\nu\triangle\overrightarrow{u}^\nu-\lambda\nabla\cdot(\nabla v^\nu\otimes\nabla v^\nu), \\ \nabla\cdot\overrightarrow{u}^\nu=0, \\ \frac{\partial}{\partial t}v^\nu+(\overrightarrow{u}^\nu\cdot\nabla)v^\nu=\gamma(\triangle v^\nu-f(v^\nu)), \end{cases} \tag{1.1}$$

where $\overrightarrow{u}^\nu$ is the velocity field, $v^\nu$ and $p^\nu$ denote the phase function and pressure, $\nu$, $\lambda$, $\gamma$ and $\varepsilon$ are positive constants, representing the kinematic viscosity, the surface tension, the

*Corresponding author. *Email address:* `xiaoqiangxie021@gmail.com` (X. Xie),

mobility and the width of the interface, respectively, and $f(x)=4(x^3-x)/\varepsilon^3$. System (1.1) can be viewed as a phase field model, which describes the motion of a mixture of two incompressible viscous fluids with the same density and viscosity (see [1])

We are interested in the asymptotic behavior of Navier-Stokes/Allen-Cahn equations in the channel $\Omega=(0,L_1)\times(0,L_2)\times(0,h)$ at small viscosity with fluids pumped into the channel at the top ($z=h$) and sucked out the channel at the bottom ($z=0$), the diffused interface has no slip on the boundary, i.e., $\overrightarrow{u}^\nu=(0,0,-U)$, $v^\nu=0$ ($U=\text{constant}>0$) at $z=0$ and $z=h$. Hence in this case the boundary which is permeable is non characteristic, i.e., it is not a stream surface. Throughout this article, we assume that all functions are periodic in $x$ (with period $L_1$) and in $y$ (with period $L_2$).

With regular initial data, $\overrightarrow{u}^\nu=\overrightarrow{u}_0$, $v^\nu=v_0$ at $t=0$, and compatibility conditions, strong solutions globally exist in 2D, but locally in 3D in short time (see for instance [2,3]).

At the limit case, namely $\nu=0$, Navier-Stokes/Allen-Cahn system formally becomes the following Euler/Allen-Cahn system:

$$\begin{cases} \dfrac{\partial}{\partial t}\overrightarrow{u}^0+(\overrightarrow{u}^0\cdot\nabla)\overrightarrow{u}^0+\nabla p^0=-\lambda\nabla\cdot(\nabla v^0\otimes\nabla v^0), \\ \nabla\cdot\overrightarrow{u}^0=0, \\ \dfrac{\partial}{\partial t}v^0+(\overrightarrow{u}^0\cdot\nabla)v^0=\gamma(\triangle v^0-f(v^0)), \end{cases} \tag{1.2}$$

with initial data

$$\overrightarrow{u}^0=\overrightarrow{u}_0,\quad v^0=v_0\qquad \text{at } t=0, \tag{1.3}$$

and boundary conditions:

$$\overrightarrow{u}^0=(0,0-U) \text{ and } v^0=0 \text{ at } z=h,\quad u_3^0=-U \text{ and } v^0=0 \text{ at } z=0. \tag{1.4}$$

We can not expect a convergence result of $\overrightarrow{u}^\nu$ to $\overrightarrow{u}^0$ in the uniform space since they do no have the same traces on the boundary.

The purpose of this paper is to present explicit boundary layer analysis of the Navier-Stokes/Allen-Cahn equations in the case when the boundary is non-characteristic. Our boundary layer analysis is performed in both $H^1$ space and physically more appealing uniform space. As a consequence we proved that the solutions of Navier-Stokes/Allen-Cahn equations can be approximated by the that of the Euler/Allen-Cahn equations uniformly away from the boundary. Our convergence rate is better controlled than the same results of the Navier-Stokes equations obtained in [4].

Our motivation is the study of the boundary layer associated with incompressible flows and the related question of vanishing viscosity (see for instance [4–17] and among many others).

The article is organized as follows. In Section 2, we show how to choose and construct the correctors for the Navier-Stokes/Allen-Cahn system. In Section 3, we present the short time result on the fully nonlinear case. The final section is devoted to $L^\infty$ boundary layer analysis to the nonlinear case in two dimensional space.

## 2 Preliminaries and the corrector

Our method to construct the corrector is based on the observation made in Ref. [4]. Utilizing a stretched coordinates, we discover that to the leading order, the difference between the solutions of the Navier-Stokes/Allen-Cahn and the Euler/Allen-Cahn system should satisfy the following Prandtl type equation

$$\begin{cases} -U\dfrac{\partial}{\partial z}\overrightarrow{\theta}^{\nu}+\nabla q^{\nu}=\nu\dfrac{\partial^2}{\partial z^2}\overrightarrow{\theta}^{\nu}, \\ \nabla\cdot\overrightarrow{\theta}^{\nu}=0, \\ \overrightarrow{\theta}^{\nu}=(0,0,-U)-\overrightarrow{u}^{0} \qquad \text{at } z=0,\ h. \end{cases} \tag{2.1}$$

We introduce a function set $X^m$ $(m\in\mathbb{N})$ in Ref. [6,18].

**Definition 2.1.** *We say a function $\theta^{\nu}\in X^m$, if and only if $\theta^{\nu}\in H^m((0,T)\times\Omega)$ and the following inequalities hold*

$$\|z^s\partial_t^{\alpha}\partial_{\tau}^{\beta}\partial_z^{\gamma}\theta^{\nu}\|_{L^2(0,T)\times\Omega}\le C\nu^{s-|\gamma|+\frac{1}{2}}, \tag{2.2}$$

*and*

$$\|z^s\partial_t^{\alpha}\partial_{\tau}^{\beta}\partial_z^{\gamma}\theta^{\nu}\|_{L^2_{t,\tau}(L^1_z)}\le C\nu^{s-|\gamma|+1}, \tag{2.3}$$

*for all $s\ge 0$, multi-index $\alpha$, $\beta$, $\gamma$ satisfying $|\alpha|+|\beta|+|\gamma|\le m$, where $\overrightarrow{\tau}$ is the simplification of $x$ and $y$ directions.*

**Theorem 2.1.** ( [18]) *If $u_0|_{z=0,z=h}\in H^{m+1}$. Then there exist $\overrightarrow{\theta}^{j}$, $j=1,\ 2,\ 3,\ 4$, such that $\overrightarrow{\theta}^{\nu}=\overrightarrow{\theta}^{1}+\nu\overrightarrow{\theta}^{2}$, $\overrightarrow{\theta}^{1}\in X^m$, $\overrightarrow{\theta}^{3}\in X^m$, $\|\overrightarrow{\theta}^{2}\|_{H^m}+\|\overrightarrow{\theta}^{4}\|_{H^m}\le C$,*

$$\begin{cases} -U\dfrac{\partial}{\partial z}\overrightarrow{\theta}^{\nu}+\nabla q^{\nu}=\nu\dfrac{\partial^2}{\partial z^2}\overrightarrow{\theta}^{\nu}+\overrightarrow{\theta}^{3}+\nu\overrightarrow{\theta}^{4}, \\ \nabla\cdot\overrightarrow{\theta}^{\nu}=0, \\ \overrightarrow{\theta}^{\nu}=(0,0,-U)-\overrightarrow{u}^{0} \ \textit{at}\ z=0,\ h. \end{cases} \tag{2.4}$$

*Moreover, $\|\theta_3^1\|_{L^2}\le C\nu^{\frac{3}{2}}$.*

**Remark 2.1.** If $m\ge 2$, it can be deduced that

$$\|z^2\frac{\partial}{\partial z}\theta^{\nu}\|_{L^\infty((0,T)\times\Omega)}\le C\nu.$$

Moreover the constant $C$ is linearly dependent on $H^2$ norm of $(0,0,-U)-\overrightarrow{u}^{0}|_{z=0,\ h}$. By the compatibility conditions, we have $(0,0,-U)-\overrightarrow{u}_{0}|_{z=0,z=h}=0$ at $t=0$. Then by continuous dependence, we deduce that there exists a $T^*>0$ such that $C\le 1/8$.

Next, consider the adjusted differences:

$$\overrightarrow{w}^{\nu}=\overrightarrow{u}^{\nu}-\overrightarrow{u}^{0}-\overrightarrow{\theta}^{\nu},\qquad \phi^{\nu}=v^{\nu}-v^{0}. \tag{2.5}$$

Hence the differences satisfy

$$\begin{aligned}&\frac{\partial}{\partial t}\overrightarrow{w}^{\nu}+(\overrightarrow{u}^{\nu}\cdot\nabla)\overrightarrow{w}^{\nu}+(\overrightarrow{w}^{\nu}\cdot\nabla)\overrightarrow{u}^{0}+(\overrightarrow{w}^{\nu}\cdot\nabla)\overrightarrow{\theta}^{\nu}+\nabla q\\&\qquad=\nu\triangle\overrightarrow{w}^{\nu}-\lambda\nabla\cdot(\nabla v^{\nu}\otimes\nabla\phi^{\nu})-\lambda\nabla\cdot(\nabla\phi^{\nu}\otimes\nabla v^{0})+\overrightarrow{g}^{\nu}, &(2.6)\\&\nabla\cdot\overrightarrow{w}^{\nu}=0, &(2.7)\\&\frac{\partial}{\partial t}\phi^{\nu}+(\overrightarrow{u}^{\nu}\cdot\nabla)\phi^{\nu}+(\overrightarrow{w}^{\nu}\cdot\nabla)v^{0}=\gamma(\triangle\phi^{\nu}-f(v^{\nu})+f(v^{0}))-(\overrightarrow{\theta}^{\nu}\cdot\nabla)v^{0}, &(2.8)\\&\overrightarrow{w}^{\nu}=0,\quad \phi^{\nu}=0\quad \text{at } z=0,\ h, &(2.9)\\&\overrightarrow{w}^{\nu}=0,\quad \phi^{\nu}=0\quad \text{at } t=0, &(2.10)\end{aligned}$$

where

$$\begin{aligned}&q=p^{\nu}-p^{0}-q^{\nu}, &(2.11)\\&\overrightarrow{g}^{\nu}=\nu\triangle\overrightarrow{u}^{0}-\frac{\partial}{\partial t}\overrightarrow{\theta}^{\nu}-(\overrightarrow{u}^{0}_{\tau}\cdot\nabla_{\tau})\overrightarrow{\theta}^{\nu}-(u_3^0+U)\frac{\partial}{\partial z}\overrightarrow{\theta}^{\nu}\\&\qquad-(\overrightarrow{\theta}^{\nu}\cdot\nabla)\overrightarrow{u}^{0}-(\overrightarrow{\theta}^{\nu}\cdot\nabla)\overrightarrow{\theta}^{\nu}-\nu\triangle_{\tau}\overrightarrow{\theta}^{\nu}-\overrightarrow{\theta}^{3}-\nu\overrightarrow{\theta}^{4}. &(2.12)\end{aligned}$$

Denote

$$\begin{aligned}&\overrightarrow{g}^{1}=-\frac{\partial}{\partial t}\overrightarrow{\theta}^{1}-(\overrightarrow{u}^{0}_{\tau}\cdot\nabla_{\tau})\overrightarrow{\theta}^{1}-(u_3^0+U)\frac{\partial}{\partial z}\overrightarrow{\theta}^{1}\\&\qquad-(\overrightarrow{\theta}^{1}\cdot\nabla)\overrightarrow{u}^{0}-(\overrightarrow{\theta}^{\nu}\cdot\nabla)\overrightarrow{\theta}^{1}-\nu\triangle_{\tau}\overrightarrow{\theta}^{1}-\overrightarrow{\theta}^{3}, &(2.13)\\&\overrightarrow{g}^{2}=\triangle\overrightarrow{u}^{0}-\frac{\partial}{\partial t}\overrightarrow{\theta}^{2}-(\overrightarrow{u}^{0}_{\tau}\cdot\nabla_{\tau})\overrightarrow{\theta}^{1}-(u_3^0+U)\frac{\partial}{\partial z}\overrightarrow{\theta}^{2}\\&\qquad-(\overrightarrow{\theta}^{2}\cdot\nabla)\overrightarrow{u}^{0}-\nu(\overrightarrow{\theta}^{\nu}\cdot\nabla)\overrightarrow{\theta}^{2}-\nu\triangle_{\tau}\overrightarrow{\theta}^{2}-\overrightarrow{\theta}^{4}. &(2.14)\end{aligned}$$

Then we have

$$\overrightarrow{g}^{\nu}=\overrightarrow{g}^{1}+\nu\overrightarrow{g}^{2},\qquad \|z\overrightarrow{g}^{1}\|_{L^2((0,T)\times\Omega)}\le C\nu^{\frac{3}{2}},\qquad \|\overrightarrow{g}^{2}\|_{L^2((0,T)\times\Omega)}\le C.$$

It is sufficient for us to check the term $z(u_3^0+U)\frac{\partial}{\partial z}\overrightarrow{\theta}^{1}$. Recalling to boundary conditions of the Euler/Allen-Cahn equations and the regularity assumptions, we have

$$\left\|z(u_3^0+U)\frac{\partial}{\partial z}\overrightarrow{\theta}^{1}\right\|_{L^2}\le\left\|\frac{u_0^3+U}{z}\right\|_{L^\infty}\left\|z^2\frac{\partial}{\partial z}\overrightarrow{\theta}^{1}\right\|_{L^2}\le C\nu^{\frac{3}{2}}. \tag{2.15}$$

## 3 A short time result in nonlinear case

We proceed with energy estimates $(2.6)\cdot\overrightarrow{w}^{\nu}-(2.8)\cdot\lambda\triangle\phi^{\nu}$, and integrating over $\Omega$. Let $|\cdot|$ and $|\cdot|_6$ denoting the norm as in $L^2(\Omega)$ and $L^6(\Omega)$, respectively. We have

$$\int_\Omega \frac{\partial}{\partial t}\overrightarrow{w}^{\nu}\cdot\overrightarrow{w}^{\nu}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\overrightarrow{w}^{\nu}|^2, \qquad \left|\int_\Omega(\overrightarrow{w}^{\nu}\cdot\nabla)\overrightarrow{u}^{0}\cdot\overrightarrow{w}^{2}\right|\le C|\overrightarrow{w}^{\nu}|^2, \tag{3.1}$$

$$\int_\Omega(\overrightarrow{u}^{\nu}\cdot\nabla)\overrightarrow{w}^{\nu}\cdot\overrightarrow{w}^{\nu}=-\frac{1}{2}\int_\Omega|\overrightarrow{w}^{\nu}|^2\nabla\cdot\overrightarrow{u}^{\nu}=0, \tag{3.2}$$

$$\begin{aligned}\left|\int_\Omega(\overrightarrow{w}^{\nu}\cdot\nabla)\overrightarrow{\theta}^{\nu}\cdot\overrightarrow{w}^{\nu}\right|&\le\left\|z^2\nabla\overrightarrow{\theta}^{\nu}\right\|_{L^\infty}\left|\frac{\overrightarrow{w}^{\nu}}{z}\right|^2, \quad \text{(thanks to Remark 2.1)}\\ &\le\frac{\nu}{8}|\nabla\overrightarrow{w}|^2, \quad \text{(by Hardy's inequality, as } t\le T^*)\end{aligned} \tag{3.3}$$

$$\int_\Omega\nabla q\cdot\overrightarrow{w}^{\nu}=0, \quad -\lambda\int_\Omega\frac{\partial}{\partial t}\phi^{\nu}\cdot\triangle\phi^{\nu}=\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\phi^{\nu}|^2, \quad -\int_\Omega\triangle\overrightarrow{w}^{\nu}\cdot\overrightarrow{w}^{\nu}=|\nabla\overrightarrow{w}^{\nu}|^2, \tag{3.4}$$

$$\lambda\left|\int_\Omega[(\overrightarrow{u}^{0}+\overrightarrow{\theta}^{\nu})\cdot\nabla\phi^{\nu}]\cdot\triangle\phi^{\nu}\right|\le\frac{\lambda\gamma}{8}|\triangle\phi^{\nu}|^2+C|\nabla\phi^{\nu}|^2, \tag{3.5}$$

$$\lambda\int_\Omega\sum_{i,j}\partial_i(\partial_i\phi^{\nu}\partial_j\phi^{\nu})\cdot w_j^{\nu}-\lambda\int_\Omega\sum_{i,j}w_j^{\nu}\partial_j\phi^{\nu}\partial_{ii}\phi^{\nu}=\lambda\sum_{i,j}\int_\Omega w_j^{\nu}\partial_i\phi^{\nu}\partial_{ij}\phi^{\nu}=0, \tag{3.6}$$

$$\begin{aligned}&\lambda\left|\int_\Omega\sum_{i,j}\partial_i(\partial_i\phi^{\nu}\partial_j v^0)\cdot w_j^{\nu}-\int_\Omega\sum_{i,j}w_j^{\nu}\partial_j v^0\partial_{ii}\phi^{\nu}\right|\\ &\quad=\lambda\left|\sum_{i,j}\int_\Omega w_j^{\nu}\partial_i\phi^{\nu}\partial_{ij}v^0\right|\le C|\overrightarrow{w}^{\nu}|^2+C|\nabla\phi^{\nu}|^2,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&\lambda\left|\int_\Omega\sum_{i,j}\partial_i(\partial_i v^0\partial_j\phi^{\nu})\cdot w_j^{\nu}\right|\\ &\quad=\lambda\left|\sum_{i,j}\int_\Omega\partial_i(\partial_{ij}v^0\phi^{\nu})\cdot\overrightarrow{w}_j^{\nu}\right|\le C|\overrightarrow{w}^{\nu}|^2+C|\phi^{\nu}|^2+C|\nabla\phi^{\nu}|^2,\end{aligned} \tag{3.8}$$

$$\lambda\gamma\int_\Omega\triangle\phi^{\nu}\cdot\triangle\phi^{\nu}=\lambda\gamma|\triangle\phi^{\nu}|^2, \qquad \int_\Omega\phi^{\nu}\triangle\phi^{\nu}=-|\nabla\phi^{\nu}|^2, \tag{3.9}$$

$$\begin{aligned}&-\int_\Omega(3|v^0|^2\phi^{\nu}+3v^0|\phi^{\nu}|^2+|\phi^{\nu}|^2\phi^{\nu})\cdot\triangle\phi^{\nu}\\ &\quad=\int_\Omega(3|v^0|^2+6v^0\phi^{\nu}+3|\phi^{\nu}|^2)|\nabla\phi^{\nu}|^2-\int_\Omega\frac{3}{2}|\phi^{\nu}|^2\triangle|v^0|^2+|\phi^{\nu}|^2\phi^{\nu}\triangle v^0,\end{aligned} \tag{3.10}$$

$$\left|\int_\Omega\frac{3}{2}|\phi^{\nu}|^2\triangle|v^0|^2+|\phi^{\nu}|^2\phi^{\nu}\triangle v^0\right|\le C|\phi^{\nu}|^2+C|\phi^{\nu}|_3^3, \tag{3.11}$$

$$\left|\int_\Omega\overrightarrow{g}^{1}\cdot\overrightarrow{w}^{\nu}\right|\le C|z\overrightarrow{g}^{1}|\cdot|\nabla\overrightarrow{w}^{\nu}|\le\frac{\nu}{8}|\nabla\phi^{\nu}|^2+C\nu^2, \tag{3.12}$$

$$\left|\int_\Omega\nu\overrightarrow{g}^{2}\cdot\overrightarrow{w}^{\nu}\right|\le C|\overrightarrow{w}^{\nu}|^2+C\nu^2. \tag{3.13}$$

It follows from Theorem 2.1 that

$$\left|(\overrightarrow{\theta}^{\nu}\cdot\nabla)v^0\right|\leq|\theta_3^{\nu}\partial_z v^0|+\left|(z\overrightarrow{\theta}_{\tau}^{\nu}\cdot\nabla_{\tau})\frac{v^0}{z}\right|\leq C\nu. \tag{3.14}$$

Combing the inequalities (3.1)-(3.14), as $t\leq T^*$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(|\overrightarrow{w}^{\nu}|^2+|\nabla\phi^{\nu}|^2)+\nu|\nabla\overrightarrow{w}^{\nu}|^2+|\triangle\phi^{\nu}|^2\\ \leq& C|\overrightarrow{w}^{\nu}|^2+C|\phi^{\nu}|^2+C|\phi^{\nu}|_3^3+|\nabla\phi^{\nu}|^2+C\nu^2.\end{aligned} \tag{3.15}$$

In view of the regularity of Navier-Stokes/Allen-Cahn equations, i. e.,

$$\|\overrightarrow{u}^{\nu}\|_{L^\infty(0,T;L^2)}\leq C, \quad \text{and} \quad \|v^{\nu}\|_{L^\infty(0,T;H^1)}\leq C, \tag{3.16}$$

and applying Galiardo-Nirenberg interpolation inequality, we have

$$\|\phi^{\nu}\|_{L^3}^3\leq C(\|v^{\nu}\|_{H^1}+\|v^0\|_{H^1})\|\phi^{\nu}\|_{H^1}^2\leq C\|\phi^{\nu}\|_{H^1}^2. \tag{3.17}$$

Using the Gronwall inequality yields

**Theorem 3.1.** *Let $\overrightarrow{u}_0$ and $v_0$ be smooth functions satisfying certain compatibility conditions such that the Euler/Allen-Cahn system* (1.2) *is well-posed* (*at least in short time*). *Then, there exists a time $T^*>0$, as $\nu\to 0$,*

$$\|\overrightarrow{u}^{\nu}-\overrightarrow{u}^0-\overrightarrow{\theta}^{\nu}\|_{L^\infty(0,T^*;L^2)}\leq C\nu, \tag{3.18}$$

$$\|v^{\nu}-v^0\|_{L^\infty(0,T^*;H^1)}\leq C\nu, \tag{3.19}$$

$$\|\overrightarrow{u}^{\nu}-\overrightarrow{u}^0-\overrightarrow{\theta}^{\nu}\|_{L^2(0,T^*;H^1)}\leq C\nu^{\frac{1}{2}}, \tag{3.20}$$

$$\|v^{\nu}-v^0\|_{L^2(0,T^*;H^2)}\leq C\nu, \tag{3.21}$$

*where the corrector $\overrightarrow{\theta}^{\nu}$ is given by Theorem* 2.1.

This is a short time result. In general, it will be violated after a sufficiently long time. However, it might be possible for all time provided that the initial date is small.

## 4 Uniform estimates in space and time in two dimension

Our aim in this section is to derive estimates of the adjusted error, which is uniform in space and time. We will rely on anisotropic Sobolev imbedding and the idea of bounding the tangential derivatives.

Differentiate the equations (2.6)-(2.8) in time and obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\overrightarrow{w}^{\nu}_t+(\overrightarrow{u}^{\nu}_t\cdot\nabla)\overrightarrow{w}^{\nu}+(\overrightarrow{u}^{\nu}\cdot\nabla)\overrightarrow{w}^{\nu}_t+(\overrightarrow{w}^{\nu}_t\cdot\nabla)\overrightarrow{u}^{0}+(\overrightarrow{w}^{\nu}\cdot\nabla)\overrightarrow{u}^{0}_t\\&+(\overrightarrow{w}^{\nu}_t\cdot\nabla)\overrightarrow{\theta}^{\nu}+(\overrightarrow{w}^{\nu}\cdot\nabla)\overrightarrow{\theta}^{\nu}_t+\nabla q_t\\&\quad=\nu\triangle\overrightarrow{w}^{\nu}_t-\lambda\nabla(\nabla v^{\nu}_t\otimes\nabla\phi^{\nu})-\lambda\nabla(\nabla v^{\nu}\otimes\nabla\phi^{\nu}_t)-\lambda\nabla(\nabla\phi^{\nu}_t\otimes\nabla v^{0})\\&\qquad-\lambda\nabla(\nabla\phi^{\nu}\otimes\nabla v^{0}_t)+\overrightarrow{g}^{\nu}_t,\end{aligned}\tag{4.1}$$

$$\nabla\cdot\overrightarrow{w}^{\nu}_t=0,\tag{4.2}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\phi^{\nu}_t+(\overrightarrow{u}^{\nu}_t\cdot\nabla)\phi^{\nu}+(\overrightarrow{u}^{\nu}\cdot\nabla)\phi^{\nu}_t+(\overrightarrow{w}^{\nu}_t\cdot\nabla)v^{0}+(\overrightarrow{w}^{\nu}\cdot\nabla)v^{0}_t\\&\quad=\gamma(\triangle\phi^{\nu}_t-f'(v^{\nu})v^{\nu}_t+f'(v^{0})v^{0}_t)-(\overrightarrow{\theta}^{\nu}_t\cdot\nabla)v^{0}-(\overrightarrow{\theta}^{\nu}\cdot\nabla)v^{0}_t,\end{aligned}\tag{4.3}$$

$$\overrightarrow{w}^{\nu}_t=0,\qquad \phi^{\nu}_t=0,\qquad \text{at } z=0,\ z=h,\tag{4.4}$$

$$\overrightarrow{w}^{\nu}_t=\nu P(\triangle\overrightarrow{u}^{0}),\qquad \phi^{\nu}_t=0,\qquad \text{at } t=0,\tag{4.5}$$

and $P$ is the Leray-Hopf projector. Next, utilize the energy estimates for $\partial_t\overrightarrow{w}^{\nu}$ and $\partial_t\phi^{\nu}$, i.e., $(4.1)\cdot\overrightarrow{w}^{\nu}_t-(4.3)\cdot\lambda\triangle\phi^{\nu}_t$ and integrating over $\Omega$. Notice that

$$\int_\Omega\frac{\partial}{\partial t}\overrightarrow{w}^{\nu}_t\cdot\overrightarrow{w}^{\nu}_t=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\overrightarrow{w}^{\nu}_t|^2,\tag{4.6}$$

$$\begin{aligned}&\left|\int_\Omega(\overrightarrow{u}^{\nu}_t\cdot\nabla)\overrightarrow{w}^{\nu}\cdot\overrightarrow{w}^{\nu}_t\right|\le\left|\int_\Omega(\overrightarrow{u}^{0}_t+\overrightarrow{\theta}^{\nu}_t)\cdot\nabla\overrightarrow{w}^{\nu}_t\cdot\overrightarrow{w}^{\nu}\right|+\left|\int_\Omega(\overrightarrow{w}^{\nu}_t\cdot\nabla)\overrightarrow{w}^{\nu}\cdot\overrightarrow{w}^{\nu}_t\right|\\&\quad\le\Big(\|\overrightarrow{u}^{0}_t+\overrightarrow{\theta}^{\nu}_t\|_{L^\infty}\Big)|\overrightarrow{w}^{\nu}||\nabla\overrightarrow{w}^{\nu}_t|+|\overrightarrow{w}^{\nu}_t|_4^2|\nabla\overrightarrow{w}^{\nu}|\\&\qquad\text{(by Soblev imbedding and Interpolation)}\\&\quad\le\frac{\nu}{16}|\nabla\overrightarrow{w}^{\nu}_t|^2+C\nu+C|\overrightarrow{w}^{\nu}_t||\nabla\overrightarrow{w}^{\nu}_t||\nabla\overrightarrow{w}^{\nu}|\le\frac{\nu}{8}|\nabla\overrightarrow{w}^{\nu}_t|^2+C\nu+\frac{C}{\nu}|\nabla\overrightarrow{w}^{\nu}|^2|\overrightarrow{w}^{\nu}_t|^2,\end{aligned}\tag{4.7}$$

$$\int_\Omega(\overrightarrow{u}^{\nu}\cdot\nabla)\overrightarrow{w}^{\nu}_t\cdot\overrightarrow{w}^{\nu}_t=0,\qquad \left|\int_\Omega(\overrightarrow{w}^{\nu}_t\cdot\nabla)\overrightarrow{u}^{0}\cdot\overrightarrow{w}^{2}_t\right|\le C|\overrightarrow{w}^{\nu}_t|^2,\tag{4.8}$$

$$\begin{aligned}&\left|\int_\Omega(\overrightarrow{w}^{\nu}_t\cdot\nabla)\overrightarrow{\theta}^{\nu}\cdot\overrightarrow{w}^{\nu}_t\right|\le\left\|z^2\nabla\overrightarrow{\theta}^{\nu}\right\|_{L^\infty}\left|\frac{\overrightarrow{w}^{\nu}_t}{z}\right|^2\le\frac{\nu}{16}|\nabla\overrightarrow{w}_t|^2,\\&\qquad\text{(thanks to Remark 2.1 and Hardy's inequality, as } t\le T^*)\end{aligned}\tag{4.9}$$

$$\left|\int_\Omega(\overrightarrow{w}^{\nu}\cdot\nabla)\overrightarrow{u}^{0}_t\cdot\overrightarrow{w}^{2}_t\right|\le C|\overrightarrow{w}^{\nu}_t|^2+C\nu^2,\tag{4.10}$$

$$\begin{aligned}&\left|\int_\Omega(\overrightarrow{w}^{\nu}\cdot\nabla)\overrightarrow{\theta}^{\nu}_t\cdot\overrightarrow{w}^{\nu}_t\right|\le\left\|z^2\nabla\overrightarrow{\theta}^{\nu}\right\|_{L^\infty}\left|\frac{\overrightarrow{w}^{\nu}_t}{z}\right|\left|\frac{\overrightarrow{w}^{\nu}}{z}\right|\le\frac{\nu}{16}|\nabla\overrightarrow{w}_t|^2+C\nu|\nabla\overrightarrow{w}^{\nu}|^2,\\&\qquad\text{(thanks to Remark 2.1 and Hardy's inequality)}\end{aligned}\tag{4.11}$$

$$\int_\Omega\nabla q_t\cdot\overrightarrow{w}^{\nu}_t=0,\quad -\lambda\int_\Omega\frac{\partial}{\partial t}\phi^{\nu}_t\cdot\triangle\phi^{\nu}_t=\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\phi^{\nu}_t|^2,\quad -\int_\Omega\triangle\overrightarrow{w}^{\nu}_t\cdot\overrightarrow{w}^{\nu}_t=|\nabla\overrightarrow{w}^{\nu}_t|^2,\tag{4.12}$$

$$\lambda\left|\int_\Omega \nabla\cdot(\nabla v_t^0\otimes\nabla\phi^\nu)\cdot\overrightarrow{w}_t^\nu\right|+\lambda\left|\int_\Omega \nabla\cdot(\nabla\phi^\nu\otimes\nabla v_t^0)\cdot\overrightarrow{w}_t^\nu\right|$$
$$\le C(|\phi^\nu|+|\triangle\phi^\nu|)|\overrightarrow{w}_t^\nu|\le C|\triangle\phi^\nu|^2+C|\overrightarrow{w}_t^\nu|^2+C\nu^2, \tag{4.13}$$

$$\lambda\left|\int_\Omega \nabla\cdot(\nabla v^0\otimes\nabla\phi_t^\nu)\cdot\overrightarrow{w}_t^\nu\right|+\lambda\left|\int_\Omega \nabla\cdot(\nabla\phi_t^\nu\otimes\nabla v^0)\cdot\overrightarrow{w}_t^\nu\right|$$
$$\le C(|\nabla\phi_t^\nu|+|\triangle\phi_t^\nu|)|\overrightarrow{w}_t^\nu|\le \frac{\lambda\gamma}{16}|\triangle\phi_t^\nu|^2+C|\overrightarrow{w}_t^\nu|^2+C|\nabla\phi_t^\nu|^2, \tag{4.14}$$

$$\lambda\left|\int_\Omega \nabla\cdot(\nabla\phi_t^\nu\otimes\nabla\phi^\nu)\cdot\overrightarrow{w}_t^\nu\right|+\lambda\left|\int_\Omega \nabla\cdot(\nabla\phi^\nu\otimes\nabla\phi_t^\nu)\cdot\overrightarrow{w}_t^\nu\right|$$
$$\le C|\triangle\phi_t^\nu||\nabla\phi^\nu|_4|\overrightarrow{w}_t^\nu|_4+C|\triangle\phi^\nu||\nabla\phi_t^\nu|_4|\overrightarrow{w}_t^\nu|_4\le C|\triangle\phi_t^\nu||\triangle\phi^\nu||\overrightarrow{w}_t^\nu|^{\frac12}|\nabla\overrightarrow{w}_t^\nu|^{\frac12}$$
$$\le\frac{\lambda\gamma}{16}|\triangle\phi_t^\nu|^2+\frac{\nu}{16}|\nabla\overrightarrow{w}_t^\nu|^2+C\frac{|\triangle\phi^\nu|^2}{\nu}|\overrightarrow{w}_t^\nu|^2, \tag{4.15}$$

$$\lambda\left|\int_\Omega[(\overrightarrow{u}^0+\overrightarrow{\theta}^\nu)\cdot\nabla\phi_t^\nu]\cdot\triangle\phi_t^\nu\right|\le\frac{\lambda\gamma}{16}|\triangle\phi_t^\nu|^2+C|\nabla\phi_t^\nu|^2, \tag{4.16}$$

$$\lambda\left|\int_\Omega[(\overrightarrow{u}_t^0+\overrightarrow{\theta}_t^\nu)\cdot\nabla\phi^\nu]\cdot\triangle\phi_t^\nu\right|\le\frac{\lambda\gamma}{16}|\triangle\phi_t^\nu|^2+C\nu^2, \tag{4.17}$$

$$\lambda\left|\int_\Omega(\overrightarrow{w}_t^\nu\cdot\nabla)v^0\cdot\triangle\phi_t^\nu\right|\le\frac{\lambda\gamma}{16}|\triangle\phi_t^\nu|^2+C|\overrightarrow{w}_t^\nu|^2, \tag{4.18}$$

$$\lambda\left|\int_\Omega(\overrightarrow{w}^\nu\cdot\nabla)v_t^0\cdot\triangle\phi_t^\nu\right|\le\frac{\lambda\gamma}{16}|\triangle\phi_t^\nu|^2+C\nu^2, \tag{4.19}$$

$$\lambda\left|\int_\Omega(\overrightarrow{w}^\nu\cdot\nabla)\phi_t^\nu\cdot\triangle\phi_t^\nu\right|\le C|\overrightarrow{w}^\nu|_4|\nabla\phi_t^\nu|_4|\triangle\phi_t^\nu|\le C|\overrightarrow{w}^\nu|^{\frac12}|\nabla\overrightarrow{w}^\nu|^{\frac12}|\nabla\phi_t^\nu|^{\frac12}|\triangle\phi_t^\nu|^{\frac32}$$
$$\le\frac{\lambda\gamma}{16}|\triangle\phi_t^\nu|^2+C\nu^2|\nabla\overrightarrow{w}^\nu|^2|\nabla\phi_t^\nu|^2, \tag{4.20}$$

$$\lambda\left|\int_\Omega(\overrightarrow{w}_t^\nu\cdot\nabla)\phi^\nu\cdot\triangle\phi_t^\nu\right|\le C|\overrightarrow{w}_t^\nu|_4|\nabla\phi^\nu|_4|\triangle\phi_t^\nu|\le C|\overrightarrow{w}_t^\nu|^{\frac12}|\nabla\overrightarrow{w}_t^\nu|^{\frac12}|\nabla\phi^\nu|^{\frac12}|\triangle\phi^\nu|^{\frac12}|\triangle\phi_t^\nu|$$
$$\le\frac{\lambda\gamma}{16}|\triangle\phi_t^\nu|^2+\frac{\nu}{16}|\nabla\overrightarrow{w}_t^\nu|^2+C\nu|\triangle\phi^\nu|^2|\overrightarrow{w}_t^\nu|^2, \tag{4.21}$$

$$\lambda\gamma\int_\Omega\triangle\phi_t^\nu\cdot\triangle\phi_t^\nu=\lambda\gamma|\triangle\phi_t^\nu|^2, \tag{4.22}$$

$$|f'(v^\nu)\phi_t^\nu|^2\le C||\phi^\nu|^2\phi_t^\nu|^2+C|\phi_t^\nu|^2\le C|\phi_t^\nu|_4|\phi^\nu|_8+C|\phi_t^\nu|^2\le C|\nabla\phi_t^\nu|^2+C\nu^2, \tag{4.23}$$

$$|(f'(v^\nu)-f'(v^0))|^2\le C|\phi^\nu|^2+C|\phi^\nu|_4^2\le C\nu^2, \tag{4.24}$$

$$\left|\int_\Omega\overrightarrow{g}_t^1\cdot\overrightarrow{w}_t^\nu\right|\le C|z\overrightarrow{g}_t^1|\cdot|\nabla\overrightarrow{w}_t^\nu|\le\frac{\nu}{16}|\nabla\phi_t^\nu|^2+C\nu^2, \tag{4.25}$$

$$\left|\int_\Omega\nu\overrightarrow{g}_t^2\cdot\overrightarrow{w}_t^\nu\right|\le C|\overrightarrow{w}_t^\nu|^2+C\nu^2,\qquad\left|(\overrightarrow{\theta}_t^\nu\cdot\nabla)v^0\right|^2+\left|(\overrightarrow{\theta}^\nu\cdot\nabla)v_t^0\right|^2\le C\nu. \tag{4.26}$$

Combing (4.6)-(4.26), as $t\le T^*$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(|\overrightarrow{w}_t^\nu|^2+|\nabla\phi_t^\nu|^2\right)+\nu|\nabla\overrightarrow{w}_t^\nu|^2+|\triangle\phi_t^\nu|^2 \qquad (4.27)\\ \le& C\left(\frac{|\nabla\overrightarrow{w}^\nu|^2+|\triangle\phi^\nu|^2}{\nu}+1\right)|\overrightarrow{w}_t^\nu|^2+C\left(\nu^2|\nabla\overrightarrow{w}^\nu|^2+1\right)|\nabla\phi_t^\nu|^2+C|\nabla\overrightarrow{w}^\nu|^2+C|\triangle\phi^\nu|^2+C\nu.\end{aligned}$$

From this we deduce, after applying the usual Gronwall inequality and Theorem 2.1, that,

$$\|\overrightarrow{w}_t^\nu\|_{L^\infty(0,T^*;L^2)}\le C\nu^{\frac{1}{2}},\qquad \|\phi_t^\nu\|_{L^\infty(0,T^*;H^1)}\le C\nu^{\frac{1}{2}}, \tag{4.28}$$
$$\|\overrightarrow{w}_t^\nu\|_{L^2(0,T^*;H^1)}\le C,\qquad \|\phi_t^\nu\|_{L^2(0,T^*;H^2)}\le C\nu^{\frac{1}{2}}. \tag{4.29}$$

Recalling (3.15), we have

$$\nu|\nabla\overrightarrow{w}^\nu|^2+|\triangle\phi^\nu|^2\le C\nu^2+C|\overrightarrow{w}^\nu||\overrightarrow{w}_t^\nu|+C|\phi^\nu||\phi_t^\nu|\le C\nu^{\frac{3}{2}}. \tag{4.30}$$

In an analogous manner, we can derive estimates on tangential derivatives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\overrightarrow{w}_x^\nu|^2+|\nabla\phi_x^\nu|^2\right)+\nu|\nabla\overrightarrow{w}_x^\nu|^2+|\triangle\phi_x^\nu|^2\le C|\overrightarrow{w}_x^\nu|^2+C|\nabla\phi_x^\nu|^2+C\nu. \tag{4.31}$$

And we have

$$\|\overrightarrow{w}_x^\nu\|_{L^\infty(0,T^*;L^2)}\le C\nu^{\frac{1}{2}},\qquad \|\phi_x^\nu\|_{L^\infty(0,T^*;H^1)}\le C\nu^{\frac{1}{2}}, \tag{4.32}$$
$$\|\overrightarrow{w}_x^\nu\|_{L^2(0,T^*;H^1)}\le C,\qquad \|\phi_x^\nu\|_{L^2(0,T^*;H^2)}\le C\nu^{\frac{1}{2}}. \tag{4.33}$$

To obtain uniform estimates in time and space on $\overrightarrow{w}^\nu$ we need to derive uniform estimates in time on $|\nabla\overrightarrow{w}_x^\nu|$ (see for instance the anisotropic Sobolev imbedding theorem). For this purpose we differentiate (4.1)-(4.3) in $x$ and obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\overrightarrow{w}_{tx}^\nu+(\overrightarrow{u}_{tx}^\nu\cdot\nabla)\overrightarrow{w}^\nu+(\overrightarrow{u}_t^\nu\cdot\nabla)\overrightarrow{w}_x^\nu+(\overrightarrow{u}_x^\nu\cdot\nabla)\overrightarrow{w}_t^\nu+(\overrightarrow{u}^\nu\cdot\nabla)\overrightarrow{w}_{tx}^\nu\\ &+(\overrightarrow{w}_{tx}^\nu\cdot\nabla)\overrightarrow{u}^0+(\overrightarrow{w}_t^\nu\cdot\nabla)\overrightarrow{u}_x^0+(\overrightarrow{w}_x^\nu\cdot\nabla)\overrightarrow{u}_t^0+(\overrightarrow{w}^\nu\cdot\nabla)\overrightarrow{u}_{tx}^0\\ &+(\overrightarrow{w}_{tx}^\nu\cdot\nabla)\overrightarrow{\theta}^\nu+(\overrightarrow{w}_t^\nu\cdot\nabla)\overrightarrow{\theta}_x^\nu+(\overrightarrow{w}_x^\nu\cdot\nabla)\overrightarrow{\theta}_t^\nu+(\overrightarrow{w}^\nu\cdot\nabla)\overrightarrow{\theta}_{tx}^\nu+\nabla q_{tx}\\ &\quad=\nu\triangle\overrightarrow{w}_{tx}^\nu-\lambda\nabla(\nabla v_{tx}^\nu\otimes\nabla\phi^\nu)-\lambda\nabla(\nabla v_t^\nu\otimes\nabla\phi_x^\nu)-\lambda\nabla(\nabla v_x^\nu\otimes\nabla\phi_t^\nu)\\ &\qquad-\lambda\nabla(\nabla v^\nu\otimes\nabla\phi_{tx}^\nu)-\lambda\nabla(\nabla\phi_{tx}^\nu\otimes\nabla v^0)-\lambda\nabla(\nabla\phi_t^\nu\otimes\nabla v_x^0)\\ &\qquad-\lambda\nabla(\nabla\phi_x^\nu\otimes\nabla v_t^0)-\lambda\nabla(\nabla\phi^\nu\otimes\nabla v_{tx}^0)+\overrightarrow{g}_{tx}^\nu, \qquad (4.34)\\ &\nabla\cdot\overrightarrow{w}_{tx}^\nu=0, \qquad (4.35)\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\phi^\nu_{tx}+(\overrightarrow{u}^\nu_{tx}\cdot\nabla)\phi^\nu+(\overrightarrow{u}^\nu_t\cdot\nabla)\phi^\nu_x+(\overrightarrow{u}^\nu_x\cdot\nabla)\phi^\nu_t+(\overrightarrow{u}^\nu\cdot\nabla)\phi^\nu_{tx}\\
&+(\overrightarrow{w}^\nu_{tx}\cdot\nabla)v^0+(\overrightarrow{w}^\nu_t\cdot\nabla)v^0_x+(\overrightarrow{w}^\nu_x\cdot\nabla)v^0_t+(\overrightarrow{w}^\nu\cdot\nabla)v^0_{tx}\\
&\quad=\gamma(\triangle\phi^\nu_{tx}-f''(v^\nu)v^\nu_tv^\nu_x+f''(v^0)v^0_tv^0_x-f'(v^\nu)v^\nu_{tx}+f'(v^0)v^0_{tx})\\
&\qquad-(\overrightarrow{\theta}^\nu_{tx}\cdot\nabla)v^0-(\overrightarrow{\theta}^\nu_t\cdot\nabla)v^0_x-(\overrightarrow{\theta}^\nu_x\cdot\nabla)v^0_t-(\overrightarrow{\theta}^\nu\cdot\nabla)v^0_{tx},
\end{aligned}\tag{4.36}
$$

$$\overrightarrow{w}^\nu_{tx}=0,\quad \phi^\nu_{tx}=0,\quad \text{at } z=0,\quad z=h,\tag{4.37}$$

$$\overrightarrow{w}^\nu_{tx}=\nu P(\triangle\overrightarrow{u}^0_x),\qquad \phi^\nu_{tx}=0,\qquad \text{at } t=0.\tag{4.38}$$

Now, $(4.34)\cdot\overrightarrow{w}^\nu_{tx}-(4.36)\cdot\lambda\triangle\phi^\nu_{tx}$, and integrate over $\Omega$. Notice that

$$\int_\Omega\frac{\partial}{\partial t}\overrightarrow{w}^\nu_{tx}\cdot\overrightarrow{w}^\nu_{tx}=\frac12\frac{\mathrm{d}}{\mathrm{d}t}|\overrightarrow{w}^\nu_{tx}|^2,\tag{4.39}$$

$$
\begin{aligned}
&\left|\int_\Omega(\overrightarrow{u}^\nu_{tx}\cdot\nabla)\overrightarrow{w}^\nu\cdot\overrightarrow{w}^\nu_{tx}\right|\le\left|\int_\Omega(\overrightarrow{u}^0_{tx}+\overrightarrow{\theta}^\nu_{tx})\cdot\nabla\overrightarrow{w}^\nu_{tx}\cdot\overrightarrow{w}^\nu\right|+\left|\int_\Omega(\overrightarrow{w}^\nu_{tx}\cdot\nabla)\overrightarrow{w}^\nu\cdot\overrightarrow{w}^\nu_{tx}\right|\\
&\quad\le C|\overrightarrow{w}^\nu||\nabla\overrightarrow{w}^\nu_{tx}|+|\overrightarrow{w}^\nu_{tx}|_4^2|\nabla\overrightarrow{w}^\nu|\quad\text{(Soblev imbedding and Interpolation)}\\
&\quad\le\frac{\nu}{32}|\nabla\overrightarrow{w}^\nu_{tx}|^2+C\nu+C|\overrightarrow{w}^\nu_{tx}||\nabla\overrightarrow{w}^\nu_{tx}||\nabla\overrightarrow{w}^\nu|\\
&\quad\le\frac{\nu}{16}|\nabla\overrightarrow{w}^\nu_{tx}|^2+C\nu+\frac{C}{\nu}|\nabla\overrightarrow{w}^\nu|^2|\overrightarrow{w}^\nu_{tx}|^2,
\end{aligned}\tag{4.40}
$$

$$
\begin{aligned}
&\left|\int_\Omega(\overrightarrow{u}^\nu_t\cdot\nabla)\overrightarrow{w}^\nu_x\cdot\overrightarrow{w}^\nu_{tx}\right|+\left|\int_\Omega(\overrightarrow{u}^\nu_x\cdot\nabla)\overrightarrow{w}^\nu_t\cdot\overrightarrow{w}^\nu_{tx}\right|\\
&\quad=\left|\int_\Omega\overrightarrow{u}^\nu_t\cdot\nabla\overrightarrow{w}^\nu_{tx}\cdot\overrightarrow{w}^\nu_x\right|+\left|\int_\Omega\overrightarrow{u}^\nu_x\cdot\nabla\overrightarrow{w}^\nu_{tx}\cdot\overrightarrow{w}^\nu_t\right|\\
&\quad\le(\|\overrightarrow{u}^0_t+\overrightarrow{\theta}^\nu_t\|_{L^\infty})|\overrightarrow{w}^\nu_x||\nabla\overrightarrow{w}^\nu_{tx}|+(\|\overrightarrow{u}^0_x+\overrightarrow{\theta}^\nu_x\|_{L^\infty})|\overrightarrow{w}^\nu_t||\nabla\overrightarrow{w}^\nu_{tx}|+2|\overrightarrow{w}^\nu_t|_4|\overrightarrow{w}^\nu_x|_4|\nabla\overrightarrow{w}^\nu_{tx}|\\
&\qquad\text{(Soblev imbedding and Interpolation)}\\
&\quad\le\frac{\nu}{32}|\nabla\overrightarrow{w}^\nu_{tx}|^2+C+C|\overrightarrow{w}^\nu_t|^{\frac12}|\nabla\overrightarrow{w}^\nu_t|^{\frac12}|\overrightarrow{w}^\nu_x|^{\frac12}|\nabla\overrightarrow{w}^\nu_x|^{\frac12}|\nabla\overrightarrow{w}^\nu_{tx}|\\
&\quad\le\frac{\nu}{16}|\nabla\overrightarrow{w}^\nu_{tx}|^2+C+C|\nabla\overrightarrow{w}^\nu_t|^2+C|\nabla\overrightarrow{w}^\nu_x|^2,
\end{aligned}\tag{4.41}
$$

$$\int_\Omega(\overrightarrow{u}^\nu\cdot\nabla)\overrightarrow{w}^\nu_{tx}\cdot\overrightarrow{w}^\nu_{tx}=0,\qquad\left|\int_\Omega\partial_{tx}[(\overrightarrow{w}^\nu\cdot\nabla)\overrightarrow{u}^0]\cdot\overrightarrow{w}^2_{tx}\right|\le C|\overrightarrow{w}^\nu_{tx}|^2+C\nu,\tag{4.42}$$

$$\left|\int_\Omega(\overrightarrow{w}^\nu_{tx}\cdot\nabla)\overrightarrow{\theta}^\nu\cdot\overrightarrow{w}^\nu_{tx}\right|\le\|z^2\nabla\overrightarrow{\theta}^\nu\|_{L^\infty}\left|\frac{\overrightarrow{w}^\nu_{tx}}{z}\right|^2\le\frac{\nu}{32}|\nabla\overrightarrow{w}_{tx}|^2,\tag{4.43}$$

(thanks to Remark 2.1 and Hardy's inequality, as $t\le T^*$)

$$
\begin{aligned}
&\left|\int_\Omega(\overrightarrow{w}^\nu_t\cdot\nabla)\overrightarrow{\theta}^\nu_x\cdot\overrightarrow{w}^\nu_{tx}\right|+\left|\int_\Omega(\overrightarrow{w}^\nu_x\cdot\nabla)\overrightarrow{\theta}^\nu_t\cdot\overrightarrow{w}^\nu_{tx}\right|+\left|\int_\Omega(\overrightarrow{w}^\nu\cdot\nabla)\overrightarrow{\theta}^\nu_{tx}\cdot\overrightarrow{w}^\nu_{tx}\right|\\
&\quad\le(\|z^2\nabla\overrightarrow{\theta}^\nu_x\|_{L^\infty}+\|z^2\nabla\overrightarrow{\theta}^\nu_t\|_{L^\infty}+\|z^2\nabla\overrightarrow{\theta}^\nu\|_{L^\infty})\cdot\left(\left|\frac{\overrightarrow{w}^\nu_t}{z}\right|+\left|\frac{\overrightarrow{w}^\nu_x}{z}\right|+\left|\frac{\overrightarrow{w}^\nu}{z}\right|\right)\left|\frac{\overrightarrow{w}^\nu_{tx}}{z}\right|
\end{aligned}
$$

(thanks to Remark 2.1 and Hardy's inequality)

$$\leq \frac{\nu}{32}|\nabla \overrightarrow{w}_{tx}|^2+C\nu|\nabla \overrightarrow{w}^\nu_t|^2+C\nu|\nabla \overrightarrow{w}^\nu_x|^2+C\nu|\nabla \overrightarrow{w}^\nu|^2, \tag{4.44}$$

$$\int_\Omega \nabla q_{tx}\cdot \overrightarrow{w}^\nu_{tx}=0, \qquad -\lambda\int_\Omega \frac{\partial}{\partial t}\phi^\nu_{tx}\cdot\triangle\phi^\nu_{tx}=\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\phi^\nu_{tx}|^2, \tag{4.45}$$

$$-\int_\Omega \triangle \overrightarrow{w}^\nu_{tx}\cdot \overrightarrow{w}^\nu_{tx}=|\nabla \overrightarrow{w}^\nu_{tx}|^2, \tag{4.46}$$

$$\begin{aligned}&\lambda\left|\int_\Omega \nabla\cdot\partial_{tx}(\nabla v^0\otimes\nabla\phi^\nu)\cdot \overrightarrow{w}^\nu_{tx}\right|+\lambda\left|\int_\Omega \nabla\cdot\partial_{tx}(\nabla\phi^\nu\otimes\nabla v^0)\cdot \overrightarrow{w}^\nu_{tx}\right|\\ &\quad\leq C(|\triangle\phi^\nu|+|\triangle\phi^\nu_t|+|\triangle\phi^\nu_x|+|\triangle\phi^\nu_{tx}|)|\overrightarrow{w}^\nu_{tx}|\\ &\quad\leq\frac{\lambda\gamma}{16}|\triangle\phi^\nu_{tx}|^2+C|\triangle\phi^\nu|^2+C|\triangle\phi^\nu_t|^2+C|\triangle\phi^\nu_x|^2+C|\overrightarrow{w}^\nu_{tx}|^2,\end{aligned} \tag{4.47}$$

$$\begin{aligned}&\lambda\left|\int_\Omega \nabla\cdot\partial_{tx}(\nabla\phi^\nu\otimes\nabla\phi^\nu)\cdot \overrightarrow{w}^\nu_{tx}\right| \\ &\quad\leq C|\triangle\phi^\nu||\nabla\phi^\nu_{tx}|_4|\overrightarrow{w}^\nu_{tx}|_4+C|\triangle\phi^\nu_t||\nabla\phi^\nu_x|_4|\overrightarrow{w}^\nu_{tx}|_4\\ &\qquad+C|\triangle\phi^\nu_x|\nabla\phi^\nu_t|_4|\overrightarrow{w}^\nu_{tx}|_4+C|\triangle\phi^\nu_{tx}||\nabla\phi^\nu|_4|\overrightarrow{w}^\nu_{tx}|_4\\ &\quad\leq C|\triangle\phi^\nu||\nabla\phi^\nu_{tx}|^{\frac12}|\triangle\phi^\nu_{tx}|^{\frac12}|\overrightarrow{w}^\nu_{tx}|^{\frac12}|\nabla\overrightarrow{w}^\nu_{tx}|^{\frac12}+C|\triangle\phi^\nu_t||\nabla\phi^\nu_x|^{\frac12}|\triangle\phi^\nu_x|^{\frac12}|\overrightarrow{w}^\nu_{tx}|^{\frac12}|\nabla\overrightarrow{w}^\nu_{tx}|^{\frac12}\\ &\qquad+C|\triangle\phi^\nu_x||\nabla\phi^\nu_t|^{\frac12}|\triangle\phi^\nu_t|^{\frac12}|\overrightarrow{w}^\nu_{tx}|^{\frac12}|\nabla\overrightarrow{w}^\nu_{tx}|^{\frac12}+C|\triangle\phi^\nu_{tx}||\nabla\phi^\nu|^{\frac12}|\triangle\phi^\nu|^{\frac12}|\overrightarrow{w}^\nu_{tx}|^{\frac12}|\nabla\overrightarrow{w}^\nu_{tx}|^{\frac12}\\ &\quad\leq\frac{\nu}{16}|\nabla\overrightarrow{w}^\nu_{tx}|^2+\frac{\lambda\gamma}{16}|\nabla\overrightarrow{w}^\nu_{tx}|^2+C|\triangle\phi^\nu_t|^2+C|\triangle\phi^\nu_x|^2+C(|\triangle\phi_t|^2+|\triangle\phi_x|^2+1)|\overrightarrow{w}^\nu_{tx}|^2,\end{aligned} \tag{4.48}$$

$$\lambda\left|\int_\Omega\partial_{tx}[(\overrightarrow{u}^0+\overrightarrow{\theta}^\nu)\cdot\nabla\phi^\nu]\cdot\triangle\phi^\nu_{tx}\right|\leq\frac{\lambda\gamma}{32}|\triangle\phi^\nu_{tx}|^2+C|\triangle\phi^\nu_t|^2+C\nu, \tag{4.49}$$

$$\lambda\left|\int_\Omega\partial_{tx}[(\overrightarrow{w}^\nu\cdot\nabla)v^0]\cdot\triangle\phi^\nu_{tx}\right|\leq\frac{\lambda\gamma}{32}|\triangle\phi^\nu_{tx}|^2+C|\overrightarrow{w}^\nu_{tx}|^2+C\nu, \tag{4.50}$$

$$\begin{aligned}&\lambda\left|\int_\Omega\partial_{tx}[(\overrightarrow{w}^\nu\cdot\nabla)\phi^\nu]\cdot\triangle\phi^\nu_{tx}\right|\\ &\quad\leq C|\overrightarrow{w}^\nu|_4|\nabla\phi^\nu_{tx}|_4|\triangle\phi^\nu_{tx}|+C|\overrightarrow{w}^\nu_x|_4|\nabla\phi^\nu_t|_4|\triangle\phi^\nu_{tx}|\\ &\qquad+C|\overrightarrow{w}^\nu_t|_4|\nabla\phi^\nu_x|_4|\triangle\phi^\nu_{tx}|+C|\overrightarrow{w}^\nu_{tx}|_4|\nabla\phi^\nu|_4|\triangle\phi^\nu_{tx}|\\ &\quad\leq C|\nabla\overrightarrow{w}^\nu||\nabla\phi^\nu_{tx}|^{\frac12}|\triangle\phi^\nu_{tx}|^{\frac32}+C|\overrightarrow{w}^\nu_x|^{\frac12}|\nabla\overrightarrow{w}^\nu_x|^{\frac12}|\nabla\phi^\nu_t|^{\frac12}|\triangle\phi^\nu_t|^{\frac12}|\triangle\phi^\nu_{tx}|\\ &\qquad+C|\overrightarrow{w}^\nu_t|^{\frac12}|\nabla\overrightarrow{w}^\nu_t|^{\frac12}|\nabla\phi^\nu_x|^{\frac12}|\triangle\phi^\nu_x|^{\frac12}|\triangle\phi^\nu_{tx}|+C|\overrightarrow{w}^\nu_{tx}|^{\frac12}|\nabla\overrightarrow{w}^\nu_{tx}|^{\frac12}|\nabla\phi^\nu|^{\frac12}|\triangle\phi^\nu|^{\frac12}|\triangle\phi^\nu_{tx}|\\ &\quad\leq\frac{\lambda\gamma}{32}|\triangle\phi^\nu_{tx}|^2+C|\nabla\overrightarrow{w}^\nu_x|^2+C|\nabla\overrightarrow{w}^\nu_t|^2+C|\triangle\phi^\nu_t|^2+C|\triangle\phi^\nu_x|^2+\frac{\nu}{32}|\nabla\overrightarrow{w}^\nu_{tx}|^2+C\nu,\end{aligned} \tag{4.51}$$

$$\lambda\gamma\int_\Omega\triangle\phi^\nu_{tx}\cdot\triangle\phi^\nu_{tx}=\lambda\gamma|\triangle\phi^\nu_{tx}|^2, \tag{4.52}$$

$$|f''(v^\nu)v^\nu_t v^\nu_x-f''(v^0)v^0_t v^0_x|^2\leq C\nu, \qquad |(f'(v^\nu)v^\nu_{tx}-f'(v^0)v^0_{tx})|^2\leq C|\nabla\phi^\nu_{tx}|^2+C\nu, \tag{4.53}$$

$$\left|\int_\Omega\overrightarrow{g}^1_{tx}\cdot\overrightarrow{w}^\nu_{tx}\right|\leq C|z\,\overrightarrow{g}^1_{tx}|\cdot|\nabla\overrightarrow{w}^\nu_{tx}|\leq\frac{\nu}{16}|\nabla\phi^\nu_{tx}|^2+C\nu^2, \tag{4.54}$$

$$\left|\int_\Omega\nu\,\overrightarrow{g}^2_{tx}\cdot\overrightarrow{w}^\nu_{tx}\right|\leq C|\overrightarrow{w}^\nu_{tx}|^2+C\nu^2, \qquad |\partial_{tx}(\overrightarrow{\theta}^\nu\cdot\nabla)v^0|^2\leq C\nu, \tag{4.55}$$

Combing (4.39)-(4.55), as $t \le T^*$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(|\overrightarrow{w}^{\nu}_{tx}|^2+|\nabla\phi^{\nu}_{tx}|^2)+\nu|\nabla\overrightarrow{w}^{\nu}_{tx}|^2+|\triangle\phi^{\nu}_{tx}|^2\\ \le&C\left(|\triangle\phi_x|^2+|\triangle\phi^{\nu}_t|^2+\frac{|\nabla\overrightarrow{w}^{\nu}|^2}{\nu}+1\right)|\overrightarrow{w}^{\nu}_{tx}|^2\\ &+C(|\nabla\overrightarrow{w}^{\nu}_t|^2+|\nabla\overrightarrow{w}^{\nu}_x|^2+|\triangle\phi^{\nu}_t|^2+|\triangle\phi^{\nu}_t|^2)+C\nu.\end{aligned} \tag{4.56}$$

From this we deduce, after applying the usual Gronwall inequality,

$$\|\overrightarrow{w}^{\nu}_{tx}\|_{L^\infty(0,T^*;L^2)}\le C,\qquad \|\overrightarrow{w}^{\nu}_{tx}\|_{L^2(0,T^*;H^1)}\le C\nu^{-\frac{1}{2}}. \tag{4.57}$$

Recalling (4.31),

$$\nu|\nabla\overrightarrow{w}^{\nu}_x|^2\le C\nu+|\overrightarrow{w}^{\nu}_x||\overrightarrow{w}^{\nu}_{tx}|+|\nabla\phi^{\nu}_x|\nabla\phi^{\nu}_{tx}|\le C\nu^{\frac{1}{2}}, \tag{4.58}$$

we have

$$\|\nabla\overrightarrow{w}^{\nu}_x\|_{L^\infty(0,T^*;L^2)}\le C\nu^{-\frac{1}{4}}. \tag{4.59}$$

Anisotropic Sobolev imbedding from [4] yields

$$\begin{aligned}\|\overrightarrow{w}^{\nu}\|_{L^\infty((0,T^*)\times\Omega)}\le&C\left(\|\overrightarrow{w}^{\nu}\|^{\frac{1}{2}}_{L^\infty(H^1)}\|\overrightarrow{w}^{\nu}_x\|^{\frac{1}{2}}_{L^\infty(L^2)}+\|\overrightarrow{w}^{\nu}\|^{\frac{1}{2}}_{L^\infty(L^2)}\|\partial_{xz}\overrightarrow{w}^{\nu}\|^{\frac{1}{2}}_{L^\infty(L^2)}\right)\\ \le&C\nu^{\frac{3}{8}},\qquad \text{(thanks to (4.30), (4.31), Theorem 2.1 and (4.59))}\end{aligned} \tag{4.60}$$

and

$$\begin{aligned}\|\phi^{\nu}\|_{L^\infty((0,T^*)\times\Omega)}\le&C\left(\|\phi^{\nu}\|^{\frac{1}{2}}_{L^\infty(H^1)}\|\phi^{\nu}_x\|^{\frac{1}{2}}_{L^\infty(L^2)}+\|\phi^{\nu}\|^{\frac{1}{2}}_{L^\infty(L^2)}\|\partial_{xz}\phi^{\nu}\|^{\frac{1}{2}}_{L^\infty(L^2)}\right)\\ \le&C\nu^{\frac{3}{4}},\qquad \text{(thanks to (4.30) and Theorem 2.1)}.\end{aligned} \tag{4.61}$$

Therefore, we supplement Theorem 2.1 with the following

**Theorem 4.1.** *Under the assumptions of Theorem* 2.1, *we have,*

$$\|\overrightarrow{u}^{\nu}-\overrightarrow{u}^{0}-\theta^{\nu}\|_{L^\infty((0,T^*)\times\Omega)}\le C\nu^{\frac{3}{8}},\qquad \|v^{\nu}-v^{0}\|_{L^\infty((0,T^*)\times\Omega)}\le C\nu^{\frac{3}{4}}. \tag{4.62}$$


## Acknowledgments

The author would like to thank the referee for helpful suggestions. This research was supported by the fund of Shanghai Second Polytechnic University, No.: A20XK11X008.

**References**


[1] Yue P., Feng J., Liu C., and Shen J., A diffuse-interface method for simulation two phase flows of complex fluids, *J. Fluid Mech.*, **515** (2004), 293-317.

[2] Lin F. H. and Liu C., Nonparabolic dissipative system modeling the flow of liquid crystals, *Comm. Pure Appl. Math.*, **48** (1995), 501-537.

[3] Zhao L., Guo B., and Huang H., Vanighing viscosity limit for a coupled Navier-Stokes/Allen-Cahn system, *J. Math. Anal. Appl.*, **384** (2011), 232-245.

[4] Temam R. and Wang X., Boundary layers associated with incompressible Navier-Stokes equations: the noncharacteristic boundary case, *J. Differential Equations*, **179** (2002), 647-686.

[5] Temam R. and Wang X., Remarks on the Prandtl type equations for a permeable wall, *ZAMM*, **80** (2000), 838-843.

[6] Xie X. and Li C., Boundary layers of the incompressible fluids for a permeable wall, *Z. Angew. Math. Mech.*, **91** (2011), 68-84.

[7] Oleinik O. A. and Samokhin V. N., Mathematical Models in Boundary Layer Theory, Applied Mathematics and Mathematical Computation, Chapman and Hall, Boca Raton, FL, 1999.

[8] Schlichting H. and Gersten K., Boundary Layer Theory, Springer Press, Berlin, New York, 2000.

[9] Prandtl L., 1905: Ueber Flussigkeitsbewegung bei sehr kleiner Reibung (on fluid motion with small friction), *Verhandlungen des III. Int. Math. Kongr., Heidelberg*, 1904 (Leipzig: Teubner), 484-491.

[10] Bona J. L. and Wu J., The zero-viscosity limit of the 2D Navier-Stokes equations, *Stud. Appl. Math.*, **109** (2002), 861-866.

[11] Alekseenko S. N., Existence and asymptotic representation of weak solutions to the flowing problem under the condition of regular slippage on solid walls, *Siberian Mathematical J.*, **35** (1994), 209-230.

[12] W. E., Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equations, *Acta Math. Sinica*, **16** (2000), 207-218.

[13] Xiao Y. and Xin Z., On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition, *Comm. Pure Appl. Math.*, **60** (2007), 1027-1055.

[14] Filho M. C. L., Mazzucato A. L., and Lopes H. J. N., Vanishing viscosity limit for incompressible flow inside a rotating circle, *Phys. D.*, **237** (2008), 1342-1333.

[15] Iftimie D. and Planas G., Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions, *Nonlinearity*, **19** (2006), 899-918.

[16] Keliher J. P., On Kato's conditions for vanishing viscosity, *Indiana Unversity Mathematics J.*, **56** (2007), 1711-1721.

[17] Wang X., Wang Y., and Xin Z., Boundary layers in incompressible Navier-Stokes equations with Navier boundary conditions for the vanishing viscosity limit, Preprint.

[18] Xie X., Asymptotic analysis of incompressible fluids with respect to the viscosity, to be preprint.

[19] Friedman A., Partial Differential Equations, Holt, Rinehart and Winston, New York, 1969.

[20] Hardy G. H., Littlewood J. E., and Polya G., Inequalities, Cambridge Univ. Press, 1934.

[21] Xie X. and Zhang L., Boundary layers associated with incompressible Navier-Stokes equations, *Chin. Ann. Math.*, **30**(A) (2009), 309-332.