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Two Regularity Criteria Via the Logarithm of the Weak Solutions to the Micropolar Fluid Equations

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Abstract. In this note, a logarithmic improved regularity criteria for the micropolar fluid equations are established in terms of the velocity field or the pressure in the homogeneous Besov space.

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1 Introduction

In this paper, we consider the following Cauchy problem for the incompressible micropolar fluid equations :

$$\begin{cases} \partial_t u+(u\cdot\nabla)u-\Delta u+\nabla\pi-\nabla\times\omega=0, \\ \partial_t\omega-\Delta\omega-\nabla\mathrm{div}\omega+2\omega+u\cdot\nabla\omega-\nabla\times u=0, \\ \nabla\cdot u=0, \\ u(x,0)=u_0(x),\ \omega(x,0)=\omega_0(x), \end{cases} \tag{1.1}$$

where $u=u(x,t)\in\mathbb{R}^3$, $\omega=\omega(x,t)\in\mathbb{R}^3$ and $\pi=\pi(x,t)$ denote the unknown velocity vector field, the micro-rotational velocity and the unknown scalar pressure of the fluid at the point $(x,t)\in\mathbb{R}^3\times(0,T)$, respectively, while u_0, ω_0 are given initial data with $\nabla\cdot u=0$ in the sense of distributions.

The global regularity of the weak solution in the 3D case is still a big open problem. Therefore it is interesting problem on the regularity criterion of the weak solutions under

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assumption of certain growth conditions on the velocity or on the pressure. As for the velocity regularity, Dong and Chen [1] (see also [2]) proved the regularity of weak solutions under the velocity condition

$$\nabla u \in L^q(0,T;\dot{B}^0_{p,r}(\mathbb{R}^3)), \qquad \frac{2}{q}+\frac{3}{p}=2, \qquad \frac{3}{2}<p\leq\infty, \qquad r\leq\frac{2p}{3}.$$

As for the pressure criterion, Yuan [3] studied the regularity of weak solutions in Lorentz spaces

$$\pi \in L^q(0,T;L^{p,\infty}(\mathbb{R}^3)), \qquad \text{for } \frac{2}{q}+\frac{3}{p}=2,\ \frac{3}{2}<p<\infty$$

or

$$\nabla\pi \in L^q(0,T;L^{p,\infty}(\mathbb{R}^3)), \qquad \text{for } \frac{2}{q}+\frac{3}{p}=3,\ 1<p<\infty.$$

Zhang et al [4] recently improved the regularity from Lorentz to Besov spaces

$$\pi \in L^q(0,T;B^r_{p,\infty}(\mathbb{R}^3)), \qquad \frac{2}{q}+\frac{3}{p}=2+r, \qquad \frac{3}{2+r}<p<\infty, \qquad -1<r\leq 1.$$

The aim of the present study is to investigate Logarithmically improved regularity criterion for the micropolar fluid equations in terms of the gradient of velocity and pressure in Besov spaces.

2 Preliminaries and main result

We recall the definition and some properties of the space we are going to use.

Definition 2.1 ([5]). *Let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity that satisfies $\widehat{\varphi}\in C_0^\infty(B_2\backslash B_{1/2})$, $\widehat{\varphi}_j(\xi)=\widehat{\varphi}(2^{-j}\xi)$ and $\sum_{j\in\mathbb{Z}}\widehat{\varphi}_j(\xi)=1$ for any $\xi\neq 0$, where B_R is the ball in $\mathbb{R}^3$ centered at the origin with radius $R>0$. The homogeneous Besov space is defined by $\dot{B}^s_{p,q}=\{f\in\mathcal{S}'/\mathcal{P}:\|f\|_{\dot{B}^s_{p,q}}<\infty\}$ with norm*

$$\|f\|_{\dot{B}^s_{p,q}}=\left(\sum_{j\in\mathbb{Z}}\left\|2^{js}\varphi_j*f\right\|^q_{L^p}\right)^{\frac{1}{q}}$$

for $s\in\mathbb{R}$, $1\leq p,q\leq\infty$, where $\mathcal{S}'$ is the space of tempered distributions and $\mathcal{P}$ is the space of polynomials.

It is easy to see the inequality

$$\|f\|_{\dot{B}^0_{\infty,\infty}}\leq C\|f\|_{BMO}\leq C\|f\|_{\dot{B}^0_{\infty,2}}$$

holds for $f\in BMO$, where BMO is the space of the bounded mean oscillations.

In the above estimates, we have used an interpolation inequality [6]:

$$\|f\|_{L^4}^2 \le C\|f\|_{L^2}\|f\|_{BMO}. \tag{2.1}$$

We will also use the following inequality, which is established in [7]

$$\|f\cdot\nabla f\|_{L^r} \le C\|f\|_{L^r}\|\nabla f\|_{BMO} \qquad \text{for } 1<r<\infty. \tag{2.2}$$

Now, we recall the following lemma due to Kozono-Ogawa-Taniuchi [8].

Lemma 2.1. *Let $s>5/2$. Then There exists a constant C such that the following estimate*

$$\|\nabla f\|_{\dot{B}^0_{\infty,2}} \le C\big(1+\|\nabla f\|_{\dot{B}^0_{\infty,\infty}}\ln^{\frac{1}{2}}(1+\|f\|_{H^s})\big) \tag{2.3}$$

holds for all $f\in H^s\left(\mathbb{R}^3\right)$.

Our main result now read as follows:

Theorem 2.1. *Suppose $T>0$, $(u_0,w_0)\in L^2(\mathbb{R}^3)\cap L^4(\mathbb{R}^3)$ and $\nabla\cdot u_0=0$ in the sense of distributions. Assume that (u,w) is a weak solution of the 3D micropolar fluid flows* (1.1) *on $(0,T)$. If either*

$$\int_0^T \frac{\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}}{\Big(1+\ln(e+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}})\Big)^{\frac{1}{2}}}\,\mathrm{d}t<\infty \tag{2.4}$$

or

$$\int_0^T \frac{\|\pi\|_{\dot{B}^0_{\infty,\infty}}}{\Big(1+\ln(e+\|\pi\|_{\dot{B}^0_{\infty,\infty}})\Big)^{\frac{1}{2}}}\,\mathrm{d}t<\infty, \tag{2.5}$$

then the weak solution (u,w) is regular on $(0,T]$.

3 Proof of Theorem 2.1

As to L^p-theory for the Navier-Stokes equations established by Kato [9] and Giga [10], it is sufficient to show the L^4-norm of the solution is bounded up to time T under (2.4). If (2.4) holds, one can deduce that for any small $\epsilon>0$, there exists $T_*<T$ such that

$$\int_{T_*}^T \frac{\|\nabla u(t)\|_{\dot{B}^0_{\infty,\infty}}}{\Big(1+\ln(e+\|\nabla u(t)\|_{\dot{B}^0_{\infty,\infty}})\Big)^{\frac{1}{2}}}\,\mathrm{d}t\le\epsilon.$$

Multiply both sides of the first equation in (1.1) by $u|u|^2$, and integrate over $\mathbb{R}^3$. After suitable integration by parts, we obtain

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^4}^4+\||\nabla u||u|(t)\|_{L^2}^2+\frac{1}{2}\left\|\nabla|u|^2(t)\right\|_{L^2}^2\\ \le&\left|\int_{\mathbb{R}^3}\nabla\pi\cdot(|u|^2u)\mathrm{d}x\right|+\int_{\mathbb{R}^3}|\omega||u|^2|\nabla u|\mathrm{d}x.\end{aligned} \tag{3.1}$$

Similarly, for the second equation of (1.1), we get

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^4}^4+\||\nabla\omega||\omega|(t)\|_{L^2}^2+\frac{1}{2}\left\|\nabla|\omega|^2(t)\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla|\omega|^2(t)\right\|_{L^2}^2 \\ +\int_{\mathbb{R}^3}|\mathrm{div}\omega|^2\mathrm{d}x+2\int_{\mathbb{R}^3}|\omega|^4\mathrm{d}x\leq\int_{\mathbb{R}^3}|u||\omega|^2|\nabla\omega|\mathrm{d}x. \tag{3.2}$$

Combining (3.1) and (3.2) together, it follows that

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4\right)+\||\nabla u||u|(t)\|_{L^2}^2+\frac{1}{2}\left\|\nabla|u|^2(t)\right\|_{L^2}^2\\&+\||\nabla\omega||\omega|(t)\|_{L^2}^2+\frac{1}{2}\left\|\nabla|\omega|^2(t)\right\|_{L^2}^2+\int_{\mathbb{R}^3}|\mathrm{div}\omega|^2\mathrm{d}x+2\int_{\mathbb{R}^3}|\omega|^4\mathrm{d}x\\ \leq&\left|\int_{\mathbb{R}^3}\nabla\pi\cdot(|u|^2u)\mathrm{d}x\right|+\int_{\mathbb{R}^3}|\omega||u|^2|\nabla u|\mathrm{d}x+\int_{\mathbb{R}^3}|u||\omega|^2|\nabla\omega|\mathrm{d}x\\ =&A_1+A_2+A_3.\end{aligned} \tag{3.3}$$

Due to Hölder's inequality and Young inequality, A_2 can be estimated as

$$A_2\leq\||\omega||u|\|_{L^2}\||u||\nabla u|\|_{L^2}\leq\frac{1}{2}\||u||\nabla u|\|_{L^2}^2+\frac{1}{4}\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right). \tag{3.4}$$

Similarly, we can bound

$$A_3\leq\frac{1}{2}\||\omega||\nabla\omega|\|_{L^2}^2+\frac{1}{4}\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right). \tag{3.5}$$

Let us now estimate the integral A_1. Before turning to estimate A_1, we recall the well-known equality given by taking ∇div on both sides of the first equation in (1.1) for smooth (u,ω,π), one can obtain

$$-\Delta(\nabla\pi)=\sum_{i,j=1}^{3}\partial_i\partial_j\left(\nabla(u_iu_j)\right).$$

The Calderón-Zygmund inequality implies

$$\|\nabla\pi\|_{L^q}\leq C\||u||\nabla u|\|_{L^q},\qquad 1<q<\infty.$$

Now, by the Hölder inequality and (2.2), we have

$$A_1\leq\|\nabla\pi\|_{L^4}\|u\|_{L^4}^3\leq C\||u||\nabla u|\|_{L^4}\|u\|_{L^4}^3\leq C\|\nabla u\|_{BMO}\|u\|_{L^4}^4. \tag{3.6}$$

Then, due to (3.3)-(3.6) and the above equality, we derive

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4\right)\leq C\|\nabla u\|_{BMO}\|u\|_{L^4}^4+C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right),$$

which implies by Lemma 2.1,

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4\right)\le& C\|\nabla u\|_{BMO}\|u\|_{L^4}^4+C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right)\\ \le& C\frac{\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}}{\left(1+\ln(e+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{1}{2}}}\left(1+\ln(e+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{1}{2}}\\ &\times\ln^{\frac{1}{2}}(1+\|u\|_{H^s})\|u\|_{L^4}^4+C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right). \qquad (3.7)\end{aligned}$$

Since it is well known that the Sobolev space $H^s(\mathbb{R}^3)$ with $s>5/2$ is continuously embedded into $L^\infty(\mathbb{R}^3)$ this yields

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4\right)\\ \le& C\frac{\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}}{\left(1+\ln(e+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{1}{2}}}(1+\ln(e+\|u\|_{H^s}))\|u\|_{L^4}^4+C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right)\\ \le& C\frac{\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}}{\left(1+\ln(e+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{1}{2}}}(1+\ln(e+y(t)))\|u\|_{L^4}^4+C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right),\end{aligned}$$

where we have used the fact that $L^\infty\subset\dot{B}^0_{\infty,\infty}$ and where $y(t)$ is defined by

$$y(t)=\sup_{T_*\le\tau\le t}\|u(\tau,.)\|_{H^s},\qquad \text{for all } T_*\le t<T.$$

Applying Gronwall's inequality on (3.7) for the interval $[T_*,t]$, one has

$$\begin{aligned}\|u(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4\le& C_0\exp(C\epsilon(1+\ln(e+y(t))))\\ \le& C_0\exp(2C\epsilon\ln(e+y(t)))\\ \le& C_0(e+y(t))^{2C\epsilon}, \qquad (3.8)\end{aligned}$$

where $C_0=\|u(\cdot,T_*)\|_{L^2}^4+\|\omega(\cdot,T_*)\|_{L^2}^4$.

Next, multiplying the first equation of (1.1) by $-\Delta u$, after integration by parts and taking the divergence free property into account, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2+\|\Delta u(t)\|_{L^2}^2=&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\mathrm{d}x\le\|u\|_{L^4}\|\nabla u\|_{L^4}\|\Delta u\|_{L^2}\\ \le& C\|u\|_{L^4}^{\frac{6}{5}}\|\Delta u\|_{L^2}^{\frac{9}{5}}\le\frac{1}{2}\|\Delta u(t)\|_{L^2}^2+C\|u(t)\|_{L^4}^{12},\end{aligned}$$

where we used

$$\|\nabla f\|_{L^4}\le C\|f\|_{L^4}^{\frac{1}{5}}\|\Delta f\|_{L^2}^{\frac{4}{5}}.$$

Integrating the above inequality over (T_*,t), we have

$$\sup_{T_*\le\tau\le t}\|\nabla u(\tau)\|_{L^2}^2\le C(e+y(\tau))^{C\epsilon}.$$

Then we go to the estimate for H^s norm. Taking the operation $\Lambda^s=(-\Delta)^{s/2}$ on both sides to the first equation of (1.1), then multiplying them by $\Lambda^s u$, after integrating over $\mathbb{R}^3$, we have (since $\nabla\cdot u=0$)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u(t)\|_{L^2}^2+\|\Lambda^{s+1}u(t)\|_{L^2}^2&=-\int_{\mathbb{R}^3}\Lambda^s(u\cdot\nabla u)\Lambda^s u\mathrm{d}x\\&=-\int_{\mathbb{R}^3}[\Lambda^s(u\cdot\nabla u)-u\cdot\Lambda^s\nabla u]\cdot\Lambda^s u\mathrm{d}x=\Pi.\end{aligned}$$

In what follows, we will use the following inequality due to Kato and Ponce [11]:

$$\|\Lambda^\alpha(fg)-f\Lambda^\alpha g\|_{L^p}\le C\left(\left\|\Lambda^{\alpha-1}g\right\|_{L^{q_1}}\|\nabla f\|_{L^{p_1}}+\|\Lambda^\alpha f\|_{L^{p_2}}\|g\|_{L^{q_2}}\right),\tag{3.9}$$

for $\alpha>1$, and $1/p=1/p_1+1/q_1=1/p_2+1/q_2$. Hence Π can be estimated as

$$\Pi\le\frac{1}{2}\|\Lambda^{s+1}u\|_{L^2}^2+C\|\nabla u\|_{L^2}^{2+\frac{(2s-3)s}{s-1}}\|\Lambda^s u\|_{L^2}^{\frac{s}{s-1}},\tag{3.10}$$

where we used (3.9) with $\alpha=s$, $p=3/2$, $p_1=q_1=p_2=q_2=3$, and the following inequalities

$$\|\nabla u\|_{L^3}\le C\|\nabla u\|_{L^2}^{\frac{2s-3}{2s-2}}\|\Lambda^s u\|_{L^2}^{\frac{1}{2s-2}},$$

and

$$\|\Lambda^s u\|_{L^3}\le C\|\nabla u\|_{L^2}^{\frac{1}{2s}}\|\Lambda^{s+1}u\|_{L^2}^{\frac{2s-1}{2s}}.$$

If we use the existing estimate (3.8) for $T_0<t<T$, (3.10) reduces to

$$\Pi\le\frac{1}{2}\|\Lambda^{s+1}u\|_{L^2}^2+C_0C(e+y(t))^{\frac{s}{s-1}+\left(2+\frac{(2s-3)s}{2s-2}\right)C\epsilon}.$$

Combining (3.8) and (3.10), we easily get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s u(t)\|_{L^2}^2\le C_0C(e+y(t))^{\frac{s}{s-1}+\left(2+\frac{(2s-3)s}{2s-2}\right)C\epsilon}.\tag{3.11}$$

Choose ϵ to be sufficiently small, then applying Gronwall's inequality to (3.11) yields

$$\sup_{T_*\le\tau\le t}\|\Lambda^s u(\tau)\|_{L^2}^2\le C.$$

We assume that the condition (2.5) holds true. We start from (3.3), we have

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4\right)+\||\nabla u||u|(t)\|_{L^2}^2+\frac{1}{2}\left\|\nabla|u|^2(t)\right\|_{L^2}^2\\&+\||\nabla\omega||\omega|(t)\|_{L^2}^2+\frac{1}{2}\left\|\nabla|\omega|^2(t)\right\|_{L^2}^2+\int_{\mathbb{R}^3}|\mathrm{div}\omega|^2\mathrm{d}x+2\int_{\mathbb{R}^3}|\omega|^4\mathrm{d}x\\\leq&\left|\int_{\mathbb{R}^3}\nabla\pi\cdot(|u|^2u)\mathrm{d}x\right|+\int_{\mathbb{R}^3}|\omega||u|^2|\nabla u|\mathrm{d}x+\int_{\mathbb{R}^3}|u||\omega|^2|\nabla\omega|\mathrm{d}x\\=&B_1+A_2+A_3.\end{aligned}\tag{3.12}$$

Let us now estimate the integral B_1. The Cauchy inequality implies that

$$\begin{aligned}B_1=&\left|\int_{\mathbb{R}^3}\nabla\pi\cdot(|u|^2u)\mathrm{d}x\right|=\left|\int_{\mathbb{R}^3}\pi\cdot\mathrm{div}(|u|^2u)\mathrm{d}x\right|\\\leq&2\int_{\mathbb{R}^3}|\pi||u|^2|\nabla u|\mathrm{d}x\leq C\|\pi u\|_{L^2}^2+\frac{1}{2}\||u||\nabla u|\|_{L^2}^2.\end{aligned}\tag{3.13}$$

Let us estimate the integral $I=\|\pi u\|_{L^2}^2$ on the right-hand side of (3.13). Before turning to estimate I, we recall the well-known inequality given by

$$\|\pi\|_{L^q}\leq C\|u\|_{L^{2q}}^2,\qquad 1<q<\infty.$$

Now, by the Hölder inequality and (2.1), we have

$$I\leq C\|\pi\|_{L^4}^2\|u\|_{L^4}^2\leq C\|\pi\|_{BMO}\|u\|_{L^4}^4.$$

The estimates for A_2 and A_3 do not change.

Then, due to (3.4), (3.5), (3.12), (3.13) and the above equality, we derive

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4\right)\leq C\|\pi\|_{BMO}\|u\|_{L^4}^4+C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right),$$

which implies by Lemma 2.1 that

$$\begin{aligned}&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|_{L^4}^4+\|\omega(t)\|_{L^4}^4\right)\\\leq&C\left(1+\|\pi\|_{\dot{B}^0_{\infty,\infty}}\ln^{\frac{1}{2}}(1+\|\pi\|_{H^{s-1}})\right)\|u\|_{L^4}^4+C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right)\end{aligned}$$

$$
\begin{aligned}
&\le C\frac{\|\pi\|_{\dot{B}^0_{\infty,\infty}}}{\left(1+\ln(e+\|\pi\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{1}{2}}}\left(1+\ln(e+\|\pi\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{1}{2}}\times\ln^{\frac{1}{2}}(1+\|\pi\|_{H^{s-1}})\|u\|_{L^4}^4\\
&\quad +C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right)\\
&\le C\frac{\|\pi\|_{\dot{B}^0_{\infty,\infty}}}{\left(1+\ln(e+\|\pi\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{1}{2}}}(1+\ln(e+\|\pi\|_{H^{s-1}}))+C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right)\\
&\le C\frac{\|\pi\|_{\dot{B}^0_{\infty,\infty}}}{\left(1+\ln(e+\|\pi\|_{\dot{B}^0_{\infty,\infty}})\right)^{\frac{1}{2}}}(1+\ln(e+\|u\|_{H^s}))+C\left(\|u\|_{L^4}^4+\|\omega\|_{L^4}^4\right),
\end{aligned}
\tag{3.14}
$$

where we used

$$
\|\pi\|_{H^{s-1}}\le C\left\||u|^2\right\|_{H^{s-1}}\le C\|u\|_{L^\infty}\|u\|_{H^{s-1}}\le C\|u\|_{H^s}^2.
$$

Using the same calculations as that in Theorem 2.1 and due to the Gronwall inequality, it follows from (3.14) that

$$
\sup_{T_*\le\tau\le t}\|\Lambda^s u(\tau)\|_{L^2}^2\le C.
$$

This completes the proof of Theorem 2.1.

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