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Global Existence and Uniqueness of Solutions to Evolution p-Laplacian Systems with Nonlinear Sources

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Abstract. This paper presents the global existence and uniqueness of the initial and boundary value problem to a system of evolution p-Laplacian equations coupled with general nonlinear terms. The authors use skills of inequality estimation and the method of regularization to construct a sequence of approximation solutions, hence obtain the global existence of solutions to a regularized system. Then the global existence of solutions to the system of evolution p-Laplacian equations is obtained with the application of a standard limiting process. The uniqueness of the solution is proven when the nonlinear terms are local Lipschitz continuous.

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1 Introduction

In this paper, we study the global existence and uniqueness of solutions to the initial and boundary value problem

$$u_{it}-\mathrm{div}(|\nabla u_i|^{p_i-2}\nabla u_i)=f_i(u_1,\cdots,u_m),\qquad (x,t)\in\Omega\times(0,T), \tag{1.1a}$$

$$u_i(x,0)=u_{i0}(x),\qquad x\in\Omega, \tag{1.1b}$$

$$u_i(x,t)=0,\qquad (x,t)\in\partial\Omega\times(0,T), \tag{1.1c}$$

where $p_i>2$, $i=1,2,\cdots,m$, $T>0$, $\Omega\subset R^n$ is an open connected bounded domain with smooth boundary $\partial\Omega$.

System (1.1a) models such as non-Newtonian fluids [1,2] and nonlinear filtration [3], etc. In the non-Newtonian fluids theory, $(p_1,p_2,\cdots,p_m)$ is a characteristic quantity of the

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fluids. The fluids with $(p_1,p_2,\cdots,p_m)>(2,2,\cdots,2)$ are called dilatant fluids and those with $(p_1,p_2,\cdots,p_m)<(2,2,\cdots,2)$ are called pseudoplastics. If $(p_1,p_2,\cdots,p_m)=(2,2,\cdots,2)$, they are Newtonian fluids.

For $p_i=2$, $i=1,2$, many authors have studied the problem above; most of them studied global existence, uniqueness, boundedness, and blowup behavior of solutions, etc(see [4–10]). Some authors have derived sufficient conditions for the nonexistence of global solutions. Such conditions are usually related to the structure of f_i, $i=1,2$. And some authors have studied the uniqueness of the global solution and blow-up of the positive solution, with nonlinearities in the form of

$$f_1(u_1,u_2)=u_1^\alpha u_2^\beta, f_2(u_1,u_2)=u_1^\gamma u_2^\delta,$$

where $\alpha,\beta,\gamma,\delta$ are nonnegative numbers.

For $p_i>2$, $i=1,2$, in [11], the authors gave local existence and uniqueness theorem of solutions for the initial and boundary value problem on $\Omega\times(0,T_1)$, where $T_1\in(0,T)(T>0)$ could be very small.

It is our goal to prove results of global existence and uniqueness for the degenerate system of m equations. Since the system is coupled with nonlinear terms, in general, the solutions of (1.1a)-(1.1c) will not exist for all time. Inspired by [12], in this paper, we study some special cases by stating constrains to nonlinear functions. The proof consists of two steps. First, we prove that the approximating problem admits a global solution; then we do some uniform estimates for these solutions. We mainly use skills of inequality estimation and the method of regularization to construct a sequence of approximation solutions, hence obtain existence of the solution to a regularized system of equations. By a standard limiting process, we obtain the existence of solutions to the system (1.1a)-(1.1c).

Systems (1.1a) degenerates when $\nabla u_i=0$. In general, there is no classical solution; therefore, we have to study the generalized solutions to the problem (1.1a)-(1.1c). The definition of generalized solutions is as follows:

Definition 1.1. *A nonnegative function $u=(u_1,\cdots,u_m)$ is called a generalized solution to the system* (1.1a)-(1.1c) *in Ω_T, $T>0$, if $u_i\in L^\infty(\Omega_T)\cap L^{p_i}(0,T;W_0^{1,p_i}(\Omega))$, $u_{it}\in L^2(\Omega_T)$, satisfying*

$$\begin{aligned}&\int_\Omega u_i(x,T)\varphi_i(x,T)\mathrm{d}x+\iint_{\Omega_T}|\nabla u_i|^{p_i-2}\nabla u_i\nabla\varphi_i\mathrm{d}x\mathrm{d}t\\ =&\iint_{\Omega_T}(f_i(u)\varphi_i+\varphi_{it}u_i)\mathrm{d}x\mathrm{d}t+\int_\Omega u_{i0}(x)\varphi_i(x,0)\mathrm{d}x,\end{aligned}\tag{1.2}$$

for any $\varphi_i\in C^1(\overline{\Omega}_T)$, s.t. $\varphi_i=0$, for $(x,t)\in\partial\Omega\times(0,T)$; and $u_i(x,t)=0,(x,t)\in\partial\Omega\times(0,T)$, where $i=1,2,\cdots,m$.

2 Main results

In order to study the problem (1.1a)-(1.1c), we make the following assumptions:

(H0) If $u_i \geq 0$, $i=1,2,\cdots,m$, $f_i(u)=f_i(u_1,\cdots,u_m)$ are smooth in R_+^m and f_i satisfies the following type of quasi-positive condition: $f_i(u)\geq 0$ for every $u=(u_1,\cdots,u_m)$ which satisfies $u_i\geq 0$ for $i=1,2,\cdots,m$.

(H1) $f_i(0)=0$.

(H2) $f_i(u)\leq \sum_j c_{ij}u_j^{\alpha_{ij}}+c_i$, in R_+^m, where c_{ij},α_{ij},c_i are constants and $\alpha_{ij}\geq 0,i,j=1,2,\cdots,m$.

In assumption (H2), we intend to give an explicit form of the growth of $f_i(u)$ for large u, furthermore to state the results that will follow; the nonlinear part $f_i(u)$ could be allowed to depend on x,t. In that case, in (H2), c_{ij}, c_i, would be functions of (x,t), each contained in same space $L^q(0,T;L^p(\Omega))$, $T>0$, where $p\geq 1$ and $q\geq 1$ would be special real numbers.

We begin by regularizing problem (1.1a)-(1.1c).

Since the nonlinear term $f_i(u)$ could be super-linear for large u, we will approximate it by a sequence of linear maps for large u. Let $\{R_q\}_{q\in N}$ be an increasing sequence of positive real numbers s.t. $\lim_{q\to+\infty}R_q=+\infty$ and f_{iq} be smooth functions that linearize for the functions f_i for $|u|>R_q$ (actually they should also satisfy the quasi-positive condition), and $f_{iq}\leq f_i$, for $u_i\geq 0$, $q\in N$.

If in (1.1b), $u_{i0}\in L^\infty(\Omega)\cap W_0^{1,p_i}(\Omega)$ and $u_{i0}\geq 0$, we can construct a sequence $\{u_{i0q}\}_{q\in N}$, s.t. $u_{i0q}\in C_0^\infty(\Omega)$, $u_{i0q}\geq 0$, $\lim_{q\to+\infty}\|u_{i0q}-u_{i0}\|_{W^{1,p_i}(\Omega)}=0$ and equilimited in L^∞ norm.

We consider the following regularizing problem for every $q\geq 1$:

$$u_{iqt}=\operatorname{div}\left(\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}\nabla u_{iq}\right)+f_{iq}\left(u_q-\frac{1}{q}\right),\quad (x,t)\in\Omega_T, \tag{2.1a}$$

$$u_{iq}(x,0)=u_{i0q}(x)+\frac{1}{q},\quad x\in\Omega, \tag{2.1b}$$

$$u_{iq}(x,t)=\frac{1}{q},\quad (x,t)\in\partial\Omega\times(0,T), \tag{2.1c}$$

where $u_q-\frac{1}{q}=\left(u_1-\frac{1}{q},u_2-\frac{1}{q},\cdots,u_m-\frac{1}{q}\right)$.

We prove the following lemma by using a similar method as in [12].

Lemma 2.1. *For every $q\geq 1$, problem* (2.1a)-(2.1c) *exists a classical global solution*

$$u_q=(u_{1q},u_{2q},\cdots,u_{mq})\quad (u_{iq}\in C^{2,1}(\overline{\Omega}_T),\ T>0)$$

and

$$u_{iq}\geq\frac{1}{q},\qquad (x,t)\in\Omega_T. \tag{2.2}$$

Proof. We consider the system

$$u_{iqt}=\operatorname{div}\left(\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}\nabla u_{iq}\right)+f_{iq}\left(\left(u_q-\frac{1}{q}\right)^+\right), \tag{2.3}$$

with

$$\Big(u_q-\frac{1}{q}\Big)^+=\Big(\Big(u_1-\frac{1}{q}\Big)^+,\Big(u_2-\frac{1}{q}\Big)^+,\cdots,\Big(u_m-\frac{1}{q}\Big)^+\Big),\qquad r^+=\max(r,0).$$

This is a quasilinear nondegenerate parabolic system. The system (2.3) with initial and boundary conditions (2.1b)-(2.1c) admits a unique classical solution

$$u_q=(u_{1q},u_{2q},\cdots,u_{mq})\quad (u_{iq}\in C^{2,1}(\overline{\Omega}_T),\ i=1,2,\cdots,m,\ T>0),$$

(see, VII, §7 [13]). Considering the structure of f and $T>0$ is arbitrary, the solution is global.

If $u_{iq}(x,t)\geq\frac{1}{q},(x,t)\in\Omega_T$, then $f_{iq}\big((u_q-\frac{1}{q})^+\big)=f_{iq}\big((u_q-\frac{1}{q})\big)$. Therefore we can conclude that system (2.3) is equivalent to (2.1a) when $u_{iq}(x,t)\geq\frac{1}{q}$. Then $u_q=(u_{1q},u_{2q},\cdots,u_{mq})$ is a classical global solution of system (2.1a)-(2.1c).

Let

$$v_{iq}(x,t)=e^{-t}\Big(u_{iq}-\frac{1}{q}\Big).$$

We will show that the functions $v_{iq}(x,t)$ are greater than zero. It is clear $v_{iq}(x,0)\geq 0$ in Ω and $v_{iq}(x,t)\geq 0$ in $\partial\Omega\times(0,T)$. Now suppose that for some $j\in\{1,2,\cdots,m\}$, $v_{jq}(x,t)$ take negative values, then it must have a negative minimum at a point (x_0,t_0); therefore, the inequality

$$v_{jqt}-\Big(\frac{1}{q}\Big)^{\frac{p_j-2}{2}}\triangle v_{jq}\leq 0, \tag{2.4}$$

is true at (x_0,t_0). On the other hand, due to (2.3),

$$v_{jqt}-\Big(\frac{1}{q}\Big)^{\frac{p_j-2}{2}}\triangle v_{jq}=-v_{jq}+e^{-t}f_{jq}\Big(\Big(u_q-\frac{1}{q}\Big)^+\Big), \tag{2.5}$$

at (x_0,t_0). If we take assumptions (H0), (H1) into account, we have

$$\begin{aligned}&f_{jq}\Big(\Big(u_q-\frac{1}{q}\Big)^+\Big)\\ =&f_{jq}\Big(\Big(u_{1q}-\frac{1}{q}\Big)^+,\cdots,\Big(u_{(j-1)q}-\frac{1}{q}\Big)^+,0,\Big(u_{(j+1)q}-\frac{1}{q}\Big)^+,\cdots,\Big(u_{mq}-\frac{1}{q}\Big)^+\Big)\geq 0,\end{aligned}$$

at (x_0,t_0).

Hence the right-hand side of equality (2.5) is positive at (x_0,t_0). This contradicts to (2.4); therefore, $v_{iq}\geq 0$, and $u_{iq}\geq\frac{1}{q}$ in Ω_T; the lemma is proved. □

We now prove some a priori estimates for the solution u_q of (2.1a)-(2.1c). We begin by proving that u_{iq} are equilimited in Ω_T, $T\geq 0$.

Lemma 2.2. *Assume that $c_{ij}>0$. If*

(1) $\alpha_{ij}<p_i-1$, $i,j=1,2,\cdots,m$,

or

(2) $\alpha_{ij}\le p_i-1$, $i,j=1,2,\cdots,m$, *and* $\mathrm{diam}(\Omega)$ *is sufficiently small,*

then the following a priori estimate

$$\|u_{iq}\|_{L^\infty(\Omega_T)}\le C_1,\qquad \forall T\ge 0, \tag{2.6}$$

is valid for $u_q=(u_{1q},u_{2q},\cdots,u_{mq})$ which is a classical solution of (2.1)-(2.3), *where c_{ij} and α_{ij} come from* (H2), *and C_1 denotes a constant independent of q.*

Proof. (1) If $u\in L^\infty(\Omega_T)$, then $\|u\|_{L^\infty(\Omega_T)}=\lim_{r\to+\infty}\|u\|_{L^r(\Omega_T)}$. We intend to prove that sequence $\|u_{iq}-\frac{1}{q}\|_{L^r(\Omega_T)}$ is equilimited by a constant independent of r and q.

Multiplying (2.1a) by $(u_{iq}-\frac{1}{q})^{r-1}$, $r>1$, and integrating by parts over Ω_T, for some $T>0$, we have

$$\begin{aligned}&\iint_{\Omega_T}\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}u_{iqt}\mathrm{d}x\mathrm{d}t\\ =&\iint_{\Omega_T}\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}\mathrm{div}\Big(\Big(|\nabla u_{iq}|^2+\frac{1}{q}\Big)^{\frac{p_i-2}{2}}\nabla u_{iq}\Big)\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}f_{iq}\Big(u_q-\frac{1}{q}\Big)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{r}\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,T)\mathrm{d}x+\iint_{\Omega_T}\Big(|\nabla u_{iq}|^2+\frac{1}{q}\Big)^{\frac{p_i-2}{2}}\nabla u_{iq}\nabla\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}\mathrm{d}x\mathrm{d}t\\ =&\iint_{\Omega_T}\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}f_{iq}\Big(u_q-\frac{1}{q}\Big)\mathrm{d}x\mathrm{d}t+\frac{1}{r}\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,0)\mathrm{d}x.\end{aligned} \tag{2.7}$$

Moreover

$$\begin{aligned}&\Big(|\nabla u_{iq}|^2+\frac{1}{q}\Big)^{\frac{p_i-2}{2}}\nabla u_{iq}\nabla\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}\\ =&(r-1)\Big(|\nabla u_{iq}|^2+\frac{1}{q}\Big)^{\frac{p_i-2}{2}}\Big(u_{iq}-\frac{1}{q}\Big)^{r-2}|\nabla u_{iq}|^2\\ \ge&(r-1)\Big(u_{iq}-\frac{1}{q}\Big)^{r-2}|\nabla u_{iq}|^{p_i}\\ =&(r-1)\frac{p_i^{p_i}}{(p_i+r-2)^{p_i}}\left|\nabla\Big(u_{iq}-\frac{1}{q}\Big)^{\frac{p_i+r-2}{p_i}}\right|^{p_i}.\end{aligned}$$

If we take assumption (H2) ($f_{iq}\le f_i$) into account, we have

$$\begin{aligned}&\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,T)\mathrm{d}x+r(r-1)\iint_{\Omega_T}\Big(u_{iq}-\frac{1}{q}\Big)^{r-2}\Big(|\nabla u_{iq}|^2+\frac{1}{q}\Big)^{\frac{p_i-2}{2}}|\nabla u_{iq}|^2\mathrm{d}x\mathrm{d}t\\ \le&\iint_{\Omega_T}\Big(\sum_j c_{ij}r\Big(u_{jq}-\frac{1}{q}\Big)^{\alpha_{ij}}\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}\Big)\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}rc_i\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}\mathrm{d}x\mathrm{d}t\\ &+\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,0)\mathrm{d}x. \end{aligned}\tag{2.8}$$

Applying Young's inequality, we have

$$\begin{aligned}&\int_0^T\int_\Omega c_{ij}r\Big(u_{jq}-\frac{1}{q}\Big)^{\alpha_{ij}}\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}\mathrm{d}x\mathrm{d}t\\ \le&\int_0^T\int_\Omega\Big(c_{ij}r\frac{\alpha_{ij}}{s}\Big(u_{jq}-\frac{1}{q}\Big)^s+c_{ij}r\frac{(s-\alpha_{ij})}{s}\Big(u_{iq}-\frac{1}{q}\Big)^{\frac{s(r-1)}{s-\alpha_{ij}}}\Big)\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{2.9}$$

where $\alpha_{ij}<s<r$ will be suitably chosen. Applying the Sobolev embedding theorem, we have

$$\begin{aligned}\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^{\frac{s(r-1)}{s-\alpha_{ij}}}\mathrm{d}x=&\int_\Omega(u_{iq}-\frac{1}{q})^{\frac{p_i+r-2}{p_i}\cdot\frac{p_i(r-1)s}{(p_i+r-2)(s-\alpha_{ij})}}\mathrm{d}x\\ &\le C\Big(\int_\Omega\Big|\nabla\Big(u_{iq}-\frac{1}{q}\Big)^{\frac{p_i+r-2}{p_i}}\Big|^{p_i}\mathrm{d}x\Big)^{\frac{(r-1)s}{(p_i+r-2)(s-\alpha_{ij})}},\end{aligned}\tag{2.10}$$

where C denotes various constants independent of r and q. In different formulae these constants will in general have different values. Choose $\alpha_{ij}<s<r$, s.t.

$$\frac{(r-1)s}{(p_i+r-2)(s-\alpha_{ij})}<1.$$

Then

$$s>\Big(\frac{p_i+r-2}{p_i-1}\Big)\alpha_{ij}>\alpha_{ij}.\tag{2.11}$$

According to assumption $\alpha_{ij}<p_i-1$, we know that $s<r$. i.e. we can choose such s.

From Young's inequality, we obtain

$$\int_\Omega\Big(u_{jq}-\frac{1}{q}\Big)^s\mathrm{d}x\le\frac{s}{r}\int_\Omega\Big(u_{jq}-\frac{1}{q}\Big)^r\mathrm{d}x+C,\tag{2.12}$$

and

$$\int_\Omega c_ir\Big(u_{iq}-\frac{1}{q}\Big)^{r-1}\mathrm{d}x\le c_irC+c_ir\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r\mathrm{d}x.\tag{2.13}$$

By (2.9)-(2.13), we get

$$\sum_i\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,T)\mathrm{d}x+\sum_i r(r-1)\frac{p_i^{p_i}}{(p_i+r-2)^{p_i}}\int_0^T\int_\Omega\left|\nabla\Big(u_{iq}-\frac{1}{q}\Big)^{\frac{p_i+r-2}{p_i}}\right|^{p_i}\mathrm{d}x\mathrm{d}t$$
$$\le\sum_{ij}Cc_{ij}r\frac{s-\alpha_{ij}}{s}\int_0^T\left(\int_\Omega\left|\nabla\Big(u_{iq}-\frac{1}{q}\Big)^{\frac{p_i+r-2}{p_i}}\right|^{p_i}\mathrm{d}x\right)^{\frac{(r-1)s}{(p_i+r-2)(s-\alpha_{ij})}}\mathrm{d}t$$
$$+\sum_{ij}c_{ij}\alpha_{ij}\iint_{\Omega_T}\Big(u_{jq}-\frac{1}{q}\Big)^r\mathrm{d}x\mathrm{d}t+T\sum_{ij}c_{ij}rC+T\sum_i c_irC$$
$$+\sum_i c_ir\iint_{\Omega_T}\Big(u_{iq}-\frac{1}{q}\Big)^r\mathrm{d}x\mathrm{d}t+\sum_i\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,0)\mathrm{d}x. \tag{2.14}$$

Therefore

$$\sum_i\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,T)\mathrm{d}x$$
$$\le Cr\sum_i\iint_{\Omega_T}\Big(u_{iq}-\frac{1}{q}\Big)^r\mathrm{d}x\mathrm{d}t+\sum_i\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,0)\mathrm{d}x+Cr. \tag{2.15}$$

Using Gronwall's lemma (see e.g. [14]) and that (2.15) is true for every $T>0$, for every $t<T$, we have

$$\sum_i\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,t)\mathrm{d}x$$
$$\le e^{Crt}\sum_i\int_\Omega\Big(u_{iq}-\frac{1}{q}\Big)^r(x,0)\mathrm{d}x+\left(\sum_{ij}\Big(\frac{CKm}{r(r-1)\beta_{ij}}\Big)^{\frac{1}{\beta_{ij}-1}}+rC\right)e^{Crt}.$$

Therefore

$$\sum_i\Big\|\Big(u_{iq}-\frac{1}{q}\Big)(x,t)\Big\|_{L^r(\Omega_T)}$$
$$\le e^{Ct}\sum_i\Big\|\Big(u_{iq}-\frac{1}{q}\Big)(x,0)\Big\|_{L^r(\Omega)}+e^{Ct}\left(\sum_{ij}\Big(\frac{CKm}{r(r-1)\beta_{ij}}\Big)^{\frac{1}{r(\beta_{ij}-1)}}+r^{\frac{1}{r}}C^{\frac{1}{r}}\right). \tag{2.16}$$

Let $r\to+\infty$, we have

$$\sum_i\Big\|\Big(u_{iq}-\frac{1}{q}\Big)(x,t)\Big\|_{L^\infty(\Omega_T)}\le e^{Ct}\sum_i\Big\|\Big(u_{iq}-\frac{1}{q}\Big)(x,0)\Big\|_{L^\infty(\Omega)}+me^{Ct},$$

from which (2.6) follows.

(2) The proof is similar to case (1) when $\max\{\alpha_{ij}\} < p_i - 1$. If $\max\{\alpha_{ij}\} = p_i - 1$, then the first part of right hand of (2.8) is as follows.

$$\begin{aligned}&\iint_{\Omega_T} \sum_j c_{ij} r \Big(u_{jq} - \frac{1}{q}\Big)^{\alpha_{ij}} \Big(u_{iq} - \frac{1}{q}\Big)^{r-1} \mathrm{d}x\mathrm{d}t \\ \le &\iint_{\Omega_T} c_{ii} r \Big(u_{iq} - \frac{1}{q}\Big)^{p_i+r-2} \mathrm{d}x\mathrm{d}t + \iint_{\Omega_T} \sum_{j\neq i} c_{ij} r \Big(u_{jq} - \frac{1}{q}\Big)^{p_i-1} \Big(u_{iq} - \frac{1}{q}\Big)^{r-1} \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.17}$$

Applying Young's inequality, we have

$$\begin{aligned}&\iint_{\Omega_T} c_{ij} r \Big(u_{jq} - \frac{1}{q}\Big)^{p_i-1} \Big(u_{iq} - \frac{1}{q}\Big)^{r-1} \mathrm{d}x\mathrm{d}t \\ \le &\int_0^T \int_\Omega c_{ij} r \bigg(\frac{p_i - 1}{p_i + r - 2} \Big(u_{jq} - \frac{1}{q}\Big)^{p_i+r-2} + \frac{r-1}{p_i+r-2} \Big(u_{iq} - \frac{1}{q}\Big)^{p_i+r-2} \bigg) \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.18}$$

Applying Poincar*é* inequality, we have

$$\begin{aligned}&\int_\Omega \Big(u_{iq} - \frac{1}{q}\Big)^{p_i+r-2} \mathrm{d}x = \int_\Omega \Big(u_{iq} - \frac{1}{q}\Big)^{\frac{p_i+r-2}{p_i}\cdot p_i} \mathrm{d}x \\ \le &C(n,p_i)(\mathrm{diam}(\Omega))^{p_i} \int_\Omega \bigg| \nabla \Big(u_{iq} - \frac{1}{q}\Big)^{\frac{p_i+r-2}{p_i}} \bigg|^{p_i} \mathrm{d}x, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned}&\int_\Omega \Big(u_{jq} - \frac{1}{q}\Big)^{p_i+r-2} \mathrm{d}x = \int_\Omega \Big(u_{jq} - \frac{1}{q}\Big)^{\frac{p_i+r-2}{p_i}\cdot p_i} \mathrm{d}x \\ \le &C(n,p_i)(\mathrm{diam}(\Omega))^{p_i} \int_\Omega \bigg| \nabla \Big(u_{jq} - \frac{1}{q}\Big)^{\frac{p_i+r-2}{p_i}} \bigg|^{p_i} \mathrm{d}x. \end{aligned} \tag{2.20}$$

Similar to case (1), we can get (2.14)-(2.16) provided that $\mathrm{diam}(\Omega)$ is sufficiently small. Then (2.6) follows. □

Lemma 2.3. *Under the assumptions of Lemma* 2.2, *we have*

$$\iint_{\Omega_T} |\nabla u_{iq}|^{p_i} \mathrm{d}x\mathrm{d}t \le C_2, \qquad (x,t)\in\Omega_T, \tag{2.21}$$

$$\iint_{\Omega_T} |u_{iqt}|^2 \mathrm{d}x\mathrm{d}t \le C_3, \qquad (x,t)\in\Omega_T, \tag{2.22}$$

where $C_j (j=2,3)$ *are constants independent of* q, $q\ge 1$.

Proof. Multiplying (2.1a) by u_{iq} and integrating over Ω_T, we have

$$\iint_{\Omega_T}\left(u_{iq}u_{iqt}-\operatorname{div}\left(\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}\nabla u_{iq}\right)u_{iq}\right)\mathrm{d}x\mathrm{d}t=\iint_{\Omega_T}f_{iq}\left(u_q-\frac{1}{q}\right)u_{iq}\mathrm{d}x\mathrm{d}t. \quad (2.23)$$

Furthermore

$$\int_\Omega\int_0^T\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u_{iq})^2\mathrm{d}t\mathrm{d}x+\iint_{\Omega_T}\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}|\nabla u_{iq}|^2\mathrm{d}x\mathrm{d}t=\iint_{\Omega_T}f_{iq}\left(u_q-\frac{1}{q}\right)u_{iq}\mathrm{d}x\mathrm{d}t,$$

i.e.

$$\begin{aligned}&\iint_{\Omega_T}\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}|\nabla u_{iq}|^2\mathrm{d}x\mathrm{d}t\\ =&\iint_{\Omega_T}f_{iq}\left(u_q-\frac{1}{q}\right)u_{iq}\mathrm{d}x\mathrm{d}t-\frac{1}{2}\int_\Omega\left((u_{iq}(x,T))^2-(u_{iq}(x,0))^2\right)\mathrm{d}x.\end{aligned}$$

By (2.6) and the property of f_{iq}, we have

$$\iint_{\Omega_T}|\nabla u_{iq}|^{p_i}\mathrm{d}x\mathrm{d}t\le\iint_{\Omega_T}\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}|\nabla u_{iq}|^2\mathrm{d}x\mathrm{d}t\le C_2', \quad (2.24)$$

where C_2' is a constant independent of q. By (2.6) and (2.24), (2.21) follows.

Multiplying (2.1a) by u_{iqt} and integrating over Ω_T, we have

$$\begin{aligned}&\iint_{\Omega_T}(u_{iqt})^2\mathrm{d}x\mathrm{d}t-\iint_{\Omega_T}\operatorname{div}\left(\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}\nabla u_{iq}\right)u_{iqt}\mathrm{d}x\mathrm{d}t\\ =&\iint_{\Omega_T}f_{iq}\left(u_q-\frac{1}{q}\right)u_{iqt}\mathrm{d}x\mathrm{d}t. \quad (2.25)\end{aligned}$$

By Hölder inequality and integrating by parts, we obtain

$$\begin{aligned}&\iint_{\Omega_T}(u_{iqt})^2\mathrm{d}x\mathrm{d}t\\ =&-\iint_{\Omega_T}\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}\nabla u_{iq}\nabla u_{iqt}\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}f_{iq}\left(u_q-\frac{1}{q}\right)u_{iqt}\mathrm{d}x\mathrm{d}t\\ \le&\frac{1}{p_i}\left|\int_\Omega\int_0^T\frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i}{2}}\mathrm{d}t\mathrm{d}x\right|+\frac{1}{2}\iint_{\Omega_T}f_{iq}^2\left(u_q-\frac{1}{q}\right)\mathrm{d}x\mathrm{d}t+\frac{1}{2}\iint_{\Omega_T}(u_{iqt})^2\mathrm{d}x\mathrm{d}t\\ \le&\frac{1}{p_i}\int_\Omega\left|\left(|\nabla u_{iq}(x,T)|^2+\frac{1}{q}\right)^{\frac{p_i}{2}}-\left(|\nabla u_{i0q}|^2+\frac{1}{q}\right)^{\frac{p_i}{2}}\right|\mathrm{d}x\\ &+\frac{1}{2}\iint_{\Omega_T}f_{iq}^2\left(u_q-\frac{1}{q}\right)\mathrm{d}x\mathrm{d}t+\frac{1}{2}\iint_{\Omega_T}(u_{iqt})^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Therefore

$$\iint_{\Omega_T}(u_{iqt})^2\mathrm{d}x\mathrm{d}t\le C_3. \tag{2.26}$$

The proof is complete. □

Now we are able to prove an existence theorem for (1.1a)-(1.1c).

Theorem 2.1. *Under the assumptions in Lemma* 2.2 *and* $u_{i0}\in L^\infty(\Omega)\cap W_0^{1,p_i}(\Omega)$, $i,j=1,2,\cdots,m$, *for every* $T>0$, *then there exists a generalized solution* $u=(u_1,\cdots,u_m)$ *to problem* (1.1a)-(1.1c) *in* Ω_T. *Furthermore,*

$$u_i\in L^\infty(\Omega_T)\cap L^{p_i}\left(0,T;W_0^{1,p_i}(\Omega)\right), \tag{2.27}$$

and

$$u_{it}\in L^2\left(0,T;L^2(\Omega)\right). \tag{2.28}$$

Proof. Due to lemma 2.2, lemma 2.3 and the property of f_{iq}, for every i, $i=1,2,\cdots,m$, there exist a function $u_i(x,t)$ and a subsequence of $\{u_{iq}\}$, which we denote again by $\{u_{iq}\}$, s.t.

$$\begin{aligned}&u_{iq}\to u_i,\quad \text{a.e.in } \Omega_T,\quad \nabla u_{iq}\rightharpoonup\nabla u_i,\quad \text{in } L^{p_i}(\Omega_T),\\ &u_{iqt}\rightharpoonup u_{it},\quad \text{in } L^2(\Omega_T),\quad |\nabla u_{iq}|^{p_i-2}u_{iqx_l}\rightharpoonup w_{ix_l},\quad \text{in } L^{\frac{p_i}{p_i-1}}(\Omega_T), \text{ for some } w_{ix_l},\end{aligned}$$

where $\rightharpoonup$ stands for weak convergence, and

$$w_{ix_l}\in L^{\frac{p_i}{p_i-1}}(\Omega_T) \text{ is the weak limit of } |\nabla u_{iq}|^{p_i-2}u_{iqx_l}.$$

We can prove that $w_{ix_l}=|\nabla u_i|^{p_i-2}u_{ix_l}$ using similar method as in [11].

Multiplying (2.1a) by $(u_{iq}-u_i)\phi_i$ and integrating over Ω_T, $\phi_i\in C^1(\overline{\Omega}_T)$, $\phi_i\ge 0$, we get

$$\begin{aligned}&\iint_{\Omega_T}\phi_i u_{iqt}(u_{iq}-u_i)\mathrm{d}x\mathrm{d}t+\iint_{\Omega_T}\phi_i\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}\nabla u_{iq}(\nabla u_{iq}-\nabla u_i)\mathrm{d}x\mathrm{d}t\\ &\quad+\iint_{\Omega_T}\left(|\nabla u_{iq}|^2+\frac{1}{q}\right)^{\frac{p_i-2}{2}}\nabla u_{iq}\nabla\phi_i(u_{iq}-u_i)\mathrm{d}x\mathrm{d}t\\ &=\iint_{\Omega_T}\phi_i f_{iq}\left(u_q-\frac{1}{q}\right)(u_{iq}-u_i)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Hence

$$\lim_{q\to+\infty}\iint_{\Omega_T}\phi_i|\nabla u_{iq}|^{p_i-2}\nabla u_{iq}(\nabla u_{iq}-\nabla u_i)\mathrm{d}x\mathrm{d}t=0. \tag{2.29}$$

On the other hand, since $\nabla u_i \in L^{p_i}(\Omega_T)$, we have

$$\lim_{q\to+\infty}\iint_{\Omega_T}|\nabla u_{iq}|^{p_i-2}\nabla u_{iq}(\nabla u_{iq}-\nabla u_i)\phi_i \mathrm{d}x\mathrm{d}t=0. \tag{2.30}$$

Note that

$$\begin{aligned}&\left(|\nabla u_{iq}|^{p_i-2}\nabla u_{iq}-|\nabla u_i|^{p_i-2}\nabla u_i\right)(\nabla u_{iq}-\nabla u_i)\\ &\geq\int_0^1|\nabla(su_{iq}+(1-s)u_i)|^{p_i-2}\mathrm{d}s|\nabla(u_{iq}-u_i)|^2.\end{aligned} \tag{2.31}$$

By (2.30) and (2.31), we have

$$\lim_{q\to+\infty}\iint_{\Omega_T}\phi_i\int_0^1|\nabla(su_{iq}+(1-s)u_i)|^{p_i-2}\mathrm{d}s|\nabla(u_{iq}-u_i)|^2\mathrm{d}x\mathrm{d}t=0.$$

Since

$$\iint_{\Omega_T}\int_0^1|\nabla(su_{iq}+(1-s)u_i)|^{p_i-2}\mathrm{d}s\mathrm{d}x\mathrm{d}t\leq C,$$

and

$$\begin{aligned}&\left||\nabla u_{iq}|^{p_i-2}u_{iqx_l}-|\nabla u_i|^{p_i-2}u_{ix_l}\right|\\ &=\left|\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left|s\nabla u_{iq}+(1-s)\nabla u_i\right|^{p_i-2}(su_{iqx_l}+(1-s)u_{ix_l})\right\}\mathrm{d}s\right|\\ &\leq\left|\int_0^1\left|s\nabla u_{iq}+(1-s)\nabla u_i\right|^{p_i-2}(u_{iqx_l}-u_{ix_l})\mathrm{d}s\right|\\ &\quad+\left|\int_0^1(p_i-2)\left|s\nabla u_{iq}+(1-s)\nabla u_i\right|^{p_i-4}(su_{iqx_l}+(1-s)u_{ix_l})(u_{iqx_l}-u_{ix_l})\mathrm{d}s\right|\\ &\leq C\left|\nabla(u_{iq}-u_i)\right|\int_0^1\left|s\nabla u_{iq}+(1-s)\nabla u_i\right|^{p_i-2}\mathrm{d}s,\end{aligned} \tag{2.32}$$

we have

$$\begin{aligned}&\left|\iint_{\Omega_T}\phi_i(\left|\nabla u_{iq}\right|^{p_i-2}u_{iqx_l}-|\nabla u_i|^{p_i-2}u_{ix_l})\mathrm{d}x\mathrm{d}t\right|\\ &\leq C\left(\iint_{\Omega_T}\phi_i\int_0^1\left|\nabla(su_{iq}+(1-s)u_i)\right|^{p_i-2}\mathrm{d}s\left|\nabla(u_{iq}-u_i)\right|^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\\ &\quad\left(\iint_{\Omega_T}\phi\int_0^1\left|\nabla(su_{iq}+(1-s)u_i)\right|^{p_i-2}\mathrm{d}s\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\to 0,\qquad q\to+\infty,\end{aligned} \tag{2.33}$$

i.e.

$$\iint_{\Omega_T}(w_{ix_l}-|\nabla u_i|^{p_i-2}u_{ix_l})\phi_i\mathrm{d}x\mathrm{d}t=0,\quad \text{for any }\ \phi_i.$$

Hence $w_{ix_l}=|\nabla u_i|^{p_i-2}u_{ix_l}$, $i=1,2,\cdots,m$.

Following a standard limiting process, we obtain that $u=(u_1,\cdots,u_m)$ satisfies the initial and boundary value conditions and the integrating expression. Thus u is a generalized solution to (1.1a)-(1.1c). ☐

Theorem 2.2. *Assume $f=(f_1,f_2,\cdots,f_m)$ is local Lipschitz continuous in u, then the solution is unique.*

Proof. Assume that $u=(u_1,\cdots,u_m)$ and $v=(v_1,\cdots,v_m)$ are two solutions to (1.1a)-(1.1c), then u,v are bounded. Considering that f is local Lipschitz continuous in u, we get that f is Lipschitz continuous on $[0,\max\{\|u\|_{L^\infty(Q_T)},\|v\|_{L^\infty(Q_T)}\}]$.

Let $\varphi_i=u_i-v_i$, then by (1.2),

$$\begin{aligned}&\int_\Omega u_i(x,T)\varphi_i(x,T)\mathrm{d}x+\iint_{\Omega_T}|\nabla u_i|^{p_i-2}\nabla u_i\nabla\varphi_i\mathrm{d}x\mathrm{d}t\\ =&\iint_{\Omega_T}(f_i(u)\varphi_i+\varphi_{it}u_i)\mathrm{d}x\mathrm{d}t+\int_\Omega u_{i0}(x)\varphi_i(x,0)\mathrm{d}x,\end{aligned} \tag{2.34a}$$

$$\begin{aligned}&\int_\Omega v_i(x,T)\varphi_i(x,T)\mathrm{d}x+\iint_{\Omega_T}|\nabla v_i|^{p_i-2}\nabla v_i\nabla\varphi_i\mathrm{d}x\mathrm{d}t\\ =&\iint_{\Omega_T}(f_i(v)\varphi_i+\varphi_{it}v_i)\mathrm{d}x\mathrm{d}t+\int_\Omega v_{i0}(x)\varphi_i(x,0)\mathrm{d}x.\end{aligned} \tag{2.34b}$$

Subtracting the two equations, we get

$$\begin{aligned}\frac{1}{2}\int_\Omega(u_i(x,T)-v_i(x,T))^2\mathrm{d}x=&-\int_0^T\int_\Omega(|\nabla u_i|^{p_i-2}\nabla u_i-|\nabla v_i|^{p_i-2}\nabla v_i)\nabla(u_i-v_i)\mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_\Omega(f_i(u)-f_i(v))(u_i-v_i)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Note that

$$\big(|\nabla u_i|^{p_i-2}\nabla u_i-|\nabla v_i|^{p_i-2}\nabla v_i\big)\nabla(u_i-v_i)\geq 0.$$

Using the previous inequality and the Lipschitz condition, a simple calculation shows that

$$\begin{aligned}&\int_\Omega(|u_1-v_1|^2+\cdots+|u_m-v_m|^2)\mathrm{d}x\\ \leq&2K\int_0^T\int_\Omega(|u_1-v_1|+\cdots+|u_m-v_m|)^2\mathrm{d}x\mathrm{d}t\\ \leq&2^mK\int_0^T\int_\Omega(|u_1-v_1|^2+\cdots+|u_m-v_m|^2)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Set

$$F(T)=\int_0^T\int_\Omega(|u_1-v_1|^2+\cdots+|u_m-v_m|^2)\mathrm{d}x\mathrm{d}t,$$

then the above inequality can be written as

$$F'(T) \leq 2^m K F(T).$$

A standard argument shows that $F(T) \equiv 0$ since $F(0) \equiv 0$, $u_i \equiv v_i$. The proof is complete. □

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