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Dynamics for Controlled 2D Generalized MHD Systems with Distributed Controls

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Abstract. We study the dynamics of a piecewise (in time) distributed optimal control problem for Generalized MHD equations which model velocity tracking coupled to magnetic field over time. The long-time behavior of solutions for an optimal distributed control problem associated with the Generalized MHD equations is studied. First, a quasi-optimal solution for the Generalized MHD equations is constructed; this quasi-optimal solution possesses the decay (in time) properties. Then, some preliminary estimates for the long-time behavior of all solutions of Generalized MHD equations are derived. Next, the existence of a solution of optimal control problem is proved also optimality system is derived. Finally, the long-time decay properties for the optimal solutions is established.

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1 Introduction

The control of viscous flows is very crucial to many technological and scientific applications. We are motivated to study the asymptotic behaviors and dynamics of solutions for the controlled Generalized MHD equations (GMHD). In this paper we study the long time behavior of the solution for optimal control problems associated with GMHD equations on the infinite time interval. The optimal control with the systems governed by Navier-Stokes and Boussinesq equations was studying by L. Hou and Y. Yan [1] and by H. Chun Lee and B. Chun Shin [2], respectively. This work is motivated by the desire to steer over time a candidate velocity field u and magnetics field b to a target velocity field

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U and magnetics field B by appropriately controlling the body force. The existence of solutions of GMHD equations was studied in [3].

We formulate here a controllability problem for the GMHD equations: find a triplet (u,b,f) such that the functional

$$J_{(0;+\infty)}(u,b,f)=\frac{\alpha}{2}\int_0^{+\infty}\int_\Omega |(u,b)-(U,B)|^2 \mathrm{d}x\mathrm{d}t+\frac{\beta}{2}\int_0^{+\infty}\int_\Omega |f-F|^2 \mathrm{d}x\mathrm{d}t, \tag{1.1}$$

is minimized subject to the 2-D GMHD equations:

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u-(b\cdot\nabla)b+\nabla p+\nu(-\triangle)^r u=f, \qquad \text{in } \Omega\times(0,\infty), \tag{1.2}$$

$$\frac{\partial b}{\partial t}+(u\cdot\nabla)b-(b\cdot\nabla)u+\theta(-\triangle)^r b=0, \qquad \text{in } \Omega\times(0,\infty), \tag{1.3}$$

$$\nabla\cdot u=0,\ \ \nabla\cdot b=0, \qquad \text{in } \Omega\times(0,\infty), \tag{1.4}$$

$$u=0, \triangle u=0,\ \ \triangle^2 u=0,\ \cdots,\ \triangle^r u=0, \qquad \text{on } \partial\Omega\times(0,\infty), \tag{1.5}$$

$$b\cdot\mathbf{n}=0,\ \ \triangle b=0,\ \ \triangle^2 b=0,\ \cdots,\ \triangle^r b=0, \qquad \text{on } \partial\Omega\times(0,\infty), \tag{1.6}$$

$$\nabla u=0,\ \ \nabla^2 u=0,\ \cdots,\ \nabla^r u=0, \qquad \text{on } \partial\Omega\times(0,\infty), \tag{1.7}$$

$$\nabla b=0,\ \ \nabla^2 b=0,\ \cdots,\ \nabla^r b=0, \qquad \text{on } \partial\Omega\times(0,\infty), \tag{1.8}$$

$$u(0)=u_0 \quad \text{and} \quad b(0)=b_0, \tag{1.9}$$

where $\mathbf{n}$ is an outward unit normal vector on $\partial\Omega$ and r a non negative integer, also $\nu>0$ and $\theta>0$ are the kinematic viscosity and conductivity parameters, respectively. Here $\alpha,\beta>0$ are given constants, Ω is a bounded, sufficiently smooth domain in $\mathbb{R}^2$ with $\partial\Omega$ denoting its boundary; U, B and F are a given desired velocity field, a given desired magnetic field and a given desired body force, respectively. Also, f is a distributed control (body force), and u,b and p denote the velocity field, the magnetic field and the pressure field, respectively.

S. S. Ravindran [4] was interesting in the standard case $(r=1)$ and he had also used a curl term. The periodic case was studying by M. Gunzburger and C. Trenchea in [5] and [6].

Intuitively, if a flow field (u,b) is close to the desired field (U,B), then the body force corresponding to the two fields (u,b) and (U,B) should also be close. Hence, in order that the optimal control solution of GMHD equations is close to the desired field (U,B), we must place some restrictions on the desired body force F involved in the cost functional (1.1). In fact, throughout this paper we will simply choose for some $P\in L_0^2(\Omega)$,

$$F:=\partial_t U+\nu(-\triangle)^r U+(U.\nabla)U+\nabla P-(B.\nabla)B \tag{1.10}$$

for the desired field (U,B) satisfying

$$\partial_t B+\theta(-\triangle)^r B+(U.\nabla)B-(B.\nabla)U=0. \tag{1.11}$$

We make the following regularity assumptions on the prescribed data U,B and F:

$$\begin{cases} (U,B)\in L^\infty\big(0,\infty;\big(\mathbf{H}^2(\Omega)\cap\mathbf{V}^r\big)\times\big(\mathbf{H}^2(\Omega)\cap\mathbf{V}^r_{\mathbf{n}}\big)\big), \\ F\in L^\infty\big(0,\infty;\mathbf{L}^2(\Omega)\big). \end{cases} \tag{A1}$$

Note that these hypotheses permit the special case of steady state (U,B). Thus one application of the optimal control problem is to match a steady state flows field through the control of external forces. Observe that (U,B) is not an optimal solution because (U,B) in general does not satisfy the initial conditions. For technical reasons, we will need the following assumptions

$$\begin{cases} 2\theta\lambda_1^{4k-2}-\dfrac{\theta}{2}-\dfrac{4}{\theta\lambda_1}|||\nabla U|||^2-\dfrac{8}{\nu\lambda_1}|||\nabla B|||^2>0, & k\in\mathbb{N}^*, \\ 2\theta\lambda_1^{4k}-\dfrac{\theta}{2}-\dfrac{4}{\theta\lambda_1}|||\nabla U|||^2-\dfrac{8}{\nu\lambda_1}|||\nabla B|||^2>0, & k\in\mathbb{N}, \\ 2\nu\lambda_1^{4k-2}-\dfrac{3\nu}{2}-\dfrac{4}{\nu\lambda_1}|||\nabla U|||^2>0, & k\in\mathbb{N}^*, \\ 2\nu\lambda_1^{4k}-\dfrac{3\nu}{2}-\dfrac{4}{\nu\lambda_1}|||\nabla U|||^2>0, & k\in\mathbb{N}. \end{cases} \tag{A2}$$

We summarize the major components of this paper as follows.

- A quasi optimizer is constructed for the optimal control problem.
- We prove the existence of a solution for the distributed optimal control problem of minimizing (1.1) subject to (1.2)-(1.9) and derive an optimality system of equations from which optimal solutions may be deduced.
- The long-time behavior (dynamics) of the optimal solution is derived and the main result is that the $L^2\big(0,\infty;\underline{\mathbf{L}}^2(\Omega)\big)$-distance $\|(u,b)(t)-(U,B)(t)\|$ between the optimal solution $(u,b)(t)$ and the desired state $(U,B)(t)$ decays to zero as time $t\to\infty$.

Our plan of the paper is as follows: Section 2 is devoted to preliminary material. In Section 3 we construct a quasi-optimal control solution and some preliminary estimates for all solutions of the GMHD equations. In Section 4 we prove the existence of an optimal solution on the finite time interval and derive an optimality system of equations from which optimal solutions may be deduced. Finally, in Section 5 we prove the decay of the controlled dynamics to the desired dynamics.

2 Preliminaries

Throughout this work, C denotes a generic constant depending only on the physical domain Ω, the viscosity constant ν and the conductivity parameter θ. We will use the standard notations for the function spaces $L^p(\Omega)$ with the norm denoted by $\|\cdot\|_{L^p(\Omega)}$ and the

Sobolev spaces $H^m(\Omega)$ with the norm denoted by $\|\cdot\|_m$. We simply denote by the norm of $L^2(\Omega)$ $\|\cdot\|$. The space $H_0^m(\Omega)$ is consisting of functions in $H^m(\Omega)$ which vanish on boundary $\partial\Omega$. The vector valued counterparts of these spaces are denoted by $\mathbf{L}^p(\Omega),\mathbf{H}^m(\Omega)$ and $\mathbf{H}_0^m(\Omega)$.

We now introduce the solenoidal spaces

$$\begin{aligned}
&\mathbf{W}^r=\Big\{u\in\mathbf{H}^{r-1}(\Omega),\ \nabla\cdot u=0\ \text{ and }\ u\cdot\mathbf{n}|_{\partial\Omega}=0\Big\},\\
&\mathbf{V}^r=\{u\in\mathbf{H}_0^r(\Omega),\ \nabla\cdot u=0\ \text{ and } \nabla u=\cdots=\nabla^r u=\triangle u=\cdots=\triangle^r u=0\ \text{ on } \partial\Omega\},\\
&\mathbf{V}_\mathbf{n}^r=\{u\in\mathbf{H}^r(\Omega),\ \nabla\cdot u=0\ \text{ and }\ u\cdot\mathbf{n}|_{\partial\Omega}=\nabla u=\cdots=\nabla^r u=\triangle u=\cdots=\triangle^r u=0\ \text{ on } \partial\Omega\}.
\end{aligned}$$

We identify the dual space of $\mathbf{W}^r$ with $\mathbf{W}^r$ itself under the $\mathbf{L}^2(\Omega)$ inner product and the dual space of $\mathbf{V}^r$ and $\mathbf{V}_\mathbf{n}^r$ is denoted by $(\mathbf{V}^r)^*$ and $(\mathbf{V}_\mathbf{n}^r)^*$, respectively. We have

$$\mathbf{V}^r\times{\mathbf{V}_\mathbf{n}}^r\subset\mathbf{V}^r\times\mathbf{W}^r\subset(\mathbf{V}^r)^*\times(\mathbf{V}_\mathbf{n}^r)^*,$$

where the injections are continuous and each space is dense in the following one. Next, we introduce the temporal-spatial function spaces $L^r(0,T;\mathbf{H}^m(\Omega))$ defined on $Q_T=\Omega\times(0,T)$ equipped with the norm

$$\|u\|_{L^p(0,T;\mathbf{H}^m)}=\left(\int_0^T\|u(t)\|_m^p\mathrm{d}t\right)^{1/p},$$

where $p\in[1,\infty)$. We simply denote Q_∞ by Q. The solenoidal temporal-spatial function space

$$\begin{aligned}
&\mathcal{H}_u^1(Q_T)=\big\{u\in L^2(0,T;\mathbf{V}^r);\ \partial_t u\in L^2(0,T;(\mathbf{V}^r)^*)\big\},\\
&\mathcal{H}_b^1(Q_T)=\big\{b\in L^2(0,T;\mathbf{V}_\mathbf{n}^r);\ \partial_t b\in L^2(0,T;(\mathbf{V}_\mathbf{n}^r)^*)\big\},
\end{aligned}$$

that associated norms are respectively given by

$$\begin{aligned}
&\|v\|_{\mathcal{H}_u^1}^2=\|v\|_{L^2(0,T;\mathbf{V}^r)}^2+\|\partial_t v\|_{L^2(0,T;(\mathbf{V}^r)^*)}^2,\\
&\|w\|_{\mathcal{H}_b^1}^2=\|w\|_{L^2(0,T;\mathbf{V}_\mathbf{n}^r)}^2+\|\partial_t w\|_{L^2(0,T;(\mathbf{V}_\mathbf{n}^r)^*)}^2.
\end{aligned}$$

For convenience we simply denote by

$$\begin{aligned}
&\underline{\mathcal{H}}^1(Q_T)=\mathcal{H}_u^1(Q_T)\times\mathcal{H}_b^1(Q_T),\quad \underline{\mathbf{H}}^m(\Omega)=\mathbf{H}^m(\Omega)\times\mathbf{H}^m(\Omega),\\
&\underline{\mathbf{H}}_0^m(\Omega)=\mathbf{H}_0^m(\Omega)\times\mathbf{H}_0^m(\Omega)\quad\text{and}\quad \underline{\mathbf{L}}^2(\Omega)=\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega).
\end{aligned}$$

We denote by $|||\cdot|||$ the simplified norm notations of $\|\cdot\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}$. This norm will be applied solely to $U,B,\nabla U$ and ∇B. For two normed spaces

$$\|(u,b)\|_{S_1\times S_2}^2=\|u\|_{S_1}^2+\|b\|_{S_2}^2,\qquad \forall(u,b)\in S_1\times S_2.$$

For a function u and b in a temporal-spatial space, we often use the notation $u(t):=u(.,t)$ and $b(t):=b(.,t)$ to stand for the restriction of u and b at time t as a function defined over the spatial domain Ω.

We introduce some standard continuous bilinear or trilinear forms:

$$\begin{aligned}
&a^{\nu}_{(2k+1)}(u,\varphi)=\nu\int_{\Omega}\nabla((-\triangle)^k u):\nabla((-\triangle)^k\varphi)\mathrm{d}x, && k\in\mathbb{N},\forall u,\varphi\in\mathbf{H}^{2k+1}(\Omega),\\
&a^{\theta}_{(2k+1)}(b,\psi)=\theta\int_{\Omega}\nabla((-\triangle)^k b):\nabla((-\triangle)^k\psi)\mathrm{d}x, && k\in\mathbb{N},\forall b,\psi\in\mathbf{H}^{2k+1}(\Omega),\\
&a^{\nu}_{2k}(u,\varphi)=\nu\int_{\Omega}((-\triangle)^k u)\cdot((-\triangle)^k\varphi)\mathrm{d}x, && k\in\mathbb{N}^*,\forall u,\varphi\in\mathbf{H}^{2k}(\Omega),\\
&a^{\theta}_{2k}(b,\psi)=\theta\int_{\Omega}((-\triangle)^k b)\cdot((-\triangle)^k\psi)\mathrm{d}x, && k\in\mathbb{N}^*,\forall u,\psi\in\mathbf{H}^{2k}(\Omega),\\
&\mathbf{c}(u,v,w)=\int_{\Omega}(u\cdot\nabla)v\cdot w\mathrm{d}x, && \forall u,v,w\in\mathbf{H}^r(\Omega),
\end{aligned}$$

where the colon notation : denotes the inner product on $\mathbb{R}^{2\times 2}$. Also, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between a Banach space and its dual. Note that for all $u,v,w\in\mathbf{H}^1(\Omega)$, $\mathbf{c}$ have the following continuity properties (see [7])

$$|\mathbf{c}(u,v,w)|\le 2^{1/4}\cdot\|u\|^{1/2}\cdot\|\nabla u\|^{1/2}\cdot\|\nabla v\|\cdot\|w\|^{1/2}\cdot\|\nabla w\|^{1/2}. \tag{2.1}$$

The trilinear form $\mathbf{c}$ have followings properties

$$\mathbf{c}(u,v,w)=-\mathbf{c}(u,w,v)\quad\text{and}\quad\mathbf{c}(u,v,v)=0,\qquad\text{for all } u,v,w\in\mathbf{H}^1(\Omega). \tag{2.2}$$

Let $\lambda_1>0$ be the greatest real number satisfying the Poincaré inequality

$$\lambda_1\|\varphi\|^2\le\|\nabla\varphi\|^2\quad\text{et}\quad\lambda_1\|\psi\|^2\le\|\nabla\psi\|^2,\quad\forall\varphi\in\mathbf{H}^r,\ \forall\psi\in\mathbf{H}^r_{\mathbf{n}}. \tag{2.3}$$

Let $\Pi:\mathbf{L}^2(\Omega)\to\mathbf{W}^r$ be the Leray operator (i.e., the orthogonal projection with respect to the $\mathbf{L}^2(\Omega)$-norm), it is well known (see [8] and [9]) that there are constants $\gamma_1>0$ and $\gamma_2>0$ depending only on Ω such that

$$\begin{aligned}
&\gamma_1\|\Pi\Delta\varphi\|\le\|\Delta\varphi\|\le\gamma_2\|\Pi\Delta\varphi\|, && \forall\varphi\in\mathbf{H}^2(\Omega)\cap\mathbf{H}^r_0(\Omega),\\
&\gamma_1\|\Pi\Delta\psi\|\le\|\Delta\psi\|\le\gamma_2\|\Pi\Delta\psi\|, && \forall\psi\in\mathbf{H}^2(\Omega)\cap\mathbf{H}^r_{\mathbf{n}}(\Omega).
\end{aligned}$$

So that $\|\Pi\Delta.\|$ is equivalent to the $\mathbf{H}^2(\Omega)$-norm on $\mathbf{H}^2(\Omega)\cap\mathbf{H}^r(\Omega)$ and on $\mathbf{H}^2(\Omega)\cap\mathbf{H}^r_{\mathbf{n}}(\Omega)$.

Definition 2.1. *Given $T\in(0,\infty)$, $(u_0,b_0)\in\mathbf{W}^r\times\mathbf{W}^r$ and $f\in L^2\big(0,T;\mathbf{L}^2(\Omega)\big)$, (u,b) is said to be a solution of the* GMHD *equations on $(0,T)$ if and only if $(u,b)\in\underline{\mathcal{H}}^1(Q_T)$ and (u,b) satisfies*

$$\begin{aligned}
&\langle\partial_t u(t),\varphi\rangle+a^{\nu}_r(u(t),\varphi)+\mathbf{c}(u(t),u(t),\varphi)-\mathbf{c}(b(t),b(t),\varphi)=\langle f(t),\varphi\rangle,\\
&\qquad\forall\varphi\in\mathbf{V}^r,\ \text{a.e. } t\in(0,\infty),
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
&\langle\partial_t b(t),\psi\rangle+a^{\theta}_r(b(t),\psi)+\mathbf{c}(u(t),b(t),\psi)-\mathbf{c}(b(t),u(t),\psi)=0,\\
&\qquad\forall\psi\in\mathbf{V}^r_{\mathbf{n}},\ \text{a.e. } t\in(0,\infty),
\end{aligned} \tag{2.5}$$

with $r=2k, k\in\mathbb{N}^*$ *or* $r=2k+1, k\in\mathbb{N}$

$$\lim_{t\to 0^+}(u(t),b(t))=(u_0,b_0),\quad \text{in } \mathbf{W}^r\times\mathbf{W}^r. \tag{2.6}$$

We point out that $(u,b)\in\underline{\mathcal{H}}^1(Q_T)$ implies $(u,b)\in C([0,T];\mathbf{W}^r\times\mathbf{W}^r)$. Hence, (2.6) makes sense.

Now for $T=\infty$, we define a solution for the GMHD equations as follows.

Definition 2.2. *Given* $(u_0,b_0)\in\mathbf{W}^r\times\mathbf{W}^r$ *and* $f\in L^2_{loc}\left(0,T;\mathbf{L}^2\right)$, (u,b) *is said to be a solution of the GMHD equations on* $(0,\infty)$ *if and only if* $(u,b)\in L^2_{loc}(0,\infty;\mathbf{V}^r\times\mathbf{V}^r_{\mathbf{n}})\cap L^\infty(0,\infty;\mathbf{W}^r\times\mathbf{W}^r)$, $\partial_t(u,b)\in L^2_{loc}(0,\infty;(\mathbf{V}^r)^*\times(\mathbf{V}^r_{\mathbf{n}})^*)$ *and* (u,b) *satisfies* (2.4)-(2.6)*with* $T=\infty$.

Now, we turn to the precise statement of the optimal control problem. For each $T\in(0,\infty]$, we define the cost functional J_T by

$$J_T(u,b,f)=\frac{\alpha}{2}\int_0^T\|(u(t),b(t))-(U(t),B(t))\|^2\mathrm{d}t+\frac{\beta}{2}\int_0^T\|f(t)-F(t\|)^2\mathrm{d}t,$$

for all $(u,b)\in(U,B)+\underline{\mathbf{L}}^2(Q_T)$ and $f\in F+\mathbf{L}^2(Q_T)$. Note that J_∞ is also simply denoted by J.

We point out that in the case of $T=\infty$, which will be considered in the sequel, if we choose the control f in the space $\mathbf{L}^2(Q_T)$, it is happen (e.g., in the case of a steady (U,B)) that the value of the cost functional $J_\infty(u,b,f)$ is always infinite for every triplet (u,b,f) under consideration. Therefore, the choice of the control set should also involve (U,B) and F. We define the admissible elements as follows with X_T and Y_T denoting respectively the functional spaces as follows:

$$\begin{aligned}
&X_T=\underline{\mathcal{H}}^1(Q_T)\quad\text{for } T\in(0,\infty),\\
&X_\infty=\Big\{(u,b)\in L^2_{loc}(0,\infty;\mathbf{V}^r\times\mathbf{H}^r_{\mathbf{n}}(\Omega))\cap L^\infty\Big(0,\infty;\mathbf{W}^r\times\mathbf{H}^{r-1}_{\mathbf{n}}(\Omega)\Big);\\
&\qquad\quad \partial_t(u,b)\in L^2_{loc}(0,\infty;(\mathbf{V}^r)^*\times(\mathbf{V}^r_{\mathbf{n}})^*)\big\},\\
&Y_T=L^2(0,T;(\mathbf{V}^r)^*)\quad\text{for } T\in(0,\infty),\\
&Y_\infty=L^2_{loc}(0,\infty;(\mathbf{V}^r)^*).
\end{aligned}$$

Definition 2.3. *For a given* $T\in(0,\infty]$, *a pair* $((u,b),f)\in X_T\times Y_T$ *is called an admissible element if* $J_T((u,b),f)<\infty$ *and* $((u,b),f)$ *satisfies* (2.4)-(2.6). *The set of all admissible elements are denoted by* $\mathcal{U}_{ad}(T)$.

Now for each $T\in(0,\infty]$, we state the optimal control problem on $(0,T)$ as follows:

$$\begin{aligned}
&\text{find a } (u,b,f)\in\mathcal{U}_{ad}(T)\text{ such that}\\
&\qquad J_T(u,b,f)\le_T(\omega,\psi,h),\qquad \forall(\omega,\psi,h)\in\mathcal{U}_{ad}(T).
\end{aligned}\tag{2.7}$$

We point out that in general, the initial state (u_0,b_0) is at a certain distance away from the desired flow, or $(u_0,b_0)\neq(U(t),B(t))$ for all t, the cost functional generally has a positive minimum. Therefore our optimal control problem has nontrivial solutions. We denote by $\bigwedge=(-\triangle)$.

Lemma 2.1. *For all $u\in\mathbf{V}^r$, we have*

$$||(\bigwedge)^r u||_{L^2}\geq\lambda_1^{2r-1}||\nabla u||_{L^2}\,, \tag{2.8}$$

where λ_1 is a constant that appears in the Poincarré inequality.

Proof. We will prove it by induction.

For $r=1$, integration by parts and the use of the Poincarré inequality (2.3) and the Schwarz inequality give

$$||\nabla u||_{L^2}^2=(\nabla u,\nabla u)=(\bigwedge u,u)\leq||\bigwedge u||_{L^2}||u||_{L^2}\leq\frac{1}{\lambda_1}||\bigwedge u||_{L^2}||\nabla u||_{L^2},$$

and then

$$||\bigwedge u||_{L^2}\geq\lambda_1||\nabla u||_{L^2}.$$

For $r=2$, thanks to the Poincarré inequality (2.3) and integration by parts, with the use of the Schwarz inequality, we get

$$\begin{aligned}\lambda_1^2||\bigwedge u||_{L^2}^2\leq&||\nabla(\bigwedge u)||_{L^2}^2=\Big(\nabla(\bigwedge u),\nabla(\bigwedge u)\Big)\\ =&\Big(-\triangle(\bigwedge u),\bigwedge u\Big)=\Big(\overset{2}{\bigwedge}u,\bigwedge u\Big)\leq||\overset{2}{\bigwedge}u||_{L^2}\|\bigwedge u||_{L^2},\end{aligned}$$

which implies

$$\|(\bigwedge)^2u\|_{L^2}\geq\lambda_1^2\|\bigwedge u\||_{L^2}\geq\lambda_1^3\|\nabla u\|_{L^2}.$$

Assume that at the level r

$$\|(\bigwedge)^r u\|_{L^2}\geq\lambda_1^{2r-1}\|\nabla u\||_{L^2},$$

then thanks to the Poincarré inequality (2.3) and integration by parts, with the use of the Schwarz inequality, we get

$$\begin{aligned}\lambda_1^2||\overset{r}{\bigwedge}u||_{L^2}^2\leq&||\nabla(\overset{r}{\bigwedge}u)||_{L^2}^2=\Big(\nabla(\overset{r}{\bigwedge}u),\nabla(\overset{r}{\bigwedge}u)\Big)\\ =&\Big(-\triangle(\overset{r}{\bigwedge}u),\overset{r}{\bigwedge}u\Big)=\Big(\overset{r+1}{\bigwedge}u,\overset{r}{\bigwedge}u\Big)\leq\|\overset{r+1}{\bigwedge}u\|_{L^2}\|\overset{r}{\bigwedge}u\|_{L^2},\end{aligned}$$

which implies

$$\|(\bigwedge)^{r+1}u\|_{L^2}\geq\lambda_1^2\|(\bigwedge)^r u||_{L^2}\geq\lambda_1^{2r+1}||\nabla u||_{L^2}=\lambda_1^{2(r+1)-1}||\nabla u||_{L^2}.$$

This finishes the proof of Lemma 2.1. ☐

Let $r=2k, k\in\mathbb{N}^*$. The system becomes

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+\nu\bigwedge^{2k} u+(u\cdot\nabla)u-(b\cdot\nabla)b+\nabla p=f, && \text{in } \Omega\times(0,\infty),\\
&\frac{\partial b}{\partial t}+\theta\bigwedge^{2k} b+(u\cdot\nabla)b-(b\cdot\nabla)u=0, && \text{in } \Omega\times(0,\infty),\\
&\nabla\cdot u=0,\ \ \nabla\cdot b=0, && \text{in } \Omega\times(0,\infty),\\
&u=0,\ \ \triangle u=0,\ \ \triangle^2 u=0,\ \cdots,\ \triangle^r u=0, && \text{on } \partial\Omega\times(0,\infty),\\
&b\cdot\mathbf{n}=0,\ \ \triangle b=0,\ \ \triangle^2 b=0,\ \cdots,\ \triangle^r b=0, && \text{on } \partial\Omega\times(0,\infty),\\
&\nabla u=0,\ \ \nabla^2 u=0,\ \cdots,\ \nabla^r u=0, && \text{on } \partial\Omega\times(0,\infty),\\
&\nabla b=0,\ \ \nabla^2 b=0,\ \cdots,\ \nabla^r b=0, && \text{on } \partial\Omega\times(0,\infty),\\
&u(0)=u_0 \quad\text{and}\quad b(0)=b_0.
\end{aligned}
$$

Using the inner product and the $2k$ integrations by parts we find

$$
\begin{cases}
\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\nu\left\|\bigwedge^k u(t)\right\|^2-\mathbf{c}(b(t),b(t),u(t))=\langle f,u(t)\rangle,\\
\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|b(t)\|^2+\theta\left\|\bigwedge^k b(t)\right\|^2-\mathbf{c}(b(t),u(t),b(t))=0,
\end{cases}
\tag{2.9}
$$

the use of the Lemma 2.1 and the Schwarz inequality gives

$$
\begin{cases}
\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\nu\lambda_1^{2(2k-1)}\|\nabla u(t)\|^2-\mathbf{c}(b(t),b(t),u(t))\le\|f\|\cdot\|u\|,\\
\dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|b(t)\|^2+\theta\lambda_1^{2(2k-1)}\|\nabla b(t)\|^2-\mathbf{c}(b(t),u(t),b(t))\le 0.
\end{cases}
\tag{2.10}
$$

Then also $\forall u\in\mathbf{V}^r$ and $\forall b\in\mathbf{V}^r_{\mathbf{n}}$,

$$
\begin{cases}
a^\nu_{2k}(u,u)=\nu\|\bigwedge^k u\|^2\ge\nu\lambda_1^{2(2k-1)}\|\nabla u\|^2,\\
a^\theta_{2k}(b,b)=\theta\|\bigwedge^k b\|^2\ge\theta\lambda_1^{2(2k-1)}\|\nabla b\|^2.
\end{cases}
\tag{2.11}
$$

Now let $r=2k+1, k\in\mathbb{N}$. The system becomes

$$
\begin{aligned}
&\frac{\partial u}{\partial t}-\nu\triangle\bigwedge^{2k} u+(u\cdot\nabla)u-(b\cdot\nabla)b+\nabla p=f, && \text{in } \Omega\times(0,\infty),\\
&\frac{\partial b}{\partial t}-\theta\triangle\bigwedge^{2k} b+(u\cdot\nabla)b-(b\cdot\nabla)u=0, && \text{in } \Omega\times(0,\infty),\\
&\nabla\cdot u=0,\ \ \nabla\cdot b=0, && \text{in } \Omega\times(0,\infty),\\
&u=0,\ \ \triangle u=0,\ \ \triangle^2 u=0,\ \cdots,\ \triangle^r u=0, && \text{on } \partial\Omega\times(0,\infty),\\
&b\cdot\mathbf{n}=0,\ \ \triangle b=0,\ \ \triangle^2 b=0,\ \cdots,\ \triangle^r b=0, && \text{on } \partial\Omega\times(0,\infty),
\end{aligned}
$$

$$\begin{aligned}&\nabla u=0,\ \nabla^2 u=0,\ \cdots,\ \nabla^r u=0, &&\text{on } \partial\Omega\times(0,\infty),\\&\nabla b=0,\ \nabla^2 b=0,\ \cdots,\ \nabla^r b=0, &&\text{on } \partial\Omega\times(0,\infty),\\&u(0)=u_0\quad\text{and}\quad b(0)=b_0.\end{aligned}$$

Using the inner product and the $2k+1$ integrations by parts give

$$\left\{\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\nu\left\|\nabla\bigwedge^k u(t)\right\|^2-\mathbf{c}(b(t),b(t),u(t))=\langle f,u(t)\rangle,\\&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b(t)\|^2+\theta\left\|\nabla\bigwedge^k b(t)\right\|^2-\mathbf{c}(b(t),u(t),b(t))=0.\end{aligned}\right. \tag{2.12}$$

The Poincarré inequality (2.3) and the use of Lemma 2.1 lead us to

$$\left\|\nabla(-\triangle)^k u(t)\right\|\geq\lambda_1\left\|(-\triangle)^k u(t)\right\|\geq\lambda_1^{2k}\|\nabla u(t)\|. \tag{2.13}$$

Then (2.3), Lemma 2.1, the Schwartz inequality and $2k+1$ integrations by parts give

$$\left\{\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+\nu\lambda_1^{4k}\|\nabla u(t)\|^2-\mathbf{c}(b(t),b(t),u(t))\leq\|f\|\cdot\|u\|,\\&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b(t)\|^2+\theta\lambda_1^{4k}\|\nabla b(t)\|^2-\mathbf{c}(b(t),u(t),b(t))\leq 0.\end{aligned}\right. \tag{2.14}$$

Hence also $\forall u\in\mathbf{V}^r$ and $\forall b\in\mathbf{V}^r_{\mathbf{n}}$,

$$a^{\nu}_{(2k+1)}(u,u)=\nu\|\nabla\bigwedge^k u\|^2\geq\nu\lambda_1^{4k}\|\nabla u\|^2, \tag{2.15a}$$

$$a^{\theta}_{(2k+1)}(b,b)=\theta\|\nabla\bigwedge^k b\|^2\geq\theta\lambda_1^{4k}\|\nabla b\|^2. \tag{2.15b}$$

Throughout this paper we denote by

$$(v,w)=(u,b)-(U,B),\qquad\text{and}\qquad g=f-F,$$

unless we specify them. Then (2.4)-(2.6) are equivalent to

$$\begin{aligned}&(v,w)\in X_T\cap L^2(0,\infty;\mathbf{V^r}\times\mathbf{V}^r_{\mathbf{n}}),\quad g\in Y_T\cap L^2\big(0,T;\mathbf{L}^2(\Omega)\big),\\&\langle\partial_t v(t),\varphi\rangle+a^{\nu}_r(v(t),\varphi)+\mathbf{c}(v(t),v(t),\varphi)+\mathbf{c}(v(t),U(t),\varphi)\\&\qquad+\mathbf{c}(U(t),v(t),\varphi)-\mathbf{c}(w(t),w(t),\varphi)-\mathbf{c}(w(t),B(t),\varphi)\\&\qquad-\mathbf{c}(B(t),w(t),\varphi)=\langle g(t),\varphi\rangle,\quad\forall\varphi\in\mathbf{V}^r\text{ a.e. }t\in(0,\infty),\end{aligned} \tag{2.16}$$

$$\begin{aligned}&\langle\partial_t w(t),\psi\rangle+a^{\theta}_r(w(t),\psi)+\mathbf{c}(v(t),w(t),\psi)+\mathbf{c}(v(t),B(t),\psi)\\&\qquad+\mathbf{c}(U(t),w(t),\psi)-\mathbf{c}(w(t),v(t),\psi)-\mathbf{c}(w(t),U(t),\psi)\\&\qquad-\mathbf{c}(B(t),v(t),\psi)=0,\qquad\forall\psi\in\mathbf{V}^r_{\mathbf{n}}\text{ a.e. }t\in(0,\infty),\end{aligned} \tag{2.17}$$

with $r=2k,k\in\mathbb{N}^*$ or $r=2k+1,\quad k\in\mathbb{N}$

$$\lim_{t\to 0^+}(u(t),b(t))=(u_0,b_0),\qquad \text{in } \mathbf{W}^r\times\mathbf{W}^r. \tag{2.18}$$

The cost functional can be rewritten as

$$\mathcal{K}_T(v,w,g)\overset{\text{def}}{=}J_T(v+U,w+B,g+F)=\frac{\alpha}{2}\int_0^{+\infty}\|(v,w)(t)\|^2\mathrm{d}t+\frac{\beta}{2}\int_0^{+\infty}\|g(t)\|^2\mathrm{d}t, \tag{2.19}$$

by defining

$$\mathcal{V}_{ad}(T)\overset{\text{def}}{=}\Big\{((v,w),g)\in X_T\cap L^2(0,T;\mathbf{V}^r\times\mathbf{V}^r_{\mathbf{n}})\times Y_T\cap L^2\big(0,T;\mathbf{L}^2(\Omega)\big),$$
$$\mathcal{K}_T(v,w,g)<\infty,\ ((v,w),g) \text{ satisfies } (2.16),(2.18)\Big\}.$$

For each $T\in(0,\infty]$, we can restate the optimization problem (2.7) in terms of the auxiliary variables (v,w,g):

$$\text{find a } (v,w,g)\in\mathcal{V}_{ad}(T) \text{ such that } \mathcal{K}_T(v,w,g)\le\mathcal{K}_T(z,k,h),\quad \forall(z,k,h)\in\mathcal{V}_{ad}(T). \tag{2.20}$$

3 Preliminary estimates for the dynamics

3.1 A quasi optimizer

To estimate the dynamics of the optimal control solution, we need to find a sharp bound for the value of $\inf_{(u,b,f)\in\mathcal{U}_{ad}(T)}J_T(u,b,f)$. It is important that this bound is uniform in T. We now construct a quasi-optimizer $(\widetilde{u},\widetilde{b},\widetilde{f})\in\mathcal{U}_{ad}(\infty)$ for $J_\infty(\cdot,\cdot)$. We can in turn derive some preliminary estimates for the optimal solutions. By a quasi-optimizer we mean an element $(\widetilde{u},\widetilde{b},\widetilde{f})\in\mathcal{U}_{ad}(\infty)$ satisfying $\big\|(\widetilde{u},\widetilde{b})(t)-(U,B)(t)\big\|\to 0$ as $t\to\infty$. The following theorem asserts the existence of such an element.

Theorem 3.1. *Assume that the assumptions* (A1) *and* (A2) (*in the introduction above*) *hold. Then there exists a pair* $\big((\widetilde{u},\widetilde{b}),\widetilde{f}\big)\in\mathcal{U}_{ad}(\infty)$ *satisfying* $\forall t\ge 0$

$$\Big\|(\widetilde{u},\widetilde{b})(t)-(U,B)(t)\Big\|^2\le\|(u_0,b_0)-(U_0,B_0)\|^2e^{-\epsilon t} \tag{3.1}$$

and $\forall T\in(0,\infty]$

$$J_T(\widetilde{u},\widetilde{b},\widetilde{f})\le\frac{\alpha\|(u_0,b_0)-(U_0,B_0)\|^2}{2\varepsilon}\Big(1-e^{-\varepsilon T}\Big) \tag{3.2}$$

with

$$\epsilon=\min\Big\{2\theta\lambda_1^{4k-2}-\frac{\theta}{2}-\frac{4}{\theta\lambda_1}|||\nabla U|||^2-\frac{8}{\nu\lambda_1}|||\nabla B|||^2,\ 2\nu\lambda_1^{4k-2}-\frac{3\nu}{2}-\frac{4}{\nu\lambda_1}|||\nabla U|||^2,$$
$$2\theta\lambda_1^{4k}-\frac{\theta}{2}-\frac{4}{\theta\lambda_1}|||\nabla U|||^2-\frac{8}{\nu\lambda_1}|||\nabla B|||^2,\ 2\nu\lambda_1^{4k}-\frac{3\nu}{2}-\frac{4}{\nu\lambda_1}|||\nabla U|||^2\Big\}. \tag{3.3}$$

Proof. For $r=2k$ ($k\in\mathbb{N}^*$), by substituting $\varphi=v(t)$ and $\psi=w(t)$ in (2.16)-(2.17), using (2.11) and summing, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\nu\lambda_1^{2(2k-1)}\|\nabla v(t)\|^2\\&\quad+\theta\lambda_1^{2(2k-1)}\|\nabla w(t)\|^2+\mathbf{c}(v(t),U(t),v(t))-\mathbf{c}(w(t),U(t),w(t))\\&\quad-\mathbf{c}(w(t),B(t),v(t))+\mathbf{c}(v(t),B(t),w(t))\leq\langle g(t),v(t)\rangle.\end{aligned}\tag{3.4}$$

The use of (2.1) and (2.3) gives

$$\begin{aligned}|\mathbf{c}(v(t),U(t),v(t))|&\leq\sqrt{2}\|v(t)\|\cdot|\|\nabla U\||\cdot\|\nabla v(t)\|\\&\leq\frac{\nu\lambda_1}{4}\|v(t)\|^2+\frac{2}{\nu\lambda_1}|\|\nabla U\||^2\cdot\|\nabla v(t)\|^2\\&\leq\frac{\nu}{4}\|\nabla v(t)\|^2+\frac{2}{\nu\lambda_1}|\|\nabla U\||^2\cdot\|\nabla v(t)\|^2,\end{aligned}\tag{3.5}$$

and

$$|\mathbf{c}(w(t),B(t),v(t))|\leq\frac{\theta}{4}\|\nabla w(t)\|^2+\frac{2}{\theta\lambda_1}|\|\nabla B\||^2\cdot\|\nabla v(t)\|^2,\tag{3.6}$$

so that combining the last inequalities give

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+2\nu\lambda_1^{2(2k-1)}\|\nabla v(t)\|^2+2\theta\lambda_1^{2(2k-1)}\|\nabla w(t)\|^2\\\leq&\frac{3\nu}{2}\|\nabla v(t)\|^2+\frac{\theta}{2}\|\nabla w(t)\|^2+\frac{4}{\nu\lambda_1}|\|\nabla U\||^2\cdot\|\nabla v(t)\|^2+\frac{4}{\nu\lambda_1}|\|\nabla U\||^2\cdot\|\nabla w(t)\|^2\\&+\frac{8}{\nu\lambda_1}|\|\nabla B\||^2\cdot\|\nabla w(t)\|^2+2|\langle g(t),v(t)\rangle|,\end{aligned}\tag{3.7}$$

hence

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|v(t)\|^2+\|w(t)\|^2\Big)+\Big(2\nu\lambda_1^{4k-2}-\frac{3\nu}{2}-\frac{4}{\nu\lambda_1}|\|\nabla U\||^2\Big)\|\nabla v(t)\|^2\\&\quad+\Big(2\theta\lambda_1^{4k-2}-\frac{\theta}{2}-\frac{4}{\theta\lambda_1}|\|\nabla U\||^2-\frac{8}{\nu\lambda_1}|\|\nabla B\||^2\Big)\|\nabla w(t)\|^2\leq2|\langle g(t),v(t)\rangle|.\end{aligned}\tag{3.8}$$

For $r=2k+1$ ($k\in\mathbb{N}$), by substituting $\varphi=v(t)$ and $\psi=w(t)$ in (2.16)-(2.17), using (2.15) and summing, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\nu\lambda_1^{4k}\|\nabla v(t)\|^2+\theta\lambda_1^{4k}\|\nabla w(t)\|^2\\&\quad+\mathbf{c}(v(t),U(t),v(t))-\mathbf{c}(w(t),U(t),w(t))-\mathbf{c}(w(t),B(t),v(t))\\&\quad+\mathbf{c}(v(t),B(t),w(t))\leq\langle g(t),v(t)\rangle.\end{aligned}\tag{3.9}$$

Hence combining the last inequality with (3.5)-(3.6) give

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2+\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+2\nu\lambda_1^{4k}\|\nabla v(t)\|^2+2\theta\lambda_1^{4k}\|\nabla w(t)\|^2\\ \le&\frac{3\nu}{2}\|\nabla v(t)\|^2+\frac{\theta}{2}\|\nabla w(t)\|^2+\frac{4}{\nu\lambda_1}|\|\nabla U\||^2\cdot\|\nabla v(t)\|^2+\frac{4}{\nu\lambda_1}|\|\nabla U\||^2\cdot\|\nabla w(t)\|^2\\ &+\frac{8}{\nu\lambda_1}|\|\nabla B\||^2\cdot\|\nabla w(t)\|^2+2|\langle g(t),v(t)\rangle|.\end{aligned}\tag{3.10}$$

Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|^2+\|w(t)\|^2\right)+\left(2\nu\lambda_1^{4k}-\frac{3\nu}{2}-\frac{4}{\nu\lambda_1}|\|\nabla U\||^2\right)\|\nabla v(t)\|^2\\ &\quad+\left(2\theta\lambda_1^{4k}-\frac{\theta}{2}-\frac{4}{\theta\lambda_1}|\|\nabla U\||^2-\frac{8}{\nu\lambda_1}|\|\nabla B\||^2\right)\|\nabla w(t)\|^2\le 2|\langle g(t),v(t)\rangle|,\end{aligned}\tag{3.11}$$

or again (3.8) and (3.11) become

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|^2+\|w(t)\|^2\right)+\frac{\epsilon}{\lambda_1}\left(\|\nabla v(t)\|^2+\|\nabla w(t)\|^2\right)\le 2|\langle g(t),v(t)\rangle|,\tag{3.12}$$

so that the use of (2.3) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(v(t),w(t))\|^2+\epsilon\|(v(t),w(t))\|^2\le 2|\langle g(t),v(t)\rangle|.\tag{3.13}$$

In particular, we let $(\widetilde{v},\widetilde{w})$ be the solution of (2.16)-(2.18) when $g\equiv 0$, i.e., $(\widetilde{u},\widetilde{b},\widetilde{f})$ satisfy (1.2)-(1.9) with $\widetilde{f}=F$. Thus we apply the Gronwall's inequality to (3.13) with $g\equiv 0$ to obtain

$$\|(\widetilde{v},\widetilde{w})(t)\|^2\le\|(v_0,w_0)\|^2e^{-\epsilon t}\le\|(u_0,b_0)-(U_0,B_0)\|^2e^{-\epsilon t},$$

which gives the conclusion (3.1). Furthermore, we see that for each $T\in(0,\infty]$, (with $f=F$)

$$\begin{aligned}J_T(\widetilde{u},\widetilde{b},\widetilde{f})&=\frac{\alpha}{2}\int_0^T\|(\widetilde{v},\widetilde{w})(t)\|^2\mathrm{d}t\\ &\le\frac{\alpha\|(u_0,b_0)-(U_0,B_0)\|^2}{2\varepsilon}\left(1-e^{-\varepsilon T}\right)\\ &\le\frac{\alpha\|(u_0,b_0)-(U_0,B_0)\|^2}{2\varepsilon}.\end{aligned}\tag{3.14}$$

This completes the proof of this theorem. □

Remark 3.1. It follows from Theorem 3.1 that

$$\lim_{T\to\infty}\min_{(\widetilde{u},\widetilde{b},\widetilde{f})\in\mathcal{U}_{ad}(T)}J_T(\widetilde{u},\widetilde{b},\widetilde{f})=0.$$

We see that a quasi optimizer $(\widetilde{u},\widetilde{b},\widetilde{f})$ has been created in the sense that

$$\left\|(\widetilde{u},\widetilde{b})(t)-(U,B)(t)\right\| \to 0, \quad \text{as } t\to\infty \text{ and } J_\infty(\widetilde{u},\widetilde{b},\widetilde{f}) \text{ is bounded.}$$

In fact, $\|(\widetilde{u},\widetilde{b})(t)-(U,B)(t)\|\to 0$ exponentially as $t\to\infty$. The true optimizer is expected to have the property $\|(\widetilde{u},\widetilde{b})(t)-(U,B)(t)\|\to 0$ as $t\to\infty$ and at the same time, minimize the work involved to realize and maintain the optimizer flow.

3.2 Estimate for the dynamics of admissible elements

In this section, we will derive some estimates for the dynamics of all solutions of (1.2)-(1.9). These estimates in turn will allow us to derive preliminary estimates for the dynamics of the optimal solutions. First we consider the $L^\infty\big(0,T;\underline{\mathbf{L}}^2(\Omega)\big)$ estimates in terms of the initial data and the functional values.

Theorem 3.2. *Let $T\in(0,\infty]$. Assume that the assumptions* (A1) *and* (A2) *hold. If $(u,b,f)\in\mathcal{U}_{ad}(T)$, then $\forall t\in[0,T]$,*

$$\|(u,b)(t)-(U,B)(t)\|^2\le\|(u_0,b_0)-(U_0,B_0)\|^2+\frac{2}{\sqrt{\alpha\beta}}J_T(u,b,f). \tag{3.15}$$

If in addition,

$$J_T(u,b,f)\le J_T(\widetilde{u},\widetilde{b},\widetilde{f}),$$

then

$$\|(u,b)(t)-(U,B)(t)\|^2\le K_0\|(u_0,b_0)-(U_0,B_0)\|^2, \tag{3.16}$$

where ϵ and $(\widetilde{u},\widetilde{b},\widetilde{f})$ are defined in Theorem 3.1 *and*

$$K_0=\left(1+\frac{1}{2\epsilon}\sqrt{\frac{\alpha}{\beta}}\right).$$

Proof. Applying the Schwarz and the Young inequalities to (3.13) we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|(v,w)(t)\|^2+\epsilon\|(v,w)(t)\|^2&\le\frac{1}{\sqrt{\alpha\beta}}(\alpha\|(v)(t)\|^2+\beta\|g(t)\|^2)\\&\le\frac{1}{\sqrt{\alpha\beta}}(\alpha\|(v,w)(t)\|^2+\beta\|g(t)\|^2).\end{aligned} \tag{3.17}$$

Multiplying both sides of this inequality by $e^{\epsilon t}$ and then integrating in t over $(0,t)$, lead us to

$$\begin{aligned}\|(v,w)(t)\|^2&\le\|(v,w)(0)\|^2e^{-\epsilon t}+\frac{1}{\sqrt{\alpha\beta}}\int_0^t\Big(\alpha\|(v,w)(s)\|^2+\beta\|g(s)\|^2\Big)e^{\epsilon(s-t)}\mathrm{d}s\\&\le\|(v,w)(0)\|^2e^{-\epsilon t}+\frac{1}{\sqrt{\alpha\beta}}\int_0^T\Big(\alpha\|(v,w)(t)\|^2+\beta\|g(t)\|^2\Big)\mathrm{d}t\\&\le\|(v,w)(0)\|^2e^{-\epsilon t}+\frac{2}{\sqrt{\alpha\beta}}J_T(u,b,f).\end{aligned}$$

This yields the inequality (3.15). Moreover combining the condition $J_T(u,b,f) \leq J_T(\widetilde{u},\widetilde{b},\widetilde{f})$ with the inequality (3.15) and the Theorem 3.1 we find the inequality (3.16). This finishes the proof of this theorem. □

Now, using the uniform Gronwall's inequality we derive $L^\infty(0,T;\underline{\mathbf{H}}^r)$ estimates.

Theorem 3.3. *Let $T \in (0,\infty]$ and $(u,b,f) \in \mathcal{U}_{ad}(T)$. Assume that the assumptions* (A1) *and* (A2) *hold and assume further that $J_T(u,b,f) \leq J_T(\widetilde{u},\widetilde{b},\widetilde{f})$. Then for each $\varepsilon > 0$, we have*

$$(u-U,b-B) \in L^2(0,T;\underline{\mathbf{H}}^r(\Omega)) \cap L^\infty(\varepsilon,T;\underline{\mathbf{H}}^r(\Omega)) \cap C([\varepsilon,T];\underline{\mathbf{H}}^r(\Omega)),$$

with

$$\int_0^T \|\nabla(u,b)(s) - \nabla(U,B)(s)\|^2 \mathrm{d}s \leq K_1 \|(u_0,b_0)-(U_0,B_0)\|^2, \tag{3.18}$$

and

$$\|\nabla(u,b)(t) - \nabla(U,B)(t)\|^2 \leq K_2 \|(u_0,b_0)-(U_0,B_0)\|^2, \quad \forall t \geq \varepsilon, \tag{3.19}$$

where

$$\begin{aligned}
K_1 &= \frac{\lambda_1}{\varepsilon}\left(1+\frac{1}{\varepsilon}\sqrt{\frac{\alpha}{\beta}}\right), \\
K_2 &= 2CK_0\left(\frac{1}{\nu^3}+\frac{2}{\theta\nu}+\frac{1}{\theta^3}\right)\|(u_0,b_0)-(U_0,B_0)\|^2, \\
K_3 &= 2(C_5+C_6).
\end{aligned}$$

Proof. Let $T \in (0,\infty]$ be given. For each $\varepsilon > 0$, it follows from the regularity results for the GMHD equations (see [3]) that $(v,w) \in L^2(0,T;\underline{\mathbf{H}}^r(\Omega)) \cap C([\varepsilon,T];\underline{\mathbf{H}}^r(\Omega))$ if $T < \infty$ so that (3.15) holds.

When $T = \infty$, we have $(v,w) \in L^2_{loc}(0,T;\underline{\mathbf{H}}^r(\Omega)) \cap C([\varepsilon,T];\underline{\mathbf{H}}^r(\Omega))$.

Applying Young and Schwarz inequalities to (3.12) and integrating by parts in t over $(0,T)$, we obtain easily (3.18).

Then we set $\varphi = -\Pi\Delta v(t)$ and $\psi = -\Pi\Delta w(t)$ in (2.16) and (2.17), respectively. Note that

$$\begin{aligned}
a_{2k}^{\nu}(v(t),-\Pi\Delta v(t)) &= \left\langle \nu(-\Delta)^{2k}v(t),-\Pi\Delta v(t)\right\rangle = \nu\left\|\Pi\nabla(-\Delta)^k v(t)\right\|^2, \\
a_{2k}^{\theta}(w(t),-\Pi\Delta w(t)) &= \left\langle \theta(-\Delta)^{2k}w(t),-\Pi\Delta w(t)\right\rangle = \theta\left\|\Pi\nabla(-\Delta)^k w(t)\right\|^2,
\end{aligned}$$

and

$$\begin{aligned}
a_{(2k+1)}^{\nu}(v(t),-\Pi\Delta v(t)) &= \left\langle -\nu\Delta(-\Delta)^{2k}v(t),-\Pi\Delta v(t)\right\rangle = \nu\left\|\Pi(-\Delta)^{k+1}v(t)\right\|^2, \\
a_{(2k+1)}^{\theta}(w(t),-\Pi\Delta w(t)) &= \left\langle -\theta\Delta(-\Delta)^{2k}w(t),-\Pi\Delta w(t)\right\rangle = \theta\left\|\Pi(-\Delta)^{k+1}w(t)\right\|^2.
\end{aligned}$$

Applying Sobolev imbedding and interpolation results, we obtain for $r=2k\,(k\in\mathbb{N}^*)$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v(t)\|^2+\nu\lambda_1^{4k}\|\Pi\nabla v(t)\|^2\\ \leq&\|v(t)\|^{1/2}\|\nabla v(t)\|\,\|\Pi\Delta v(t)\|^{3/2}\\ &+\|v(t)\|^{1/2}\|\nabla v(t)\|^{1/2}\|\nabla U(t)\|^{1/2}\|\Delta U(t)\|^{1/2}\|\Pi\Delta v(t)\|\\ &+\|U(t)\|^{1/2}\|\nabla U(t)\|^{1/2}\|\nabla v(t)\|^{1/2}\|\Pi\Delta v(t)\|^{3/2}\\ &+\|w(t)\|^{1/2}\|\nabla w(t)\|\,\|\Pi\Delta w(t)\|^{1/2}\|\Pi\Delta v(t)\|\\ &+\|w(t)\|^{1/2}\|\nabla w(t)\|^{1/2}\|\nabla B(t)\|^{1/2}\|\Delta B(t)\|^{1/2}\|\Pi\Delta v(t)\|\\ &+\|B(t)\|^{1/2}\|\nabla B(t)\|^{1/2}\|\nabla w(t)\|^{1/2}\|\Pi\Delta w(t)\|^{1/2}\|\Pi\Delta v(t)\|\\ &+\|g(t)\|\,\|\Pi\Delta v(t)\|,\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w(t)\|^2+\theta\lambda_1^{4k}\|\Pi\nabla v(t)\|^2\\ \leq&\|v(t)\|^{1/2}\|\nabla v(t)\|^{1/2}\|\nabla w(t)\|^{1/2}\|\Pi\Delta w(t)\|^{3/2}\\ &+\|v(t)\|^{1/2}\|\nabla v(t)\|^{1/2}\|\nabla B(t)\|^{1/2}\|\Delta B(t)\|^{1/2}\|\Pi\Delta w(t)\|\\ &+\|U(t)\|^{1/2}\|\nabla U(t)\|^{1/2}\|\nabla w(t)\|^{1/2}\|\Pi\Delta w(t)\|^{3/2}\\ &+\|w(t)\|^{1/2}\|\nabla w(t)\|^{1/2}\|\nabla v(t)\|^{1/2}\|\Pi\Delta v(t)\|^{1/2}\|\Pi\Delta w(t)\|\\ &+\|w(t)\|^{1/2}\|\nabla w(t)\|^{1/2}\|\nabla U(t)\|^{1/2}\|\Delta U(t)\|^{1/2}\|\Pi\Delta w(t)\|\\ &+\|B(t)\|^{1/2}\|\nabla B(t)\|^{1/2}\|\nabla v(t)\|^{1/2}\|\Pi\Delta v(t)\|^{1/2}\|\Pi\Delta w(t)\|.\end{aligned}\tag{3.21}$$

Applying now the Young and the Poincarré inequalities and summing each inequality, we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(v,w)(t)\|^2\leq&\frac{C}{\nu^3}\|v(t)\|^2\|\nabla v(t)\|^4+\frac{C}{\nu\theta}\|w(t)\|^2\|\nabla w(t)\|^4\\ &+\frac{C}{\theta^3}\|v(t)\|^2\|\nabla v(t)\|^2\|\nabla w(t)\|^2+\frac{C}{\theta\nu}\|w(t)\|^2\|\nabla w(t)\|^2\|\nabla v(t)\|^2\\ &+C_5\|\nabla v(t)\|^2+C_6\|\nabla w(t)\|^2+\frac{1}{\nu}\|g(t)\|^2,\end{aligned}\tag{3.22}$$

with

$$\begin{aligned}C_5=\Big(&\frac{C}{\nu^3}\|U(t)\|^2\|\nabla U(t)\|^2+\frac{C}{\nu\sqrt{\lambda_1}}\|\nabla U(t)\|\,\|\Delta U(t)\|\\ &+\frac{C}{\nu\theta}\|B(t)\|^2\|\nabla B(t)\|^2+\frac{C}{\theta\sqrt{\lambda_1}}\|\nabla B(t)\|\,\|\Delta B(t)\|-\nu\lambda_1^{4k}\Big),\end{aligned}$$

$$C_6=\bigg(\frac{C}{\theta^3}\|U(t)\|^2\|\nabla U(t)\|^2+\frac{C}{\theta\sqrt{\lambda_1}}\|\nabla U(t)\|\|\Delta U(t)\|\\+\frac{C}{\nu\theta}\|B(t)\|^2\|\nabla B(t)\|^2+\frac{C}{\nu\sqrt{\lambda_1}}\|\nabla B(t)\|\|\Delta B(t)\|-\theta\lambda_1^{4k}\bigg).$$

Likewise, for $r=2k+1$ $(k\in\mathbb{N})$, we use the same technique and we obtain (3.22). But in this case C_5 and C_6 are given by:

$$C_5'=\bigg(\frac{C}{\nu^3}\|U(t)\|^2\|\nabla U(t)\|^2+\frac{C}{\nu\sqrt{\lambda_1}}\|\nabla U(t)\|\|\Delta U(t)\|\\+\frac{C}{\nu\theta}\|B(t)\|^2\|\nabla B(t)\|^2+\frac{C}{\theta\sqrt{\lambda_1}}\|\nabla B(t)\|\|\Delta B(t)\|-\nu\lambda_1^{4k-2}\bigg),$$

$$C_6'=\bigg(\frac{C}{\theta^3}\|U(t)\|^2\|\nabla U(t)\|^2+\frac{C}{\theta\sqrt{\lambda_1}}\|\nabla U(t)\|\|\Delta U(t)\|\\+\frac{C}{\nu\theta}\|B(t)\|^2\|\nabla B(t)\|^2+\frac{C}{\nu\sqrt{\lambda_1}}\|\nabla B(t)\|\|\Delta B(t)\|-\theta\lambda_1^{4k-2}\bigg).$$

The use of (3.16) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(v,w)(t)\|^2\le K_2\|\nabla(v,w)(t)\|^2\|\nabla(v,w)(t)\|^2+K_3\|\nabla(v,w)(t)\|^2+\frac{2}{\nu}\|g(t)\|^2,$$

where

$$K_2=2CK_0\left(\frac{1}{\nu^3}+\frac{2}{\theta\nu}+\frac{1}{\theta^3}\right)\|(u_0,b_0)-(U_0,B_0)\|^2,$$

and

$$K_3=2(C_5+C_6),\qquad \text{or}\qquad K_3=2(C_5'+C_6').$$

To apply the uniform Gronwall's inequality to this last inequality, we need the following estimates which followed from (3.2), for each $\varepsilon>0$, we have

$$\int_t^{t+\varepsilon}\|\nabla(u,b)(s)-\nabla(U,B)(s)\|^2\mathrm{d}s\le K_1\|(u_0,b_0)-(U_0,B_0)\|^2,$$

and

$$\int_t^{t+\varepsilon}\left(K_2\|\nabla(u,b)(s)-\nabla(U,B)(s)\|^2+\frac{2}{\nu}\|g(s)\|^2\right)\mathrm{d}s\\\le\left(K_1K_2+\frac{2\alpha}{\nu\varepsilon}\right)\|(u_0,b_0)-(U_0,B_0)\|^2.$$

This completes the proof. □

An immediate consequence of Theorems 3.2 and 3.3 is the following preliminary estimates for the optimal solutions.

Theorem 3.4. *Assume that the assumptions* (A1) *and* (A2) *hold. Let* $T\in(0,\infty]$ *and* $(\widehat{u},\widehat{b},\widehat{f})\in\mathcal{U}_{ad}(T)$ *be an optimal solution for* (2.7). *Then*

$$\left\|\left(\widehat{u},\widehat{b}\right)(t)-(U,B)(t)\right\|^2\leq K_0\|(u_0,b_0)-(U_0,B_0)\|^2, \tag{3.23}$$

$$\int_0^T\left\|\nabla(\widehat{u},\widehat{b})(s)-\nabla(U,B)(s)\right\|^2\mathrm{d}s\leq K_1\|(u_0-U_0,b_0-B_0)\|^2, \tag{3.24}$$

and

$$\left\|\nabla(\widehat{u},\widehat{b})(t)-\nabla(U,B)(t)\right\|^2\leq K_2(\varepsilon)\|(u_0-U_0,b_0-B_0)\|^2,\qquad\forall t\geq\varepsilon, \tag{3.25}$$

where all constants are as defined in Theorems 3.2 *and* 3.3.

4 Existence of an optimal control

4.1 The case of finite time interval

In this section, we first give the existence of an optimal solution for (2.7) with $T<\infty$ and we give also an optimality system. We then derive some estimates for the adjoint state.

4.1.1 Existence of an optimal control

Theorem 4.1. *Let* $T\in(0,\infty)$. *Then there exists an optimal solution* $(\widehat{u},\widehat{b},\widehat{f})\in\mathcal{U}_{ad}(T)$ *for the problem* (2.7), *i.e. there exists at least an element* $\widehat{f}\in L^2\left(0,T;\mathbf{L}^2(\Omega)\right)$ *and* $(\widehat{u},\widehat{b})\in C([0,T];\mathbf{W}^r\times\mathbf{W}^r)\cap L^2(0,T;\mathbf{V}^r\times\mathbf{V}^r_{\mathbf{n}})$ *such that the functional* $J_T(u,b,f)$ *attains its minimum at* $(\widehat{u},\widehat{b},\widehat{f})$ *and* $(\widehat{u},\widehat{b})$ *satisfies* (2.4)-(2.6) *with* $\widehat{f}=f$.

Proof. Note first that, since T is finite, assumption $(A1)$ yields

$$(U,B)\in L^2(0,T;\mathbf{H}^r(\Omega)\times\mathbf{H}^r_{\mathbf{n}}(\Omega)),\ \ F\in L^2\left(0,T;\mathbf{L}^2(\Omega)\right).$$

Let $(u_n,b_n,f_n)\in\mathcal{U}_{ad}(T)$ be a minimizing sequence for the problem (2.7). Hence

$$\lim_{n\to\infty}J_T(u_n,b_n,f_n)=\inf_{(u,b,f)\in\mathcal{U}_{ad}(T)}J_T(u,b,f). \tag{4.1}$$

The cost functional verifies

$$J_T(u,b,f)\geq\frac{\beta}{2}\int_0^T\|f(t)\|^2\mathrm{d}t-\frac{\beta}{2}\int_0^T\|F(t)\|^2\mathrm{d}t,$$

so that f_n is bounded in $L^2\left(0,T;\mathbf{L}^2(\Omega)\right)$. Indeed, suppose that f_n is not bounded. Then, it would exist a sub-sequence of (f_{n_k}) again noted f_n such that

$$\lim_{n\to\infty}\|f_n\|_{L^2(0,T;\mathbf{L}^2(\Omega))}=+\infty$$

and then since $F \in L^2\left(0,T;\mathbf{L}^2(\Omega)\right)$,

$$\lim_{n\to\infty} J_T(u_n,b_n,f_n) = +\infty.$$

This contradicts (4.1). Therefore f_n is bounded in $L^2\left(0,T;\mathbf{L}^2(\Omega)\right)$.

Using the estimates in Theorem 3.4, for $\widehat{v} = (u_n,b_n)$, (u_n,b_n) remains bounded in $C([0,T];\mathbf{H}^r \times \mathbf{H}^r_{\mathbf{n}}) \cap L^2(0,T;\mathbf{H}^r(\Omega)\times \mathbf{H}^r_{\mathbf{n}}(\Omega))$ whenever $u_0,b_0,U_0,B_0 \in L^2\left(0,T;\mathbf{L}^2(\Omega)\right)$.

Note that (u,b) remains bounded in $L^2(0,T;\underline{\mathbf{H}}^r(\Omega))$ see [3], then thanks to (1.2)-(1.3) and the Schwarz inequality, we obtain:

$$\begin{aligned}
\left\|\frac{\partial u}{\partial t}\right\| &\leq \|u\|\cdot\|\nabla u\| + \|b\|\cdot\|\nabla b\| + \|\nabla p\| + \nu\|u\|_{\mathbf{H}^r} + \|f\|,\\
\left\|\frac{\partial b}{\partial t}\right\| &\leq \|u\|\cdot\|\nabla b\| + \|b\|\cdot\|\nabla u\| + \theta\|b\|_{\mathbf{H}^r}.
\end{aligned}$$

Also

$$\begin{aligned}
\|u\|_{\mathbf{H}^r} &\leq \sup_{[0,T]} \|u\|_{\mathbf{H}^r} \leq \|u\|_{L^\infty(0,T,\mathbf{L}^2(\Omega)},\\
\|b\|_{\mathbf{H}^r} &\leq \sup_{[0,T]} \|b\|_{\mathbf{H}^r} \leq \|b\|_{L^\infty(0,T,\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Hence $\frac{\partial u}{\partial t}$ and $\frac{\partial b}{\partial t}$ are bounded in $L^2(0,T,\mathbf{L}^2(\Omega)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_n,b_n) \ \text{ is bounded in } \ L^\infty\left(0,T;\mathbf{L}^2(\Omega)\right),$$

and consequently in $L^2\left(0,T;\mathbf{L}^2(\Omega)\right)$, since $L^\infty(0,T;\underline{\mathbf{H}}^r(\Omega)) \subset L^2(0,T;\underline{\mathbf{H}}^r(\Omega))$. Therefore we can find a pair $(\widehat{u},\widehat{b},\widehat{f})$ and a subsequence, still denoted by (u_n,b_n,f_n), such that

$$\begin{cases}
f_n \longrightarrow \widehat{f} \text{ in } L^2\left(0,T;\mathbf{L}^2(\Omega)\right) \text{ weakly },\\
u_n \longrightarrow \widehat{u} \ \text{ in } L^2(0,T;\mathbf{V}^r) \text{ weakly and in } L^\infty(0,T;(\mathbf{V}^r)^*)^* - \text{weakly},\\
b_n \longrightarrow \widehat{b} \ \text{ in } L^2(0,T;\mathbf{V}^r_{\mathbf{n}}) \text{ weakly and in } L^\infty(0,T;(\mathbf{V}^r_{\mathbf{n}})^*)^* - \text{weakly},\\
\partial_t u_n \longrightarrow \partial_t\widehat{u} \text{ in } L^2\left(0,T;\mathbf{L}^2(\Omega)\right) \text{ weakly},\\
\partial_t b_n \longrightarrow \partial_t\widehat{b} \text{ in } L^2\left(0,T;\mathbf{L}^2(\Omega)\right) \text{ weakly}.
\end{cases}$$

Using the fact that, from compactness of the inclusion of $L^2(0,T;\mathbf{V}^r)$ in $L^2(0,T;\mathbf{W}^r)$, $u_n \longrightarrow \widehat{u}$ and $b_n \longrightarrow \widehat{b}$ strongly in $L^2(0,T;\mathbf{W}^r)$, we set

$$\begin{aligned}
&(\partial_t u_n,\varphi) \rightharpoonup (\partial_t\widehat{u},\varphi), \qquad &&(\partial_t b_n,\varphi) \rightharpoonup \left(\partial_t\widehat{b},\varphi\right), \qquad &&a_r^\nu(u_n,\varphi) \rightharpoonup a_r^\nu(\widehat{u},\varphi),\\
&a_r^\theta(b_n,\varphi) \rightharpoonup a_r^\theta\left(\widehat{b},\varphi\right), &&c(u_n,u_n,\varphi) \rightharpoonup c(\widehat{u},\widehat{u},\varphi), &&c(b_n,b_n,\varphi) \rightharpoonup c\left(\widehat{b},\widehat{b},\varphi\right),
\end{aligned}$$

so that $(\widehat{u},\widehat{b})$ is a solution of (2.4)-(2.6) with $f=\widehat{f}$.

Moreover, using lower semicontinuity yields that:

$$\int_0^T \|\widehat{u}(t)-U(t)\|^2 \mathrm{d}t \le \lim_{n\to\infty}\int_0^T \|u_n(t)-U(t)\|^2 \mathrm{d}t,$$
$$\int_0^T \left\|\widehat{b}(t)-B(t)\right\|^2 \mathrm{d}t \le \lim_{n\to\infty}\int_0^T \|b_n(t)-B(t)\|^2 \mathrm{d}t,$$
$$\int_0^T \left\|\widehat{f}(t)-F(t)\right\|^2 \mathrm{d}t \le \lim_{n\to\infty}\int_0^T \|f_n(t)-F(t)\|^2 \mathrm{d}t,$$

which implies

$$J_T(\widehat{u},\widehat{b},\widehat{f}) \le J_T(u_n,b_n,f_n).$$

and therefore the proof is completed. □

4.1.2 Optimality system

For numerical needs and thanks to [10], let us give here some results of an optimality system, consisting in the forward GMHD equations given by (2.4)-(2.5) with initial data

$$\lim_{t\to 0^+} (u(t),b(t)) = (u_0,b_0),$$

the backward in the time adjoint system

$$\begin{aligned}&-\langle \partial_t \xi(t),\varphi\rangle + a_r^\nu(\xi(t),\varphi) + \mathbf{c}(u(t),\varphi,\xi(t)) + \mathbf{c}(\varphi,u(t),\xi(t)) - \mathbf{c}(b(t),\varphi,\eta(t))\\ &\qquad -\mathbf{c}(\varphi,b(t),\eta(t)) = \langle \alpha(u-U)(t),\varphi\rangle, \qquad \forall \varphi\in \mathbf{V}^r \text{ a.e. } t\in(0,\infty),\end{aligned} \tag{4.2}$$
$$\begin{aligned}&-\langle \partial_t \eta(t),\psi\rangle + a_r^\theta(\eta(t),\psi) + \mathbf{c}(u(t),\psi,\eta(t)) + \mathbf{c}(\psi,u(t),\eta(t))) - \mathbf{c}(b(t),\psi,\xi(t))\\ &\qquad -\mathbf{c}(\psi,b(t),\xi(t)) = \langle \alpha(b-B)(t),\psi\rangle, \qquad \forall \psi\in \mathbf{V}_{\mathbf{n}}^r \text{ a.e. } t\in(0,\infty),\end{aligned} \tag{4.3}$$

with final condition

$$\lim_{t\to T^-} (\xi(t),\eta(t)) = (0,0).$$

The above system of equations is a weak formulation of the system:

$$\begin{aligned}&\frac{\partial u}{\partial t} + \nu(-\triangle)^r u + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla p = f,\\ &\frac{\partial b}{\partial t} + \theta(-\triangle)^r b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0,\\ &\nabla\cdot u = 0, \quad \nabla\cdot b = 0,\\ &-\frac{\partial \xi}{\partial t} + \nu(-\triangle)^r \xi - (u\cdot\nabla)\xi + (\nabla u)^T\xi + (b\cdot\nabla)\eta - (\nabla b)^T\eta + \nabla\pi = \alpha(u-U),\\ &-\frac{\partial \eta}{\partial t} + \theta(-\triangle)^r \eta - (u\cdot\nabla)\eta + (\nabla u)^T\eta + (b\cdot\nabla)\xi - (\nabla b)^T\xi = \alpha(b-B),\\ &\nabla\cdot\xi = 0, \quad \nabla\cdot\eta = 0,\\ &f = F - \beta^{-1}\xi,\end{aligned}$$

in Q_T with the same initial, final and boundary conditions. We see that the optimal solution $(\hat{u},\hat{b},\hat{f})$ along with the Lagrange multiplier $(\hat{\xi},\hat{\mu})$.

4.2 The case of the infinite time interval

We prove now the existence of an optimal solution for (2.7) on the infinite time interval $(0,\infty)$. We will make use of the existence results on finite time intervals.

Theorem 4.2. *There exists an optimal solution* $(\widehat{u},\widehat{b},\widehat{f})\in\mathcal{U}_{ad}(T)$ *for* (2.7) *with* $T=\infty$.

Proof. For each $T\in(0,\infty)$, we may use Theorem 4.1 to choose a $(u_T,b_T,f_T)\in\mathcal{U}_{ad}(T)$ which solves (2.7) and satisfies

$$J_T(u_T,b_T,f_T)=\inf_{(\omega,\psi,h)\in\mathcal{U}_{ad}(T)}J_T(\omega,\psi,h), \tag{4.4}$$

$$\langle\partial_t u_T(t),\varphi\rangle+a_r^\nu(u_T(t),\varphi)+\mathbf{c}(u_T(t),u_T(t),\varphi)-\mathbf{c}(b_T(t),b_T(t),\varphi)=\langle f(t),\varphi\rangle,$$
$$\forall\varphi\in\mathbf{V}^r \text{ a.e. } t\in(0,\infty), \tag{4.5}$$

$$\langle\partial_t b_T(t),\psi\rangle+a_r^\theta(b_T(t),\psi)+\mathbf{c}(u_T(t),b_T(t),\psi)-\mathbf{c}(b_T(t),u_T(t),\psi)=0,$$
$$\forall\psi\in\mathbf{V}_{\mathbf{n}}^r \text{ a.e. } t\in(0,\infty) \tag{4.6}$$

and

$$\lim_{t\to0^+}(u_T(t),b_T(t))=(u_0,b_0),\qquad \text{in } \mathbf{W}^r\times\mathbf{W}^r. \tag{4.7}$$

The fact that $\mathcal{U}_{ad}(\infty)|_{(0,\infty)}\subset\mathcal{U}_{ad}(T)$ for each finite T yields that

$$J_T(u_T,b_T,f_T)\le J_T(w,\psi,h)<J_\infty(w,\psi,h),\qquad \text{for all } (\omega,\psi,h)\in\mathcal{U}_{ad}(\infty).$$

Using the bound of $J_\infty(\widetilde{u},\widetilde{b},\widetilde{f})$ for a quasi-optimizer $(\widetilde{u},\widetilde{b},\widetilde{f})$ constructed in Section 3-1, we have $(\widetilde{u},\widetilde{b},\widetilde{f})\in\mathcal{U}_{ad}(\infty)$ and then

$$J_T(u_T,b_T,f_T)\le\inf_{(\omega,\psi,h)\in\mathcal{U}_{ad}(\infty)}J_\infty(\omega,\psi,h)<\infty. \tag{4.8}$$

For each integer $l>0$, we denote by (u_l,b_l,f_l) a solution of (4.4)-(4.7) for $T=l$. We set $(v_l,w_l,g_l)=(u_l-U,b_l-B,f_l-F)$. Then, (v_l,w_l,g_l) satisfies (2.16)-(2.18) with $T=l$. Using (4.8) and the standard estimates for the MHD and GMHD equations on finite time interval, we obtain that $\|g_l\|_{L^2(0,l;\mathbf{L}^2(\Omega))}$, $\|(u_l,b_l)\|_{\underline{\mathcal{H}}^1(Q_l)}$ and $\|(u_l,b_l)\|_{L^\infty(0,l;\mathbf{W}^r\times\mathbf{W}^r)}$ are uniformly bounded for all l. Hence, by induction we may choose successive subsequences of positive integers $\{l_n^{(m)}\}_{n=1}^\infty$ for $m=1,2,\cdots$ such that

$$\left\{l_n^{(1)}\right\}_{n=1}^\infty\supset\left\{l_n^{(2)}\right\}_{n=1}^\infty\supset\left\{l_n^{(3)}\right\}_{n=1}^\infty\supset\cdots$$

and

$$\begin{aligned}
&(v_{l_n^{(m)}}, w_{l_n^{(m)}}) \rightharpoonup (v^{(m)}, w^{(m)}), \text{ in } \underline{\mathcal{H}}^1(Q_m) \text{ as } n\to\infty,\\
&(v_{l_n^{(m)}}, w_{l_n^{(m)}}) \overset{*}{\rightharpoonup} (v^{(m)}, w^{(m)}), \text{ in } L^\infty(0,m;\mathbf{W}^r\times\mathbf{W}^r) \text{ as } n\to\infty,\\
&g_{l_n^{(m)}} \rightharpoonup g^{(m)}, \text{ in } L^2\big(0,m;\mathbf{L}^2(\Omega)\big) \text{ as } n\to\infty,
\end{aligned}$$

for some $(v^{(m)},w^{(m)})\in\underline{\mathcal{H}}^1(Q_m)$ and $g^{(m)}\in L^2\big(0,m;\mathbf{L}^2(\Omega)\big)$. Hence, by extracting the diagonal subsequence, we have that for each m',

$$(v_{l_m^{(m)}}, w_{l_m^{(m)}}) \rightharpoonup (v^{(m')}, w^{(m')}), \text{ in } \underline{\mathcal{H}}^1(Q_{m'}) \text{ as } m\to\infty, \tag{4.9}$$

$$(v_{l_m^{(m)}}, w_{l_m^{(m)}}) \overset{*}{\rightharpoonup} (v^{(m')}, w^{(m')}), \text{ in } L^\infty\big(0,m';\mathbf{W}^r\times\mathbf{W}^r\big) \text{ as } m\to\infty, \tag{4.10}$$

$$g_{l_m^{(m)}} \rightharpoonup g^{(m)}, \text{ in } L^2\big(0,m';\mathbf{L}^2(\Omega)\big) \text{ as } m\to\infty. \tag{4.11}$$

For each integer $m'>0$, (4.9)-(4.11) and standard techniques for the MHD equations, compactness results and density arguments (see [7, 11]) allows us to pass to the limit as $m\to\infty$ in the equation

$$\begin{aligned}
&\int_0^{m'}\Big\{\Big\langle\partial_t v_{l_n^{(m)}}(t),\varphi\Big\rangle\chi(t)+a_r^\nu\Big(v_{l_n^{(m)}}(t),\varphi\Big)\chi(t)\\
&\qquad+\mathbf{c}\Big(v_{l_n^{(m)}}(t),v_{l_n^{(m)}}(t),\varphi\Big)\chi(t)+\mathbf{c}\Big(v_{k_n^{(m)}}(t),U(t),\varphi\Big)\chi(t)\\
&\qquad+\mathbf{c}\Big(U(t),v_{{}_n^{(m)}}(t),\varphi\Big)\chi(t)-\mathbf{c}\Big(w_{l_n^{(m)}}(t),w_{l_n^{(m)}}(t),\varphi\Big)\chi(t)\\
&\qquad-\mathbf{c}\Big(w_{l_n^{(m)}}(t),B(t),\varphi\Big)\chi(t)-\mathbf{c}\Big(B(t),w_{l_n^{(m)}}(t),\varphi\Big)\chi(t)\Big\}\mathrm{d}t\\
=&\int_0^{m'}\Big\langle g_{l_n^{(m)}}(t),\varphi\Big\rangle\chi(t)\mathrm{d}t,\quad \forall\varphi\in\mathbf{V}^r,\chi\in\mathcal{C}\left[0,m'\right] \text{ with } \boldsymbol{\emptyset}(\mathbf{m}')=0,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{m'}\Big\{\Big\langle\partial_t w_{l_n^{(m)}}(t),\psi\Big\rangle\chi(t)+a_r^\theta\Big(w_{l_n^{(m)}}(t),\psi\Big)\chi(t)\\
&\qquad+\mathbf{c}\Big(v_{l_n^{(m)}}(t),w_{l_n^{(m)}}(t),\psi\Big)\chi(t)+\mathbf{c}\Big(v_{l_n^{(m)}}(t),B(t),\psi\Big)\chi(t)\\
&\qquad+\mathbf{c}\Big(U(t),w_{l_n^{(m)}}(t),\psi\Big)\chi(t)-\mathbf{c}\Big(w_{l_n^{(m)}}(t),v_{l_n^{(m)}}(t),\psi\Big)\chi(t)\\
&\qquad-\mathbf{c}\Big(w_{l_n^{(m)}}(t),U(t),\psi\Big)\chi(t)-\mathbf{c}\Big(B(t),v_{l_n^{(m)}}(t),\psi\Big)\chi(t)\Big\}\mathrm{d}t\\
=&0,\quad \forall\psi\in\mathbf{V}_\mathbf{n}^r,\ \chi\in\mathcal{C}\left[0,m'\right] \text{ with } \boldsymbol{\emptyset}(m')=\mathbf{0},
\end{aligned}$$

to obtain

$$\begin{aligned}&\int_0^{m'}\Big\{\Big\langle\partial_t v^{(m')}(t),\varphi\Big\rangle\chi(t)+a_r^{\nu}\Big(v^{(m')}(t),\varphi\Big)\chi(t)\\&\quad+\mathbf{c}\Big(v^{(m')}(t),v^{(m')}(t),\varphi\Big)\chi(t)+\mathbf{c}\Big(v^{(m')}(t),U(t),\varphi\Big)\chi(t)\\&\quad+\mathbf{c}\Big(U(t),v^{(m')}(t),\varphi\Big)\chi(t)-\mathbf{c}\Big(w^{(m')}(t),w^{(m')}(t),\varphi\Big)\chi(t)\\&\quad-\mathbf{c}\Big(w^{(m')}(t),B(t),\varphi\Big)\chi(t)-\mathbf{c}\Big(B(t),w^{(m')}(t),\varphi\Big)\chi(t)\Big\}\mathrm{d}t\\&=\int_0^{m'}\Big\langle g^{(m')}(t),\varphi\Big\rangle\chi(t)\mathrm{d}t,\quad\forall\varphi\in\mathbf{V}^r,\ \chi\in\mathcal{C}\left[0,m'\right]\text{ with }\mathbf{Ø}(m')=0,\end{aligned}$$

and

$$\begin{aligned}&\int_0^{m'}\Big\{\Big\langle\partial_t w^{(m')}(t),\psi\Big\rangle\chi(t)+a_r^{\theta}\Big(w^{(m')}(t),\psi\Big)\chi(t)\\&\quad+\mathbf{c}\Big(v^{(m')}(t),w^{(m')}(t),\psi\Big)\chi(t)+\mathbf{c}\Big(v^{(m')}(t),B(t),\psi\Big)\chi(t)\\&\quad+\mathbf{c}\Big(U(t),w^{(m')}(t),\psi\Big)\chi(t)-\mathbf{c}\Big(w^{(m')}(t),v^{(m')}(t),\psi\Big)\chi(t)\\&\quad-\mathbf{c}\Big(w^{(m')}(t),U(t),\psi\Big)\chi(t)-\mathbf{c}\Big(B(t),v^{(m')}(t),\psi\Big)\chi(t)\Big\}\mathrm{d}t\\&=0,\quad\forall\psi\in\mathbf{V}^r_{\mathbf{n}},\ \chi\in\mathcal{C}\left[0,m'\right]\text{ with }\mathbf{Ø}(m')=\mathbf{0},\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\Big\langle\partial_t u^{(m')}(t),\varphi\Big\rangle+a_r^{\nu}\Big(u^{(m')}(t),\varphi\Big)+\mathbf{c}\Big(u^{(m')}(t),u^{(m')}(t),\varphi\Big)\\&\quad-\mathbf{c}\Big(b^{(m')}(t),b^{(m')}(t),\varphi\Big)=\Big\langle f^{(m')}(t),\varphi\Big\rangle,\ \forall\varphi\in\mathbf{V}^r\text{ a.e. }t\in(0,\infty),\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}&\Big\langle\partial_t b^{(m')}(t),\psi\Big\rangle+a_r^{\theta}\Big(b^{(m')}(t),\psi\Big)+\mathbf{c}\Big(u^{(m')}(t),b^{(m')}(t),\psi\Big)\\&\quad-\mathbf{c}\Big(b^{(m')}(t),u^{(m')}(t),\psi\Big)=0,\quad\forall\psi\in\mathbf{V}^r_{\mathbf{n}}\text{ a.e. }t\in(0,\infty),\end{aligned}\tag{4.13}$$

where

$$\Big(u^{(m')},b^{(m')}\Big)=(v^{(m')},w^{(m')})+(U,B).$$

By uniqueness of weak limits, we have that

$$(v^{(m_1)},w^{(m_1)})\Big|_{(0,m_2)}=(v^{(m_2)},w^{(m_2)}),$$

and

$$g^{(m_1)}\Big|_{(0,m_2)}=g^{(m_2)},$$

for all m_1,m_2 with $m_1<m_2$. Thus, the functions $(\widehat{v}(t),\widehat{w}(t)):=(v^{(m)}(t),w^{(m)}(t))$ if $t\le m$ and $\widehat{g}(t):=g^{(m)}(t)$ if $t\le m$ are well defined on $(0,\infty)$ and furthermore, $(\widehat{v},\widehat{w})\in\underline{\mathcal{H}}^1_{Loc}(Q)$

and $\widehat{g}\in L^2\left(0,\infty;\mathbf{L}^2(\Omega)\right)$. Upon setting $(\widehat{u},\widehat{b})=(\widehat{v},\widehat{w})+(U,B)$ and $\widehat{f}=\widehat{g}+F$ and noting that m' is arbitrary in (4.12)-(4.13), we are easily led to

$$\begin{aligned}&\langle\partial_t\widehat{u}(t),\varphi\rangle+a_r^\nu(\widehat{u}(t),\varphi)+\mathbf{c}(\widehat{u}(t),\widehat{u}(t),\varphi)-\mathbf{c}\left(\widehat{b}(t),\widehat{b}(t),\varphi\right)\\&=\left\langle f^{(m')}(t),\varphi\right\rangle,\quad\forall\varphi\in\mathbf{V}^r\text{ a.e. }t\in(0,\infty),\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}&\left\langle\partial_t\widehat{b}(t),\psi\right\rangle+a_r^\theta\left(\widehat{b}(t),\psi\right)+\mathbf{c}\left(\widehat{u}(t),\widehat{b}(t),\psi\right)\\&\quad-\mathbf{c}\left(\widehat{b}(t),\widehat{u}(t),\psi\right)=0,\quad\forall\psi\in\mathbf{V}^r_{\mathbf{n}}\text{ a.e. }t\in(0,\infty).\end{aligned}\tag{4.15}$$

Now we examine the initial condition for $(\widehat{u},\widehat{b})$. The continuous embedding

$$\underline{\mathcal{H}}^1(Q_T)\hookrightarrow\mathcal{C}([0,T];\mathbf{W}^r\times\mathbf{W}^r)$$

implies that $(\hat{u}(0),\hat{b}(0))$ is well defined in $\mathbf{W}^r\times L^2(\Omega)$. Let χ be a continuously differentiable function in $[0,\infty)$ with a bounded support, integrating by parts, using the fact that $(u_{l_m^{(m)}}(0),b_{l_m^{(m)}}(0))=(u_0,b_0)$ and then passing to the limit, we obtain

$$\begin{aligned}&\int_0^\infty\Big\{-\left\langle u_{l_n^{(m)}}(t),\varphi\right\rangle\chi'(t)+a_r^\nu\left(u_{l_n^{(m)}}(t),\varphi\right)\chi(t)+\mathbf{c}\left(u_{l_n^{(m)}}(t),\widehat{u}(t),\varphi\right)\chi(t)\\&\qquad-\mathbf{c}\left(b_{l_n^{(m)}}(t),b_{l_n^{(m)}}(t),\varphi\right)\chi(t)\Big\}\mathrm{d}t\\&=\int_0^\infty\left\langle f^{(m')}(t),\varphi\right\rangle\chi(t)\mathrm{d}t+\langle u_0,\varphi\rangle\chi(0),\qquad\forall\varphi\in\mathbf{V}^r,\end{aligned}\tag{4.16}$$

and

$$\begin{aligned}&\int_0^\infty\Big\{-\left\langle b_{l_n^{(m)}}(t),\psi\right\rangle\chi'(t)+a_r^\theta\left(b_{l_n^{(m)}}(t),\psi\right)\chi(t)+\mathbf{c}\left(u_{l_n^{(m)}}(t),b_{l_n^{(m)}}(t),\psi\right)\chi(t)\\&\qquad-\mathbf{c}\left(b_{l_n^{(m)}}(t),u_{l_n^{(m)}}(t),\psi\right)\chi(t)\Big\}\mathrm{d}t=\langle b_0,\psi\rangle\chi(0),\qquad\forall\psi\in\mathbf{V}^r_{\mathbf{n}}.\end{aligned}\tag{4.17}$$

Thus, by passing to the limit in the last equations we obtain

$$\begin{aligned}&\int_0^\infty\Big\{-\langle\widehat{u}(t),\varphi\rangle\chi'(t)+a_r^\nu(\widehat{u}(t),\varphi)\chi(t)+\mathbf{c}(\widehat{u}(t),\widehat{u}(t),\varphi)\chi(t)\\&\quad-\mathbf{c}\left(\widehat{b}(t),\widehat{b}(t),\varphi\right)\chi(t)\Big\}\mathrm{d}t=\int_0^\infty\left\langle f^{(m')}(t),\varphi\right\rangle\chi(t)\mathrm{d}t+\langle u_0,\varphi\rangle\chi(0),\quad\forall\varphi\in\mathbf{V}^r,\end{aligned}\tag{4.18}$$

and

$$\begin{aligned}&\int_0^\infty\Big\{-\left\langle\widehat{b}(t),\psi\right\rangle\chi'(t)+a_r^\theta\left(\widehat{b}(t),\psi\right)\chi(t)+\mathbf{c}\left(\widehat{u}(t),\widehat{b}(t),\psi\right)\chi(t)\\&\quad-\mathbf{c}\left(\widehat{b}(t),\widehat{u}(t),\psi\right)\chi(t)\Big\}\mathrm{d}t=\langle b_0,\psi\rangle\chi(0),\quad\forall\psi\in\mathbf{V}^r_{\mathbf{n}}.\end{aligned}\tag{4.19}$$

On the other and, by multiplying (4.14) and (4.15) by $\chi(t)$ and integrating by parts we obtain

$$\int_0^\infty \Big\{ -\langle \widehat{u}(t),\varphi\rangle \chi'(t) + a_r^\nu(\widehat{u}(t),\varphi)\chi(t) + \mathbf{c}(\widehat{u}(t),\widehat{u}(t),\varphi)\chi(t) \\ -\mathbf{c}\Big(\widehat{b}(t),\widehat{b}(t),\varphi\Big)\chi(t)\Big\}\mathrm{d}t = \int_0^\infty \Big\langle f^{(m')}(t),\varphi\Big\rangle \chi(t)\mathrm{d}t + \langle \hat{u}_0,\varphi\rangle\chi(0), \quad \forall \varphi \in \mathbf{V}^r, \tag{4.20}$$

and

$$\int_0^\infty \Big\{ -\Big\langle \widehat{b}(t),\psi\Big\rangle \chi'(t) + a_r^\theta\Big(\widehat{b}(t),\psi\Big)\chi(t) + \mathbf{c}\Big(\widehat{u}(t),\widehat{b}(t),\psi\Big)\chi(t) \\ -\mathbf{c}\Big(\widehat{b}(t),\widehat{u}(t),\psi\Big)\chi(t)\Big\}\mathrm{d}t = \langle \hat{b}_0,\psi\rangle\chi(0), \quad \forall \psi \in \mathbf{V}_{\mathbf{n}}^r. \tag{4.21}$$

A comparison of (4.18)-(4.19) and (4.20)-(4.21) yields $(\hat{u}_0,\hat{b}_0)=(u_0,b_0)$ in $\mathbf{W}^r\times\mathbf{W}^r$. Finally, using the lower semicontinuity of the functional $J_T(.,.,.)$ and the fact that $(\widehat{v},\widehat{w})=(\widehat{u},\widehat{b})-(U,B)\in L^2(0,\infty;\mathbf{V}^r\times{\mathbf{V}_{\mathbf{n}}}^r)$ and $\widehat{g}=\widehat{f}-F\in L^2\left(0,\infty;\mathbf{L}^2(\Omega)\right)$, we obtain

$$J_{l_m^{(m)}}(\widehat{u},\widehat{b},\widehat{f}) \le \liminf_{m\to\infty} J_{l_m^{(m)}}(u_{l_m^{(m)}},b_{l_m^{(m)}},f_{l_m^{(m)}}) \le J_\infty(\omega,\psi,h), \quad \forall(\omega,\psi,h)\in\mathcal{U}_{ad}(\infty),$$

so that by letting $m\to\infty$,

$$J_\infty(\widehat{u},\widehat{b},\widehat{f}) \le J_\infty(\omega,\psi,h), \quad \forall(\omega,\psi,h)\in\mathcal{U}_{ad}(\infty).$$

Hence we have proved that $(\widehat{u},\widehat{b},\widehat{f})$ is the desired optimizer for (2.7) with $T=\infty$. □

5 Dynamics of optimal control solutions on the infinite time interval

For many feedback control models, the controlled flow exponentially decays to the desired flow. For our optimal control system, Theorems 3.3 and 3.4 gave some preliminary results as $\|(u,b)(t)-(U,B)(t)\|$ stays bounded. We will prove much stronger results in this Section: $\|(u,b)(t)-(U,B)(t)\|$ approach zero as t goes to ∞. We point out that these last results are not unique to the solutions of the optimal control system; these results can be proved under weaker conditions.

Lemma 5.1. *Let $T\in(0,\infty)$. Assume that $(u,b,f)\in\mathcal{U}_{ad}(T)$ and $\lambda_1>1$. If*

$$\|(u,b)(t)-(U,B)(t)\|>0$$

for all $t\in(t_1,t_2)\subset[0,T]$, then

$$\|(u,b)(t_2)-(U,B)(t_2)\| \le \|(u,b)(t_1)-(U,B)(t_1)\| + K_4\sqrt{t_2-t_1}\,(J_T(u,b,f))^{1/2},$$

with

$$K_4=\left(\frac{1}{\alpha}\left(\frac{2}{\nu}+\frac{2}{\theta}\right)^2\left(|||\nabla U|||^2+|||\nabla B|||^2\right)^2+\frac{1}{\beta}\right)^{1/2}.$$

If in addition, the assumptions (A1) *and* (A2) *hold and* $J_T(u,b,f)\le J_T(\widetilde{u},\widetilde{b},\widetilde{f})$*, where* $(\widetilde{u},\widetilde{b},\widetilde{f})$ *is defined in Theorem* 3.1*, then*

$$\begin{aligned}\|(u,b)(t_2)-(U,B)(t_2)\|\le&\|(u,b)(t_1)-(U,B)(t_1)\|\\&+K_4\sqrt{t_2-t_1}\|(u_0,b_0)-(U_0,B_0)\|\sqrt{\frac{\alpha}{2\varepsilon}}.\end{aligned}\tag{5.1}$$

Proof. By setting $\varphi=v(t)$ and $\psi=w(t)$ in (2.16)-(2.17) we obtain, for $r=2k\,(k\in\mathbb{N}^*)$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2+\nu\Big\|\bigwedge^k v(t)\Big\|^2=&\mathbf{c}(w(t),w(t),v(t))-\mathbf{c}(v(t),U(t),v(t))\\&+\mathbf{c}(w(t),B(t),v(t))+\mathbf{c}(B(t),w(t),v(t))+\langle g(t),v(t)\rangle,\end{aligned}\tag{5.2}$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\theta\Big\|\bigwedge^k w(t)\Big\|^2=&\mathbf{c}(w(t),v(t),w(t))-\mathbf{c}(v(t),B(t),w(t))\\&+\mathbf{c}(w(t),U(t),w(t))+\mathbf{c}(B(t),v(t),w(t)),\end{aligned}\tag{5.3}$$

summing, thanks to (2.11) and using Sobolev imbedding and interpolation results

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|^2+\|w(t)\|^2\right)+\nu\lambda_1^{2(2k-1)}\|\nabla v(t)\|^2+\theta\lambda_1^{2(2k-1)}\|\nabla w(t)\|^2\\ \le&\sqrt{2}\|\nabla v(t)\|\cdot|||\nabla U(t)|||\cdot\|v(t)\|+\sqrt{2}\|\nabla w(t)\|\cdot|||\nabla B(t)|||\cdot\|v(t)\|\\&+\sqrt{2}\|\nabla w(t)\|\cdot|||\nabla U(t)|||\cdot\|w(t)\|+\sqrt{2}\|\nabla v(t)\|\cdot|||\nabla B(t)|||\cdot\|w(t)\|+\|g(t)\|\cdot\|v(t)\|,\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(v(t),w(t))\|^2+\left(\lambda_1^{2(2k-1)}-1\right)\left(\nu\|\nabla v(t)\|^2+\theta\|\nabla w(t)\|^2\right)\\ \le&\|g(t)\|\cdot\|v(t)\|+\left(\frac{1}{\nu}|||\nabla U(t)|||^2+\frac{1}{\theta}|||\nabla B(t)|||^2\right)\cdot\|v(t)\|^2\\&+\left(\frac{1}{\theta}|||\nabla U(t)|||^2+\frac{1}{\nu}|||\nabla B(t)|||^2\right)\cdot\|w(t)\|^2,\end{aligned}$$

using the Poincarré inequality

$$\begin{aligned}&\|(v(t),w(t))\|\frac{\mathrm{d}}{\mathrm{d}t}\|(v(t),w(t))\|+\lambda_1\left(\lambda_1^{2(2k-1)}-1\right)\left(\nu\|v(t)\|^2+\theta\|w(t)\|^2\right)\\ \le&\left(\frac{1}{\nu}+\frac{1}{\theta}\right)\left(|||\nabla U(t)|||^2+|||\nabla B(t)|||^2\right)\cdot\|v(t),w(t)\|^2+\|g(t)\|\cdot\|v(t),w(t)\|.\end{aligned}\tag{5.4}$$

For $r=2k+1\,(k\in\mathbb{N})$, using the same method, we obtain

$$\begin{aligned}&\|(v(t),w(t))\|\frac{\mathrm{d}}{\mathrm{d}t}\|(v(t),w(t))\|+\lambda_1\left(\lambda_1^{4k}-1\right)\left(\nu\|v(t)\|^2+\theta\|w(t)\|^2\right)\\ \leq&\left(\frac{1}{\nu}+\frac{1}{\theta}\right)\left(|\|\nabla U(t)\||^2+|\|\nabla B(t)\||^2\right)\cdot\|v(t),w(t)\|^2+\|g(t)\|\cdot\|v(t),w(t)\|.\end{aligned}\tag{5.5}$$

The inequalities (5.4) and (5.5) become

$$\begin{aligned}&\|(v(t),w(t))\|\frac{\mathrm{d}}{\mathrm{d}t}\|(v(t),w(t))\|+\varepsilon_1\|(v(t),w(t))\|^2\\ \leq&C_0\cdot\|(v(t),w(t))\|^2+\|g(t)\|\cdot\|(v(t),w(t))\|,\end{aligned}$$

where

$$\varepsilon_1=\min\left\{\nu\lambda_1\left(\lambda_1^{2(2k-1)}-1\right),\theta\lambda_1\left(\lambda_1^{2(2k-1)}-1\right),\nu\lambda_1\left(\lambda_1^{4k}-1\right),\theta\lambda_1\left(\lambda_1^{4k}-1\right)\right\},$$

and

$$C_0=\left(\frac{1}{\nu}+\frac{1}{\theta}\right)\left(|\|\nabla U\||^2+|\|\nabla B\||^2\right).$$

If $\|v(t),w(t)\|>0$ for all $t\in(t_1,t_2)$, then we may divide this inequality by $\|v(t),w(t)\|$ to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|(v,w)(t)\|+\varepsilon_1\|(v,w)(t)\|\leq C_0\|(v,w)(t)\|+\|g(t)\|\\ \leq&\left(\frac{1}{\alpha}C_0^2+\frac{1}{\beta}\right)^{1/2}\left(\alpha\|(v,w)(t)\|^2+\beta\|g(t)\|^2\right)^{1/2},\end{aligned}$$

for all $t\in(t_1,t_2)$. Multiplying the last inequality by $e^{\varepsilon_1 t}$ and then integrating over (t_1,t_2), we are led to

$$\begin{aligned}&\|(v,w)(t_2)\|e^{\varepsilon_1 t_2}\\ \leq&\|(v,w)(t_1)\|e^{\varepsilon_1 t_1}+\left(\frac{1}{\alpha}C_0^2+\frac{1}{\beta}\right)^{1/2}\int_{t_1}^{t_2}\left(\alpha\|(v,w)(t)\|^2+\beta\|g(t)\|^2\right)^{1/2}e^{\varepsilon_1 t}\mathrm{d}t,\end{aligned}$$

we have

$$\begin{aligned}&\|(v,w)(t_2)\|\\ \leq&\|(v,w)(t_1)\|e^{-\varepsilon_1(t_2-t_1)}\\ &+\left(\frac{1}{\alpha}C_0^2+\frac{1}{\beta}\right)^{1/2}\left(\int_{t_1}^{t_2}(\alpha\|(v,w)(t)\|^2+\beta\|g(t)\|^2)\mathrm{d}t\right)^{1/2}\left(\int_{t_1}^{t_2}e^{-2\varepsilon_1(t_2-t)}\mathrm{d}t\right)^{1/2},\end{aligned}$$

with $e^{-\varepsilon_1(t_2-t_1)}<1$,

$$\begin{aligned}\|(v,w)(t_2)\| &\le \|(v,w)(t_1)\| + \left(\frac{1}{\alpha}C_0^2+\frac{1}{\beta}\right)^{1/2}(J_T(u,b,f))^{1/2}.\left(\int_{t_1}^{t_2} e^{-2\varepsilon_1(t_2-t)}\mathrm{d}t\right)^{1/2}\\ &\le \|(v,w)(t_1)\| + \left(\frac{1}{\alpha}C_0^2+\frac{1}{\beta}\right)^{1/2}(J_T(u,b,f))^{1/2}\left(\frac{1-e^{-2\varepsilon_1(t_2-t_1)}}{2\varepsilon_1}\right)^{1/2}\\ &\le \|(v,w)(t_1)\| + \sqrt{t_2-t_1}\left(\frac{1}{\alpha}C_0^2+\frac{1}{\beta}\right)^{1/2}(J_T(u,b,f))^{1/2},\end{aligned}$$

where we have used the fact that $1-e^{-y}\le y$ for $y\ge 0$. Hence, we have shown (5.1) and (5.1) simply follows from the bound (3.2) so that applying the mean value theorem to the last factor we have the result. □

We are now prepared to establish the asymptotic decay property of

$$\|(u(t),b(t))-(U,B(t))\|$$

as $t\to\infty$ for any $(u,b,f)\in\mathcal{U}_{ad}(\infty)$.

Theorem 5.1. *Assume that*

$$(u,b,f)\in\mathcal{U}_{ad}(T).$$

Then

$$\lim_{t\to\infty}\|(u(t),b(t))-(U(t),B(t))\|=0. \tag{5.6}$$

Proof. If $J_\infty(u,b,f)=0$, then the theorem is trivial. Thus we assume

$$J_\infty(u,b,f)>0,$$

and proceed to prove (5.6) by contradiction. Assume that (5.6) is false. For given $\epsilon>0$ we set

$$\delta=\frac{\epsilon^2}{4(t_2-t_1)(K_4)^2J_\infty(u,b,f)}>0. \tag{5.7}$$

Then we may choose a sequence $\{t_n\}$ such that $t_n\to\infty$, $t_{n+1}-t_n\ge\delta$ and

$$\|(u(t_n),b(t_n))-(U(t_n),B(t_n))\|\ge\epsilon>0,$$

then we can show that

$$\|(u(t),b(t))-(U(t),B(t))\|>0,\quad \forall t\in(t_n-\delta,t_n). \tag{5.8}$$

Indeed, we set

$$\bar{t}=\sup\{t\in(t_{n-1},t_n):\|-(U(t),B(t))\|=0\}$$

and assume that $t_n - \bar{t} < \delta$, i.e., $\bar{t} \in (t_n - \delta, t_n)$. Then we have

$$\|(u(t),b(t)) - (U(t),B(t))\| > 0, \qquad \text{on } (\bar{t}, t_n),$$

so that by (5.1),

$$\begin{aligned}\|\left(u(\bar{t}),b(\bar{t})\right) - \left(U(\bar{t}),B(\bar{t})\right)\| &\geq \|(u(t_n),b(t_n)) - (U(t_n),B(t_n))\| - K_4\delta^{1/2}(J_T(u,b,f))^{1/2}\\ &\geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2},\end{aligned}$$

which contradicts $\|\left(u(\bar{t}),b(\bar{t})\right) - \left(U(\bar{t}),B(\bar{t})\right)\| = 0$. This proves the assertion (5.8). Now using (5.1) again, we have

$$\begin{aligned}\|(u(t),b(t)) - (U(t),B(t))\| &\geq \|(u(t_n),b(t_n)) - (U(t_n),B(t_n))\| - K_4\delta^{1/2}(J_T(u,b,f))^{1/2}\\ &\geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}, \quad \forall t \in (t_n - \delta, t_n),\end{aligned}$$

and we are led to

$$J_\infty(u,b,f) \geq \frac{\alpha}{2}\sum_{n=2}^{\infty}\int_{t_n-\delta}^{t_n} \|(u(t),b(t)) - (U(t),B(t))\|^2 \geq \frac{\alpha}{2}\sum_{n=2}^{\infty}\frac{\epsilon}{2}\delta = \infty,$$

which contradicts the assumption $J_\infty(u,b,f) < \infty$. Hence, (5.6) is true. □

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