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Multiple Positive Solutions for Semilinear Elliptic Equations Involving Subcritical Nonlinearities in $\mathbb{R}^{\mathbb{N}}$

KHADEMLOO Somayeh* and MOHSENHI Rahelh

Department of Basic Sciences, Babol Noushirvani University of Technology, Babol, Iran.

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Abstract. In this paper, we study how the shape of the graph of $a(z)$ affects on the number of positive solutions of

$$-\Delta v+\mu b(z)v=a(z)v^{p-1}+\lambda h(z)v^{q-1}, \qquad \text{in } \mathbb{R}^N. \tag{0.1}$$

We prove for large enough $\lambda,\mu>0$, there exist at least $k+1$ positive solutions of the this semilinear elliptic equations where $1\leq q<2<p<2^*=2N/(N-2)$ for$N\geq 3$.

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1 Introduction

For $N\geq 3$, $1\leqslant q<2<p<2^*=2N/(N-2)$, we suppose the semilinear elliptic equations

$$\begin{cases} -\Delta v+\mu b(z)v=a(z)v^{p-1}+\lambda h(z)v^{q-1}, & \text{in } \mathbb{R}^N; \\ v\in H^1(\mathbb{R}^N), \end{cases} \tag{$E_{\lambda,\mu}$}$$

where $\lambda,\mu>0$. Suppose a, b and h satisfy the following conditions:
(a_1) a is a positive continuous function in $\mathbb{R}^N$ and $\lim_{|z|\to\infty}a(z)=a_\infty>0$.
(a_2) There are k points $a^1,a^2,\cdots,a^k$ in $\mathbb{R}^N$ such that $a(a^i)=a_{\max}=\max_{z\in\mathbb{R}^N}a(z)$; for $1\leq i\leq k$ and $a_\infty<a_{\max}$.

*Corresponding author. *Email addresses:* s.khademloo@nit.ac.ir (S. Khademloo), rm_omran78@yahoo.com (R. Mohsenhi)

(h_1) $h\in L^{\frac{p}{p-q}}(\mathbb{R}^{\mathbb{N}})\cap L^{\infty}(\mathbb{R}^{\mathbb{N}})$ and $h\gneqq 0$.
(b_1) b is a bounded and positive continuous function in $\mathbb{R}^N$.

For $\mu=1$, $\lambda=0$, $a(z)=b(z)=1$ for all $z\in\mathbb{R}^N$, we assume the semilinear elliptic equation

$$\begin{cases} -\Delta u+u=u^{p-1}, & \text{in } \mathbb{R}^{\mathbb{N}};\\ u\in H^1(\mathbb{R}^N), \end{cases} \tag{E_0}$$

where

$$\| u\|_H^2=\int_{\mathbb{R}^{\mathbb{N}}}(|\nabla u|^2+u^2)\mathrm{d}z \quad \text{is the norm in } H^1(\mathbb{R}^N),$$

and the energy functional

$$J_0^{\infty}(u)=\frac{1}{2}\| u\|_H^2-\frac{1}{p}\| u_+\|_{L^p}^p, \qquad \text{where } u_+=\max\{u,0\}\geqslant 0.$$

We consider the semilinear elliptic equation

$$\begin{cases} -\Delta u+u=a(z)u^{p-1}+\lambda h(z)u^{q-1}, & \text{in } \mathbb{R}^{\mathbb{N}};\\ u\in H^1(\mathbb{R}^{\mathbb{N}}), \end{cases}$$

have been studied by Huei-li Lin [1] ($b(z)=1$, $\mu=1$ and for $N\geq 3$, $1\leqslant q<2<p<2^*=2N/(N-2)$) and she studied the effect of the coefficient a(z) of the subcritical nonlinearity in $\mathbb{R}^{\mathbb{N}}$, Ambrosetti [2] ($a\equiv 1$ and $1<q<2<p\leq 2^*=2N/(N-2)$ and Wu [3] ($a\in C(\overline{\Omega})$ and changes sign, $1<q<2<p<2^*$). They showed that this equation has at least two positive solutions for small enough $\lambda>0$. In [4], Hsu and Lin have studied that there are four positive solutions of the general cases

$$-\Delta v+v=a(z)v^{p-1}+\lambda h(z)v^{q-1}, \qquad \text{in } \mathbb{R}^N;$$

for small enough $\lambda>0$.

In this paper, we study the existence and multiplicity of positive solutions of the equation $(E_{\lambda,\mu})$ in $\mathbb{R}^{\mathbb{N}}$. By the change of variables

$$\mu=\frac{1}{\varepsilon^2} \qquad \text{and} \qquad u(z)=\varepsilon^{\frac{2}{p-2}}v(\varepsilon z),$$

Eq. $(E_{\lambda,\mu})$ is converted to

$$\begin{cases} -\Delta u+b(\varepsilon z)u=a(\varepsilon z)u^{p-1}+\lambda h(\varepsilon z)u^{q-1}, & \text{in } \mathbb{R}^{\mathbb{N}};\\ u\in H^1(\mathbb{R}^N). \end{cases} \tag{$E_{\varepsilon,\lambda}$}$$

Based on Eq. $(E_{\varepsilon,\lambda})$, we consider the C^1-functional $J_{\varepsilon,\lambda}$, for $u\in H^1(\mathbb{R}^{\mathbb{N}})$.

$$J_{\varepsilon,\lambda}(u)=\frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}}(|\nabla u|^2+b(\varepsilon z)u^2)\mathrm{d}z-\frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\mathrm{d}z-\frac{1}{q}\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(\varepsilon z)u_+^q\mathrm{d}z,$$

where

$$\|u\|_b^2=\int_{\mathbb{R}^{\mathbb{N}}}(|\nabla u|^2+b(\varepsilon z)u^2)\mathrm{d}z$$

is the norm in $H^1(\mathbb{R}^N)$. In fact that $d=\max\{1,b(\varepsilon z)\}$ then $\|u\|_H\leq\|u\|_b\leq d\|u\|_H$, i.e., $\|u\|_b$ is an equivalent norm by $\|u\|_H$. We know that the nonnegative weak solutions of Eq. $(E_{\varepsilon,\lambda})$ are equivalent to the critical points of $J_{\varepsilon,\lambda}$. Here we study the existence and multiplicity of positive solutions of Eq. $(E_{\varepsilon,\lambda})$ in $\mathbb{R}^{\mathbb{N}}$.

We organize this paper in this way. In Section 2, we apply the argument of Tarantello [5] to divide the Nehari manifold $M_{\varepsilon,\lambda}$ into two parts $M^+_{\varepsilon,\lambda}$ and $M^-_{\varepsilon,\lambda}$. In Section 3, we show that the existence of a positive ground state solution $u_0\in M^+_{\varepsilon,\lambda}$ of Eq. $(E_{\varepsilon,\lambda})$. In Section 4, there are at least k critical points $u_1,\cdots,u_k\in M^-_{\varepsilon,\lambda}$ of $J_{\varepsilon,\lambda}$ such that $J_{\varepsilon,\lambda}(u_i)=\beta^i_{\varepsilon,\lambda}((PS)$-value) for $1\leq i\leq k$. Let

$$S=\sup_{u\in H^1(\mathbb{R}^{\mathbb{N}}),\|u\|_H=1}\|u\|_{L^p},$$

then $\|u\|_{L^p}\leq S\|u\|_H$ for every $u\in H^1(\mathbb{R}^{\mathbb{N}})\setminus\{0\}$.

2 Main results

Theorem 2.1. *Under assumptions a_1 and h_1, if*

(a)

$$0<\lambda<\Lambda=(p-2)\left(\frac{2-q}{a_{\max}}\right)^{\frac{2-q}{p-2}}\left((p-q)S^2\right)^{\frac{q-p}{p-2}}\|h\|_{\#}^{-1},$$

where $\|h\|_{\#}$ is the norm in $L^{\frac{p}{p-q}}(\mathbb{R}^N)$, then Eq. $(E_{\varepsilon,\lambda})$ accepts at least a positive ground state solution, (see Theorem 3.4).

(b) *Under assumptions a_1,a_2 and h_1, if λ is large enough, then Eq. $(E_{\lambda,\mu})$ archives at least $k+1$ positive solutions, (see Theorem 4.10).*

For the semilinear elliptic equations

$$\begin{cases}-\Delta u+u=a(\varepsilon z)u^{p-1}, & \text{in } \mathbb{R}^{\mathbb{N}};\\ u\in H^1(\mathbb{R}^{\mathbb{N}}),\end{cases}$$

if $a=a_{\max}$ and $\Omega=\{u\in H^1(\mathbb{R}^{\mathbb{N}})\setminus\{0\}|u_+\not\equiv 0$ and $\langle I'_{max}(u),u\rangle=0\}$. We define the energy functional

$$I_{\max}=\frac{1}{2}\|u\|_H^2-\frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}}a_{\max}(\varepsilon z)u_+^p\mathrm{d}z,$$

then $\gamma_{\max}=\inf_{u\in\Omega}I_{\max}(u)$.

Lemma 2.1. *We have*

$$\gamma_{\max}=\frac{p-2}{2p}(a_{\max}S^p)^{\frac{-2}{(p-2)}}>0.$$

Proof. If

$$I_{\max}=\frac{1}{2}\|u\|_H^2-\frac{1}{P}\int_{\mathbb{R}^{\mathbb{N}}}a_{\max}u_+^p\mathrm{d}z,$$

then

$$\gamma_{\max}=\gamma_{\max}(\Omega)=\Big(\frac{1}{2}-\frac{1}{p}\Big)\gamma(\Omega)^{\frac{2p}{2-p}};$$

$$\gamma(\Omega)=\sup\left\{\int_{\mathbb{R}^{\mathbb{N}}}a_{\max}u^p\,\middle|\,u\in H^1(\mathbb{R}^N)\text{ and }\|u\|_H=1\right\}=a_{\max}^{\frac{1}{p}}.$$

Moreover $\gamma_{\max}=(\frac{1}{2}-\frac{1}{p})(a_{\max}^{\frac{1}{p}}S)^{\frac{2p}{p-2}}>0.$ □

Definition 2.1. *We define the Palais-Smale (denoted by (PS))-sequences, (PS)-value, and (PS)-conditions in $H^1(\mathbb{R}^N)$ for $J_{\varepsilon,\lambda}$ as follows.*

(i) *For $\beta\in\mathbb{R}$, a sequence $\{u_n\}$ is a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^N)$ for $J_{\varepsilon,\lambda}$ if $J_{\varepsilon,\lambda}(u_n)=\beta+o_n(1)$ and $J'_{\varepsilon,\lambda}(u_n)=o_n(1)$ strongly in $H^{-1}(\mathbb{R}^N)$ as $n\longrightarrow\infty$, where $H^{-1}(\mathbb{R})^N$ is the dual space of $H^1(\mathbb{R}^N)$;*

(ii) *$\beta\in\mathbb{R}$ is a (PS)-value in $H^1(\mathbb{R}^{\mathbb{N}})$ for $J_{\varepsilon,\lambda}$ if there is a $(PS)_\beta$-sequence in $H^1(\mathbb{R})$ for $J_{\varepsilon,\lambda}$;*

(iii) *$J_{\varepsilon,\lambda}$ satisfy the $(PS)_\beta$-condition in $H^1(\mathbb{R}^N)$ if every $(PS)_\beta$-sequence in $H^1(\mathbb{R}^N)$ for $J_{\varepsilon,\lambda}$ includes a convergent subsequence.*

Next, since $J_{\varepsilon,\lambda}$ is not bounded form below in $H^1(\mathbb{R}^N)$, we consider the Nehari manifold

$$M_{\varepsilon,\lambda}=\{u\in H^1(\mathbb{R}^N)\setminus 0|u_+\not\equiv 0,\qquad\text{and}\qquad \langle J'_{\varepsilon,\lambda}(u),u\rangle=0\},\tag{2.1}$$

where

$$\langle J'_{\varepsilon,\lambda}(u),u\rangle=\|u\|_H^2-\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\mathrm{d}z-\lambda\int_{\mathbb{R}^{\mathbb{N}}}h(\varepsilon z)u_+^q\mathrm{d}z.$$

Notice $M_{\varepsilon,\lambda}$ includes all nonnegative solutions of Eq. $(E_{\lambda,\mu})$.

Lemma 2.2. *The energy functional $J_{\varepsilon,\lambda}$ is coercive and bounded from below on $M_{\varepsilon,\lambda}$.*

Proof. For $u\in M_{\varepsilon,\lambda}$, the Holder inequality $(p_1=p/(p-q),p_2=p/q)$ and the Sobolev embedding we get

$$\begin{aligned}J_{\varepsilon,\lambda}(u)&=\Big(\frac{1}{2}-\frac{1}{p}\Big)\|u\|_H^2-\Big(\frac{1}{q}-\frac{1}{p}\Big)\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(\varepsilon z)u_+^q\mathrm{d}z\\&\geq\Big(\frac{1}{2}-\frac{1}{p}\Big)\|u\|_H^2-\Big(\frac{1}{q}-\frac{1}{p}\Big)\lambda\|h\|_\# S^q\|u\|_H^q\\&\geq\frac{\|u\|_H^q}{p}\left[\frac{p-2}{2}\|u\|_H^{2-q}-(\frac{p-q}{q})\lambda\|h\|_\# S^q\right]\geq 0,\end{aligned}$$

where

$$C_1=(p-2)/2>0 \qquad \text{and} \qquad C_2=((p-q)/q)\lambda\,\|h\|_\# S^q>0,$$

i.e, we have that $J_{\varepsilon,\lambda}$ is coercive and bounded from below on $M_{\varepsilon,\lambda}$. □

Definition 2.2. *Define* $\psi_{\varepsilon,\lambda}(u)=\langle J'_{\varepsilon,\lambda}(u),u\rangle$.

Under assumptions for $u\in M_{\varepsilon,\lambda}$, we get

$$\begin{aligned}\langle \psi'_{\varepsilon,\lambda}(u),u\rangle &=2\|u\|_H^2-p\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\,\mathrm{d}z-\lambda q\int_{\mathbb{R}^{\mathbb{N}}}h(\varepsilon z)u_+^q\,\mathrm{d}z\\ &=(2-p)\,\|u\|_H^2+(p-q)\lambda\int_{\mathbb{R}^{\mathbb{N}}}h(\varepsilon z)u_+^q\,\mathrm{d}z\\ &=(2-q)\,\|u\|_H^2+(q-p)\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\,\mathrm{d}z.\end{aligned} \tag{2.2}$$

We apply the method in Tarantello [5], suppose

$$\begin{aligned}M^+_{\varepsilon,\lambda}&=\{u\in M_{\varepsilon,\lambda}\,|\,\langle\psi'_{\varepsilon,\lambda}(u),u\rangle>0\};\\ M^0_{\varepsilon,\lambda}&=\{u\in M_{\varepsilon,\lambda}\,|\,\langle\psi'_{\varepsilon,\lambda}(u),u\rangle=0\};\\ M^-_{\varepsilon,\lambda}&=\{u\in M_{\varepsilon,\lambda}|\langle\psi'_{\varepsilon,\lambda}(u),u\rangle<0\}.\end{aligned}$$

Lemma 2.3. *Under assumptions* a_1,a_2 *and* h_1, *if* $0<\lambda<\Lambda$, *then* $M^0_{\varepsilon,\lambda}=\emptyset$.

Proof. On the contrary, there is a number $\lambda_0\in\mathbb{R}$ and $0<\lambda_0<\Lambda$ such that $M^0_{\lambda_0}=\emptyset$. Then for $u\in M^0_{\lambda_0}$, by (2.2), we have

$$\|u\|_H^2=\frac{p-q}{p-2}\lambda_0\int_{\mathbb{R}^{\mathbb{N}}}h(\varepsilon z)u_+^q\,\mathrm{d}z=\frac{p-q}{2-q}\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\,\mathrm{d}z.$$

By the Holder and the Sobolev embedding theorem, we obtain

$$\|u\|_H\ge\left[\frac{(2-q)}{(p-q)a_{\max}}S^{-p}\right]^{\frac{1}{(p-2)}} \qquad \text{and} \qquad \|u\|_H\le\left(\frac{p-q}{p-2}\lambda_0\,\|h\|_\# S^q\right)^{\frac{1}{2-q}}.$$

Thus,

$$\lambda_0\ge(p-2)\Big(\frac{2-q}{a_{\max}}\Big)^{\frac{2-q}{p-2}}\big((p-q)S^2\big)^{\frac{q-p}{p-2}}\,\|h\|_\#^{-1}=\Lambda.$$

This makes a contradiction. □

Lemma 2.4. *Suppose that u is a local minimizer for* $J_{\varepsilon,\lambda}$ *on* $M_{\varepsilon,\lambda}$ *and* $u\in M^0_{\varepsilon,\lambda}$. *Then* $J'_{\varepsilon,\lambda}(u)=0$ *in* $H^{-1}(\mathbb{R}^{\mathbb{N}})$.

Proof. See [6, Theorem 2.3]. □

Lemma 2.5. *For each $u \in M^{+}_{\varepsilon,\lambda}$, we have*

$$\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z) u_{+}^{q}\, dz > 0, \qquad \text{and} \qquad \| u \|_{H} < \left(\frac{p-q}{p-2} \lambda \| h \|_{\neq} S^{q} \right)^{\frac{1}{(2-q)}}.$$

Proof. For $u \in M^{+}_{\varepsilon,\lambda}$, we get

$$\begin{aligned}
&(2-p) \| u \|_{H}^{2} + (p-q)\lambda \int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z) u_{+}^{q}\, dz > 0, \\
&(p-q)\lambda \int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z) u_{+}^{q}\, dz > (2-p) \| u \|_{H}^{2}, \\
&\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z) u_{+}^{q}\, dz > \frac{(2-p)}{\lambda(p-q)} \| u \|_{H}^{2} > 0.
\end{aligned}$$

For every $u \in M^{+}_{\varepsilon,\lambda} \subset M_{\varepsilon,\lambda}$, by (2.2), we apply the Holder inequality $(p_1 = p/(p-q), p_2 = p/q)$ to obtain that

$$\begin{aligned}
&0 < (p-q) \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z) \| u_{+}^{q}\, dz - (p-2) \| u \|_{H}^{2} \leq (p-q)\lambda \| h \|_{\#} S^{q} \| u \|_{H}^{q} - (p-2) \| u \|_{H}^{2}, \\
&\|u\|_{H} \leq \left(\frac{p-q}{p-2} \lambda \| h \|_{\#} S^{q} \right)^{\frac{1}{2-q}}.
\end{aligned}$$

This completes the proof. □

Lemma 2.6. *For each $u \in M^{-}_{\varepsilon,\lambda}$, we have*

$$\| u \|_{H} > \left[\frac{2-q}{(p-q)a_{\max}} S^{p} \right]^{\frac{1}{p-2}}.$$

Proof. For every $u \in M^{-}_{\varepsilon,\lambda}$, by (2.2), we have that

$$\| u \|_{H}^{2} < \frac{p-q}{2-q} \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z) u_{+}^{p}\, dz \leq \frac{p-q}{2-q} S^{p} \| u \|_{H}^{p} a_{\max}, \qquad \| u \|_{H} \geq \left[\frac{(2-q)}{(p-q)a_{\max}} S^{-p} \right]^{\frac{1}{(p-2)}}.$$

This completes the proof. □

Lemma 2.7. *If $0 < \lambda < \frac{q\Lambda}{2}$ and $u \in M^{-}_{\varepsilon,\lambda}$, then $J_{\varepsilon,\lambda}(u) > 0$.*

Proof. For $u \in M^{-}_{\varepsilon,\lambda}$, we have

$$\begin{aligned}
J_{\varepsilon,\lambda}(u) &= \left(\frac{1}{2} - \frac{1}{p} \right) \| u \|_{H}^{2} - \left(\frac{1}{q} - \frac{1}{p} \right) \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z) u_{+}^{q}\, dz \\
&\geq \frac{\| u \|_{H}^{q}}{p} \left[\frac{P-2}{2} \| u \|_{H}^{2-q} - \frac{p-q}{q} \lambda \| h \|_{\#} S^{q} \right] \\
&> \frac{1}{p} \left(\frac{2-p}{(p-q)a_{\max} S^{p}} \right)^{\frac{q}{p-2}} \left(\frac{p-2}{2} \left(\frac{2-q}{(p-q)a_{\max} S^{p}} \right)^{\frac{2-q}{p-2}} - \frac{p-q}{q} \lambda \| h \|_{\#} S^{q} \right).
\end{aligned}$$

So $J_{\varepsilon,\lambda}(u) \geqslant d_0 > 0$ for some $d_0 = d_0(\varepsilon, p, q, S, \lambda, \| h \|_{\#}, a_{\max})$. □

For $u\in H^1(\mathbb{R}^N)\setminus\{0\}$ and $u_+\not\equiv 0$, let

$$\bar{l}=\bar{l}(u)=\left[\frac{(2-q)\parallel u\parallel_H^2}{(p-q)\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\mathrm{d}z}\right]^{\frac{1}{p-2}}>0.$$

Lemma 2.8. *For every $u\in H^1(\mathbb{R}^{\mathbb{N}})\setminus\{0\}$ and $u_+\not\equiv 0$, we have that, if*

$$\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(z)u_+^q\mathrm{d}z=0,$$

then there is a unique positive number $l^-=l^-(u)>\bar{l}$ such that $l^-u\in M_{\varepsilon,\lambda}^-$ and $J_{\varepsilon,\lambda}(l^-u)=\sup_{l\geqslant 0}J_{\varepsilon,\lambda}(lu)$.

Proof. For every $u\in H^1(\mathbb{R}^{\mathbb{N}})\setminus\{0\}$ and $u_+\not\equiv 0$, define

$$k(l)=k_u(l)=l^{2-q}\parallel u\parallel_H^2-l^{p-q}\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\mathrm{d}z,\qquad \text{for } l\geqslant 0.$$

Clearly, we get that $k(0)=0$ and $k(l)\to-\infty$ as $l\to\infty$ since

$$k'(l)=\frac{1}{l^{q+1}}\left[(2-q)\parallel lu\parallel_H^2-(p-q)\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)(lu_+)^p\mathrm{d}z\right],\qquad \text{for } l\geqslant 0,$$

then $k'(\bar{l})=0$, $k'(l)>0$ for $0<l<\bar{l}$, and $k'(l)<0$ for $l>\bar{l}$. Thus, $k(l)$ get its maximum at $\bar{l}$. Furthermore, by the Sobolev embedding theorem, we have that

$$\begin{aligned}k(\bar{l})=&\left(\frac{(2-q)\parallel u\parallel_H^2}{(p-q)\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\mathrm{d}z}\right)^{\frac{(2-q)}{(p-2)}}\parallel u\parallel_H^2\\&-\left(\frac{(2-q)\parallel u\parallel_H^2}{(p-q)\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\mathrm{d}z}\right)^{\frac{(p-q)}{(}p-2)}\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\mathrm{d}z\\&\geq(p-2)(2-q)^{\frac{2-q}{p-2}}(p-q)^{\frac{q-p}{p-2}}S^{\frac{p(q-2)}{p-2}}\parallel u\parallel_H^q, \qquad (2.3)\\&\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(z)u_+^q\mathrm{d}z=0.\end{aligned}$$

There is a unique positive number $l^-=l^-(u)>\bar{l}$ such that

$$k(l^-)=\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(z)u_+^q\mathrm{d}z=0,$$

and $k'(l^-)>0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}l}J_{\varepsilon,\lambda}(lu)=\frac{1}{l}\left(\parallel lu\parallel_H^2-\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)(lu_+)^p\mathrm{d}z-\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(\varepsilon z)(lu_+)^q\mathrm{d}z\right)|_{l=l^-}=0,$$

$$\frac{\mathrm{d}^2}{\mathrm{d}l^2}J_{\varepsilon,\lambda}(lu)=\frac{1}{l^2}\left[\parallel lu\parallel_H^2-(p-1)\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)(lu_+)^p\mathrm{d}z-(q-1)\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(\varepsilon z)(lu_+)^q\mathrm{d}z\right]|_{l=l^-}<0,$$

and

$$J_{\varepsilon,\lambda}(lu)\to-\infty,\qquad \text{as } l\to\infty.$$

Furthermore, it is not difficult to find that $l^-u\in M^-_{\varepsilon,\lambda}$ and $J_{\varepsilon,\lambda}(l^-u)=\sup_{l\geq 0}J_{\varepsilon,\lambda}(lu)$. □

Lemma 2.9. *If $0<\lambda<\Lambda$ and $\int_{\mathbb{R}^N}\lambda h(\varepsilon z)u_+^q\,dz>0$, then there is unique positive number $l^+=l^+(u)<\bar{l}<l^-=l^-(u)$ such that $l^+u\in M^-_{\varepsilon,\lambda}$, and*

$$J_{\varepsilon,\lambda}(l^+u)=\inf_{0\leq l\leq\bar{l}}J_{\varepsilon,\lambda}(lu),\qquad J_{\varepsilon,\lambda}(l^-u)=\sup_{l\geq\bar{l}}J_{\varepsilon,\lambda}(lu).$$

Proof. Since $0<\lambda<\Lambda$ and $\int_{\mathbb{R}^N}\lambda h(\varepsilon z)u_+^q\,dz>0$, by (2.3), then

$$\begin{aligned}k(0)=0<\lambda\int_{\mathbb{R}^N}h(\varepsilon z)u_+^q\,dz&\leq\lambda\,\|h\|_\#S^q\,\|u\|_H^q\\&<(P-2)(2-q)^{\frac{2-q}{p-2}}(p-q)^{\frac{q-p}{p-2}}S^{\frac{p(q-2)}{p-2}}\,\|u\|_H^q\leq k(\bar{l}).\end{aligned}$$

It follows that there are unique positive number $l^+=l^+(u)$ and $l^-=l^-(u)$ such that

$$l^+<\bar{l}<l^-,\qquad k(l^+)=\int_{\mathbb{R}^N}\lambda h(\varepsilon z)u_+^q\,dz=k(l^-)\qquad\text{and}\qquad k'(l^-)<0<k'(l^+).$$

We also have that

$$l^+u\in M^+_{\varepsilon,\lambda},\quad l^-u\in M^-_{\varepsilon,\lambda},\quad J_{\varepsilon,\lambda}(l^+u)\leq J_{\varepsilon,\lambda}(lu)\leq J_{\varepsilon,\lambda}(l^-u)$$

for every $l\in[l^+,l^-]$, and $J_{\varepsilon,\lambda}(l^+u)\leq J_{\varepsilon,\lambda}(lu)$ for every $l\in[0,\bar{l}]$. Hence,

$$J_{\varepsilon,\lambda}(l^+u)=\inf_{0\leq l\leq\bar{l}}J_{\varepsilon,\lambda}(lu),\qquad J_{\varepsilon,\lambda}(l^-u)=\sup_{l\geq\bar{l}}J_{\varepsilon,\lambda}(lu).$$

This completes the proof. □

Applying Lemma 2.6 ($M^0_{\varepsilon,\lambda}=\emptyset$ for $0<\lambda<\Lambda$). We have $M_{\varepsilon,\lambda}=M^+_{\varepsilon,\lambda}\cup M^-_{\varepsilon,\lambda}$, where

$$\begin{aligned}M^+_{\varepsilon,\lambda}&=\left\{u\in M_{\varepsilon,\lambda}\,\middle|\,(2-q)\,\|u\|_H^2-(p-q)\int_{\mathbb{R}^N}a(\varepsilon z)u_+^p\,dz>0\right\},\\M^-_{\varepsilon,\lambda}&=\left\{u\in M_{\varepsilon,\lambda}\,\middle|\,(2-q)\,\|u\|_H^2-(p-q)\int_{\mathbb{R}^N}a(\varepsilon z)u_+^p\,dz<0\right\}.\end{aligned}$$

Define

$$\alpha_{\varepsilon,\lambda}=\inf_{u\in M_{\varepsilon,\lambda}}J_{\varepsilon,\lambda}(u);\qquad\alpha^+_{\varepsilon,\lambda}=\inf_{u\in M^+_{\varepsilon,\lambda}}J_{\varepsilon,\lambda}(u);\qquad\alpha^-_{\varepsilon,\lambda}=\inf_{u\in M^-_{\varepsilon,\lambda}}J_{\varepsilon,\lambda}(u).$$

Lemma 2.10. *If $0<\lambda<\Lambda$, then $\alpha_{\varepsilon,\lambda}\leq\alpha^+_{\varepsilon,\lambda}<0$.*

Proof. Suppose $u \in M^+_{\varepsilon,\lambda}$, by (2.2) we get that

$$(p-2)\,\|u\|_H^2<(p-q)\lambda\int_{\mathbb{R}^{\mathbb{N}}} h(z)u_+^q\,\mathrm{d}z.$$

Then

$$\begin{aligned} J_{\varepsilon,\lambda}(u)&=\Big(\frac{1}{2}-\frac{1}{p}\Big)\,\|u\|_H^2-\Big(\frac{1}{q}-\frac{1}{p}\Big)\lambda\int h(\varepsilon z)u_+^q\,\mathrm{d}z\\ &<\Big[\Big(\frac{1}{2}-\frac{1}{p}\Big)-\Big(\frac{1}{q}-\frac{1}{p}\Big)\frac{p-2}{p-q}\Big]\,\|u\|_H^2\\ &=-\frac{(2-q)(p-2)}{2pq}\,\|u\|_H^2<0. \end{aligned}$$

By the definition $\alpha_{\varepsilon,\lambda}$ and $\alpha^+_{\varepsilon,\lambda}$, we conclude that $\alpha_{\varepsilon,\lambda}\le\alpha^+_{\varepsilon,\lambda}<0$. □

Lemma 2.11. *If $0<\lambda<q\Lambda/2$, then $\alpha^-_{\varepsilon,\lambda}\ge d_0>0$ for some $d_0=d_0(\varepsilon,\lambda,p,q,S,\|h\|_\#)$.*

Proof. See [4, Lemma 2.5]. □

Lemma 2.12. *We conclude*

(a) *There is a $(PS)_{\alpha_{\varepsilon,\lambda}}$-sequence $\{u_n\}$ in $M_{\varepsilon,\lambda}$ for $J_{\varepsilon,\lambda}$;*

(b) *There is a $(PS)_{\alpha^+_{\varepsilon,\lambda}}$-sequence $\{u_n\}$ in $M^+_{\varepsilon,\lambda}$ for $J_{\varepsilon,\lambda}$;*

(c) *There is a $(PS)_{\alpha^-_{\varepsilon,\lambda}}$-sequence $\{u_n\}$ in $M^-_{\varepsilon,\lambda}$ for $J_{\varepsilon,\lambda}$.*

3 Existence of a ground state solution

At first, we show that $J_{\varepsilon,\lambda}$ satisfy the $(PS)_\beta$-condition in $H^1(\mathbb{R}^{\mathbb{N}})$ for $\beta\in(-\infty,\gamma_{\max}-C_0\lambda^{\frac{2}{2-q}})$, where

$$C_0=(2-q)\,[(p-q)\,\|h\|_\#\,S^q]^{\frac{2}{2-q}}\Big/\Big[2pq(p-2)^{\frac{q}{2-q}}\Big].$$

Lemma 3.1. *Under some assumptions a_1,a_2,h_1 and $0<\lambda<\Lambda$. If $\{u_n\}$ is a $(PS)_\beta$- sequence in $H^1(\mathbb{R}^N)$ for $J_{\varepsilon,\lambda}$ with $u_n\rightharpoonup u$ weakly in $H^1(\mathbb{R}^{\mathbb{N}})$, then $J'_{\varepsilon,\lambda}(u)=0$ in $H^{-1}(\mathbb{R}^{\mathbb{N}})$.*

Proof. Suppose $\{u_n\}$ be a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^N)$ for $J_{\varepsilon,\lambda}$ such that $J_{\varepsilon,\lambda}(u_n)=\beta+o_n(1)$

and $J'_{\varepsilon,\lambda}(u_n)=o_n(1)$ in $H^{-1}(\mathbb{R}^N)$. Then

$$\begin{aligned}|\beta|+o_n(1)+\frac{d_n\|u_n\|_H}{p}&\geq J_{\varepsilon,\lambda}(u_n)-\frac{1}{p}\langle J'_{\varepsilon,\lambda}(u_n),u_n\rangle\\&=\Big(\frac{1}{2}-\frac{1}{p}\Big)\|u_n\|_H^2-\Big(\frac{1}{q}-\frac{1}{p}\Big)\int_{\mathbb{R}^\mathbb{N}}\lambda h(\varepsilon z)(u_n)_+^q\mathrm{d}z\\&\geq\frac{p-2}{2p}\|u_n\|_H^2-\frac{p-q}{pq}\lambda\|h\|_\# S^q\|u_n\|_H^q\\&\geq\frac{p-2}{2p}\|u_n\|_H^2,\end{aligned}$$

then

$$\|u_n\|\geq 2p(|\beta|+o_n(1))/(2d_n-(p-2)),$$

where $d_n=o_n(1)$ as $n\to\infty$. It follows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^\mathbb{N})$. Furthermore there are a subsequence $\{u_n\}$ and $u\in H^1(\mathbb{R}^\mathbb{N})$ such that $J'_{\varepsilon,\lambda}(u)=0$ in $H^{-1}(\mathbb{R}^\mathbb{N})$. □

Lemma 3.2. *Under some assumptions a_1,a_2,h_1 and $0<\lambda<\Lambda$. If $\{u_n\}$ is a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^N)$ for $J_{\varepsilon,\lambda}$ with $u_n\rightharpoonup u$ weakly in $H^1(\mathbb{R}^\mathbb{N})$, $J_{\varepsilon,\lambda}(u)\geqslant -C_0\lambda^{\frac{2}{2-q}}\geqslant -C'_0$, where*

$$C'_0=\Big((p-2)(2-q)^{\frac{p}{p-2}}\Big)\Big/\Big(2pq(a_{\max}(p-q))^{\frac{2}{p-2}}S^{\frac{2p}{p-2}}\Big).$$

Proof. we have $\langle J'_{\varepsilon,\lambda}(u),u\rangle=0$, that is,

$$\int_{\mathbb{R}^\mathbb{N}}a(\varepsilon z)u_+^p\mathrm{d}z=\|u\|_H^2-\int_{\mathbb{R}^\mathbb{N}}\lambda h(\varepsilon z)u_+^q\mathrm{d}z.$$

Hence, by the Young inequality ($p_1=\frac{2}{q}$ and $p_2=\frac{2}{2-q}$).

$$\begin{aligned}J_{\varepsilon,\lambda}(u)&=\left(\frac{1}{2}-\frac{1}{p}\right)\|u\|_H^2-\left(\frac{1}{q}-\frac{1}{p}\right)\int_{\mathbb{R}^\mathbb{N}}\lambda h(\varepsilon z)u^q\mathrm{d}z\\&\geq\frac{p-2}{2p}\|u\|_H^2-\frac{p-q}{pq}\lambda\|h\|_\# S^q\|u\|_H^q\\&\geq\frac{p-2}{2p}\|u\|_H^2-\frac{p-2}{pq}\left[\frac{q\|u\|_H^2}{2}+\Big(\frac{p-q}{p-2}\lambda\|h\|_\# S^q\Big)^{\frac{2}{2-q}}\frac{2-q}{2}\right]\\&=-\lambda^{\frac{2}{2-q}}(2-q)[(p-q)\|h\|_\# S^q]^{\frac{2}{2-q}}\Big/\Big[2pq(p-2)^{\frac{q}{2-q}}\Big]\\&\geq-\frac{(p-2)(2-q)^{\frac{p}{p-2}}}{2pq[a_{\max}(p-q)]^{\frac{2}{p-2}}S^{\frac{2p}{p-2}}}\\&=-C'_0.\end{aligned}$$

This completes the proof. □

Lemma 3.3. *Assume that a, b and h satisfy a_1 and h_1. If $0<\lambda<\Lambda$. Then $J_{\varepsilon,\lambda}$ satisfy the $(PS)_\beta$-condition in $H^1(\mathbb{R}^{\mathbb{N}})$ for $\beta\in(-\infty,\gamma_{\max}-C_0\lambda^{\frac{2}{2-q}})$.*

Proof. Suppose $\{u_n\}$ be a $(PS)_\beta$-sequence in $H^1(\mathbb{R}^N)$ for $J_{\varepsilon,\lambda}$ such that

$$J_{\varepsilon,\lambda}(u_n)=\beta+o_n(1),$$

and $J'_{\varepsilon,\lambda}(u_n)=o_n(1)$ in $H^{-1}(\mathbb{R}^N)$. Then it follows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^{\mathbb{N}})$. Moreover, there are a subsequence $\{u_n\}$ and $u\in H^1(\mathbb{R}^{\mathbb{N}})$ such that $J'_{\varepsilon,\lambda}(u)=0$ in $H^{-1}(\mathbb{R}^{\mathbb{N}})$. $u_n\rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $u_n\to u$ a.e. in $\mathbb{R}^N$,$u_n\rightharpoonup u$ strongly in $L^s_{\rm loc}(\mathbb{R}^{\mathbb{N}})$ for every $1\le s<2^*$. Next, claim that

$$\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)|u_n-u|^q\mathrm{d}z\to 0,\qquad \text{as } n\to\infty. \tag{3.1}$$

Using the Brezis-Lieb lemma to get

$$\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)(u_n-u)^q_+\mathrm{d}z=\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)(u_n)^q_+\mathrm{d}z-\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)u^q dz+o_n(1).$$

For every $\sigma>0$, there is $r>0$ so that

$$\int_{[B^N(0;r)]^c} h(\varepsilon z)^{\frac{p}{p-q}}\mathrm{d}z<\sigma.$$

By the Holder inequality and the Sobolev embedding theorem, we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)\,|\,u_n-u\,|^q\,\mathrm{d}z\right|\\ \le&\int_{B^N(0;r)} h(\varepsilon z)\,|\,u_n-u\,|^q\,\mathrm{d}z+\int_{[B^N(0;r)]^c} h(\varepsilon z)\,|\,u_n-u\,|^q\,\mathrm{d}z\\ \le&\,\|\,h\,\|_\#\left(\int_{\mathbb{R}^{\mathbb{N}}}|\,u_n-u\,|^p\,\mathrm{d}z\right)^{\frac{q}{p}}+s^q\left(\int_{\mathbb{R}^{\mathbb{N}}} h(\varepsilon z)^{\frac{p}{p-q}}\mathrm{d}z\right)^{\frac{p-q}{p}}\|\,u_n-u\,\|^q_H\\ \le&\, o_n(1)+\sigma C'.\end{aligned}$$

$\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $u_n\to u$ in $L^q_{\rm loc}(\mathbb{R}^N)$. Applying a_1 and $u_n\to u$ in $L^q_{\rm loc}(\mathbb{R}^N))$, we get that

$$\int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)(u_n-u)^p_+\mathrm{d}z=\int_{\mathbb{R}^{\mathbb{N}}} a_{\max}(u_n-u)^p_+\mathrm{d}z+o_n(1). \tag{3.2}$$

Let $p_n = u_n - u$. Suppose $p_n \nrightarrow 0$ strongly in $H^1(\mathbb{R}^N)$. By (3.1), (3.2), we conclude that

$$\begin{aligned}\| p_n \|_H^2 =& \| u_n \|_H^2 - \| u \|_H^2 + o_n(1)\\ =& \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)(u_n)_+^p \mathrm{d}z - \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z)(u_n)_+^q \mathrm{d}z\\ &- \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z) u^p \mathrm{d}z + \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z) u^q \mathrm{d}z + o_n(1)\\ =& \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)(u_n - u)_+^p \mathrm{d}z + o_n(1)\\ =& \int_{\mathbb{R}^{\mathbb{N}}} a_{\max}(p_n)_+^p \mathrm{d}z + o_n(1),\end{aligned}$$

also

$$I_{\max}(u) = \frac{1}{2} \| u_n \|_H^2 - \frac{1}{p} \int_{\mathbb{R}^{\mathbb{N}}} a_{\max} u_+^p \mathrm{d}z,$$

then

$$I_{\max}(p_n) = \frac{1}{2} \| p_n \|_H^2 - \frac{1}{p} \int_{\mathbb{R}^{\mathbb{N}}} a_{\max}(p_n)_+^p \mathrm{d}z = \Big(\frac{1}{2} - \frac{1}{p}\Big) \| p_n \|_H^2 + o_n(1) > 0.$$

By Theorem 4.3 in Wang [7], there is a sequence $\{s_n\} \subset \mathbb{R}^+$ such that

$$s_n = 1 + o_n(1), \qquad \{s_n p_n\} \subset \Omega, \qquad \text{and } I_{\max}(s_n p_n) = I_{\max}(p_n) + o_n(1).$$

It follows that

$$\begin{aligned}\gamma_{\max} \le& I_{\max}(s_n p_n) = I_{\max}(p_n) + o_n(1) = J_{\varepsilon,\lambda}(u_n) - J_{\varepsilon,\lambda}(u) + o_n(1)\\ =& \beta - J_{\varepsilon,\lambda}(u) + o_n(1) = J_{\varepsilon,\lambda}(u_n) - J_{\varepsilon,\lambda}(u)\\ =& J_{\varepsilon,\lambda}(p_n) \to o_n(1) < \gamma_{\max},\end{aligned}$$

which is a contradiction. Hence, $u_n \to u$ strongly in $H^1(\mathbb{R}^{\mathbb{N}})$. □

Theorem 3.1. *Under some assumptions a_1, a_2, h_1 and $0<\lambda<\Lambda$, then there is at least one positive ground state solution u_0 of Eq. $(E_{\varepsilon,\lambda})$ in $\mathbb{R}^N$. Moreover, we have that $u_0 \in M_{\varepsilon,\lambda}^+$ and*

$$J_{\varepsilon,\lambda}(u_0) = \alpha_{\varepsilon,\lambda} = \alpha_{\varepsilon,\lambda}^+ \ge -C_0 \lambda^{\frac{2}{2-q}}.$$

Proof. There is a minimizing sequence $\{u_n\} \subset M_{\varepsilon,\lambda}$ for $J_{\varepsilon,\lambda}$ such that

$$J_{\varepsilon,\lambda}(u_n) = \alpha_{\varepsilon,\lambda} + o_n(1), \qquad \text{and } J'_{\varepsilon,\lambda}(u_n) = o_n(1) \qquad \text{in } H^{-1}(\mathbb{R}^{\mathbb{N}}).$$

By Lemma 3.2 (i), there is a subsequence $\{u_n\}$ and $u_0 \in H^1(\mathbb{R}^{\mathbb{N}})$. We claim that

$$u_0 \in M_{\varepsilon,\lambda}^+ \ (M_{\varepsilon,\lambda}^0 = \emptyset \text{ for } 0 < \lambda < \Lambda) \qquad \text{and} \qquad J_{\varepsilon,\lambda}(u_0) = \alpha_{\varepsilon,\lambda}.$$

On the contrary that $u_0 \in M^-_{\varepsilon,\lambda}$, we get that

$$\int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z)(u_0)^q_+ \mathrm{d}z > 0.$$

Otherwise,

$$\begin{aligned}\| u_n \|^2_H - \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z)(u_n)^p_+ \mathrm{d}z &= \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z)(u_n)^q_+ \mathrm{d}z \\ &= \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z)(u_0)^q_+ \mathrm{d}z + o_n(1) = o_n(1).\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \Big(\frac{1}{2} - \frac{1}{p}\Big) \| u_n \|^2_H = \alpha_{\varepsilon,\lambda};$$

that contradicts to $\alpha_{\varepsilon,\lambda} < 0$. By Lemma 2.11 (ii), then there are positive numbers $l^+ < \bar{l} < l^- = 1$ such that $l^+ u_0 \in M^+_{\varepsilon,\lambda}$, $l^- u_0 \in M^-_{\varepsilon,\lambda}$ and that is a contradiction. Hence,

$$u_0 \in M^+_{\varepsilon,\lambda}, \qquad -C_0 \lambda^{\frac{2}{2-q}} \le J_{\varepsilon,\lambda}(u_0) = \alpha_{\varepsilon,\lambda} = \alpha^+_{\varepsilon,\lambda}.$$

This completes the proof. □

4 Existence of multiple solutions

From this time, we assume that a and h satisfy a_1, a_2 and h_1. Suppose $w \in H^1(\mathbb{R}^{\mathbb{N}})$ be the positive ground state solution of Eq. (E_0) in $\mathbb{R}^{\mathbb{N}}$ for $a \equiv a_{\max}$.

(i) $w \in L^\infty(\mathbb{R}^{\mathbb{N}}) \cap C^{2,\theta}_{loc}(\mathbb{R}^{\mathbb{N}})$ for some $0 < \theta < 1$ and $\lim_{|z|\to\infty} w(z) = 0$.

(ii) For every $\varepsilon > 0$, there are positive numbers C_1, C^ε_2 and C^ε_3 such that for all

$$z \in \mathbb{R}^{\mathbb{N}} C^\varepsilon_2 \exp(-(1+\varepsilon)|z|) \le w(z) \le C_1 \exp(-|z|),$$

and

$$|\nabla w(z)| \le C^\varepsilon_3 \exp(-(1-\varepsilon)|z|).$$

For $1 \le i \le k$, we define

$$w^i_\varepsilon(z) = w\Big(z - \frac{a^i}{\varepsilon}\Big), \qquad \text{where } \ a(a^i) = a_{\max}.$$

Clearly, $w^i_\varepsilon(z) \in H^1(\mathbb{R}^{\mathbb{N}})$. By Lemma 2.11 (ii) there is a unique number $(l^i_\varepsilon)^- > 0$ so that $(l^i_\varepsilon)^- w^i_\varepsilon \in M^-_{\varepsilon,\lambda} \subset M_{\varepsilon,\lambda}$, where $1 \le i \le k$.

Lemma 4.1. *There is a number $t_0 > 0$ such that for $0 \le t < t_0$ and every $\varepsilon > 0$, we have that*

$$J_{\varepsilon,\lambda}(t w^i_\varepsilon) < \gamma_{\max}, \qquad \text{uniformly in } i$$

Proof. For every $\varepsilon > 0$, we have

$$J_{\varepsilon,\lambda}(tw_\varepsilon^i) = \frac{t^2}{2} \| w_\varepsilon^i \|_H^2 - \frac{t^p}{p} \int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z)(w_\varepsilon^i)^p \mathrm{d}z - \frac{t^q}{q} \int_{\mathbb{R}^\mathbb{N}} \lambda h(\varepsilon z)(w_\varepsilon^i)^q \mathrm{d}z.$$

Since $J_{\varepsilon,\lambda}$ is continuous in $H^1(\mathbb{R}^\mathbb{N})$, $\{w_\varepsilon^i\}$ is uniformly bounded in $H^1(\mathbb{R}^\mathbb{N})$ for every $\varepsilon > 0$ and $\gamma_{\max} > 0$ there is $t_0 > 0$ such that for $0 \le t \le t_0$ and every $\varepsilon > 0$

$$J_{\varepsilon,\lambda}(tw_\varepsilon^i) < \gamma_{\max}.$$

This completes the proof. □

Lemma 4.2. *There are positive numbers t_1 and ε_1 such that for every $t > t_1$ and $\varepsilon < \varepsilon_1$, we have that*

$$J_{\varepsilon,\lambda}(tw_\varepsilon^i) < 0, \qquad \text{uniformly in } i.$$

Proof. There is an $r_0 > 0$ such that $a(z) \ge a_{\max}/2$ for $z \in B^N(a^i : r_0)$ uniformly in i. Then is $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$

$$\begin{aligned} J_{\varepsilon,\lambda}(tw_\varepsilon^i) &= \frac{t^2}{2} \| w_\varepsilon^i \|_H^2 - \frac{t^p}{p} \int_{\mathbb{R}^\mathbb{N}} a(\varepsilon z)(w_\varepsilon^i)^p \mathrm{d}z - \frac{t^q}{q} \int_{\mathbb{R}^\mathbb{N}} \lambda h(\varepsilon z)(w_\varepsilon^i)^q \mathrm{d}z \\ &\le \frac{t^2}{2} \int_{\mathbb{R}^\mathbb{N}} [| \nabla w |^2 + w^2] - \frac{t^p}{2p} [| \nabla w |^2 + w^2] - \frac{t^p}{2p} \int_{\mathbb{R}^\mathbb{N}} a_{\max} w^p \mathrm{d}z. \end{aligned}$$

Thus, there is $t_1 > 0$ such that for every $t > t_1$ and $\varepsilon < \varepsilon_1$

$$J_{\varepsilon,\lambda}(tw_\varepsilon^i) < 0, \qquad \text{uniformly in } i.$$

This completes the proof. □

Lemma 4.3. *Suppose that a_1,a_2, and h_1 hold. If $0 < \lambda < q\Lambda/2$, then*

$$\lim_{\varepsilon \to 0^+} \sup_{t \ge 0} J_{\varepsilon,\lambda}(tw_\varepsilon^i) \le < \gamma_{\max}, \qquad \text{uniformly in } i.$$

Proof. By Lemma 4.1 we just try to indicate

$$\lim_{\varepsilon \to 0^+} \sup_{t_0 \le t \le t_1} J_{\varepsilon,\lambda}(tw_\varepsilon^i) \le \gamma_{\max}$$

uniformly in i; we learn that $\sup_{t\geq 0} I_{\max}(tw)=\gamma_{\max}$. For $t_0\leq t\leq t_1$, we get

$$\begin{aligned}
J_{\varepsilon,\lambda}(tw^i_\varepsilon)=&\frac{1}{2}\parallel tw^i_\varepsilon\parallel^2_H-\frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)(tw^i_\varepsilon)^p\mathrm{d}z-\frac{1}{q}\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(\varepsilon z)(tw^i_\varepsilon)^q\mathrm{d}z\\
=&\frac{t^2}{2}\int_{\mathbb{R}^{\mathbb{N}}}\Big[\Big|\bigtriangledown w\Big(z-\frac{a^i}{\varepsilon}\Big)\Big|^2+w\Big(z-\frac{a^i}{\varepsilon}\Big)^2\Big]\mathrm{d}z\\
&\quad-\frac{t^p}{p}\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)w\Big(z-\frac{a^i}{\varepsilon}\Big)^p\mathrm{d}z-\frac{t^q}{q}\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(\varepsilon z)w\Big(z-\frac{a^i}{\varepsilon}\Big)^q\mathrm{d}z\\
=&\Big\{\frac{t^2}{2}\int_{\mathbb{R}^{\mathbb{N}}}[|\bigtriangledown w|^2+w^2]\mathrm{d}z-\frac{t^p}{p}\Big\}\\
&\quad+\frac{t^p}{p}\int_{\mathbb{R}^{\mathbb{N}}}(a_{\max}-a(\varepsilon z)w\Big(z-\frac{a^i}{\varepsilon}\Big)^p\mathrm{d}z-\frac{t^q}{q}\lambda\int_{\mathbb{R}^{\mathbb{N}}}h(\varepsilon z)w\Big(z-\frac{a^i}{\varepsilon}\Big)^q\mathrm{d}z\\
\leq&\gamma_{\max}\frac{t_1^p}{p}\int_{\mathbb{R}^{\mathbb{N}}}(a_{\max}-a(\varepsilon z))w\Big(z-\frac{a^i}{\varepsilon}\Big)^p\mathrm{d}z-\frac{t_0^q}{q}\lambda\int_{\mathbb{R}^{\mathbb{N}}}h(\varepsilon z)w\Big(z-\frac{a^i}{\varepsilon}\Big)^q\mathrm{d}z.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^{\mathbb{N}}}(a_{\max}-a(\varepsilon z))w\Big(z-\frac{a^i}{\varepsilon}\Big)^p\mathrm{d}z=\int_{\mathbb{R}^{\mathbb{N}}}\Big[a_{\max}-a(\varepsilon z+a^i)\Big]w^p\mathrm{d}z=o(1)$$

as $\varepsilon\to 0^+$ uniformly in i. And

$$\lambda\int_{\mathbb{R}^{\mathbb{N}}}h(\varepsilon z)w\Big(z-\frac{a^i}{\varepsilon}\Big)^q\mathrm{d}z\leq\lambda\parallel h\parallel_\# S^q\parallel w\parallel^q_H=o(1)\quad\text{as }\ \varepsilon\to 0^+.$$

then

$$\lim_{\varepsilon\to 0^+}\sup{}_{t_0\leq t\leq t_1}J_{\varepsilon,\lambda}(tw^i_\varepsilon)\leq\gamma_{\max},\qquad\lim_{\varepsilon\to 0^+}\sup{}_{t\geq 0}J_{\varepsilon,\lambda}(tw^i_\varepsilon)\leq\gamma_{\max},$$

uniformly in i. □

Remark 4.1. Applying the results of Lemma 4.3, we can conclude that

$$0<d_0\leq\alpha^-_{\varepsilon,\lambda}\leq\gamma_{\max}+0(1),\qquad\text{as }\varepsilon\to 0^+.$$

Since there is $\varepsilon_0>0$ such that

$$\left\{\begin{array}{ll}0<\gamma_{\max}-C_0\lambda^{\frac{2}{2-q}}, & \text{for any }\varepsilon<\varepsilon_0,\\ \overline{B^N_{\rho_0}(a^i)}\cap\overline{B^N_{\rho_0}(a^j)}=\emptyset, & \text{for }1\leq i\neq j\leq k;\end{array}\right.\tag{4.1}$$

where

$$\overline{B^N_{\rho_0}(a^i)}=\{z\in\mathbb{R}\,||z-a^i|\leq\rho_0\}\qquad\text{and}\qquad a(a^i)=a_{\max}.$$

Define

$$\mathbf{k}=\{a^i\,|\,1\le i\le k\}\quad\text{and}\quad \mathbf{K}_{\frac{\rho_0}{2}}=\cup_{i=1}^k\overline{B^N_{\frac{\rho_0}{2}}(a^i)},$$

choosing $0\le\rho_0<1$. Suppose $\cup_{i=1}^k\overline{B^N_{\rho_0}(a^i)}\subset B^N_{r_0}(0)$ for some $r_0>0$. Let $Q_\varepsilon:H^1(\mathbb{R}^\mathbb{N})\setminus\{0\}\to\mathbb{R}^\mathbb{N}$ be given by

$$Q_\varepsilon(u)=\frac{\int_{\mathbb{R}^\mathbb{N}}\chi(\varepsilon z)\,|u|^p\,\mathrm{d}z}{\int_{\mathbb{R}^\mathbb{N}}|u|^p\,\mathrm{d}z},$$

where $\chi:\mathbb{R}^\mathbb{N}\to\mathbb{R}^\mathbb{N}$, $\chi(z)=z$ for $|z|\le r_0$, and $\chi(z)=r_0z/|z|$ for $|z|>r_0$. For every $1\le i\le k$, define

$$\begin{aligned}
&O^i_\varepsilon=\{u\in M^-_{\varepsilon,\lambda}\,|\,|Q_\varepsilon(u)-a^i|<\rho_0\};\\
&\partial O^i_\varepsilon=\{u\in M^-_{\varepsilon,\lambda}\,|\,|Q_\varepsilon(u)-a^i|=\rho_0\};\\
&\beta^i_{\varepsilon,\lambda}=\inf_{u\in O^i_\varepsilon}J_{\varepsilon,\lambda}(u)\qquad\text{and}\qquad \overline{\beta}^i_{\varepsilon,\lambda}=\inf_{u\in\partial O^i_\varepsilon}J_{\varepsilon,\lambda}(u).
\end{aligned}$$

By Lemma 4.3, there is $t^i_\varepsilon>0$ such that $t^i_\varepsilon w^i_\varepsilon>0\in M_{\varepsilon,\lambda}$ for every $1\le i\le k$.

Lemma 4.4. *There is $0<\varepsilon^0\le\varepsilon_0$ such that if $\varepsilon<\varepsilon^0$, then $Q_\varepsilon((t^i_\varepsilon)^-w^i_\varepsilon)\in\mathbf{K}_{\frac{\rho_0}{2}}$ for every $1\le i\le k$.*

Proof. Since

$$\begin{aligned}
Q_\varepsilon((t^i_\varepsilon)^-w^i_\varepsilon)&=\frac{\int_{\mathbb{R}^\mathbb{N}}\chi(\varepsilon z)\,|w(z-\frac{a^i}{\varepsilon})|^p\,\mathrm{d}z}{\int_{\mathbb{R}^\mathbb{N}}|w(z-\frac{a^i}{\varepsilon})|^p\,\mathrm{d}z}\\
&=\frac{\int_{\mathbb{R}^\mathbb{N}}\chi(\varepsilon Z+a^i)\,|w(z)|^p\,\mathrm{d}z}{\int}\,{}_{\mathbb{R}^\mathbb{N}}|w(z)|^p\,\mathrm{d}z\to a^i\qquad\text{as }\ \varepsilon\to0^+.
\end{aligned}$$

There is $\varepsilon^0>0$ such that

$$Q_\varepsilon((t^i_\varepsilon)^-w^i_\varepsilon)\in\mathbf{K}_{\frac{\rho_0}{2}},\quad\text{for every }\varepsilon<\varepsilon^0\text{ and every }1\le i\le k.$$

This completes the proof. □

Lemma 4.5. *There is a number $\delta>0$ such that if $u\in\Omega$ and $I_{\max}(u)\le\gamma_{\max}+\delta$ then $Q_\varepsilon(u)\in\mathbf{K}_{\frac{\rho_0}{2}}$ for every $0<\varepsilon<\varepsilon^0$.*

Proof. On the contrary, there exist the sequences $\{\varepsilon_n\}\subset\mathbb{R}^+$ and $\{u_n\}\in\Omega$ such that $\varepsilon_n\to0^+$. $I_{\varepsilon_n}(u_n)=\gamma_{\max}(>0)+o_n(1)$ as $n\to\infty$ and $Q_{\varepsilon_n}(u_n)\notin\mathbf{K}_{\frac{\rho_0}{2}}$ for all $n\in\mathbb{N}$. It is not difficult to find that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^\mathbb{N})$. Suppose that

$$\int_{\mathbb{R}^\mathbb{N}}|u_n|^p\,\mathrm{d}z\to0,\quad\text{as }n\to\infty,\qquad u_n\to0,$$

strongly in $L^p(\mathbb{R}^{\mathbb{N}})$. Since

$$\| u_n \|_H^2 = \int_{\mathbb{R}} a(\varepsilon_n z)(u_n)_+^p \mathrm{d}z, \quad \text{for every } n \in \mathbb{N},$$

then

$$I_{\varepsilon_n}(u_n) = \Big(\frac{1}{2} - \frac{1}{p}\Big) \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon_n z)(u_n)^p \mathrm{d}z = \gamma_{\max}(>0) + o_n(1) \leq o_n(1).$$

That is a contradiction. Then

$$\int_{\mathbb{R}^{\mathbb{N}}} | u |^p \,\mathrm{d}z \nrightarrow 0, \qquad \text{as } n \to \infty.$$

Thus $u_n \nrightarrow 0$ strongly in $L^p(\mathbb{R}^{\mathbb{N}})$. Also the concentration - compactness principle (see Wang [7, Lemma 2.16],then there is a fixed $d_0 > 0$ and a sequence $\{\overline{z_n}\} \subset \mathbb{R}^{\mathbb{N}}$ such that

$$\int_{B^N(\overline{z}_n:1)} | u_n(z) |^2 \,\mathrm{d}z \geq d_0 > 0. \tag{4.2}$$

Suppose $\nu_n(z) = u_n(z + \overline{z_n})$ then there a subsequence $\{\nu_n\}$ and $\nu \in H^1(\mathbb{R}^{\mathbb{N}})$ such that $\nu_n \rightharpoonup \nu$ weakly in $H^1(\mathbb{R}^{\mathbb{N}})$. Using the same computation in Lemma 2.11. There is a sequence $\{s_{\max}^n\} \subset \mathbb{R}^+$ such that $\overline{\nu_n} = s_{\max}^n \nu_n \in \Omega$ and

$$0 < \gamma_{\max} \leq I_{\max}(\overline{\nu_n}) \leq I_{\varepsilon_n}(s_{\max}^n u_n) \leq I_{\varepsilon_n}(u_n) = \gamma_{\max}(>0) + o_n(1)$$

as $n \to \infty$.

We conclude that a convergent subsequence $\{s_{\max}^n\}$ satisfy $s_{\max}^n \to s_0 > 0$. Then there are subsequences $\{\overline{\nu_n}\}$ and $\overline{\nu} \in H^1(\mathbb{R}^{\mathbb{N}})$ such that $\overline{\nu_n} \rightharpoonup \overline{\nu}(= s_0 \nu)$ weakly in $H^1(\mathbb{R}^{\mathbb{N}})$. By (4.2), then $\overline{\nu} \neq 0$. Furthermore, we can obtain that $\overline{\nu_n} \to \overline{\nu}$ strongly in $H^1(\mathbb{R}^{\mathbb{N}})$, and $I_{\max}(\overline{\nu}) = \gamma_{\max}$. Now, we try to indicate that there is a subsequence $\{z_n\} = \{\varepsilon_n \overline{z_n}\}$ such that $z_n \to z_0 \in \mathbf{K}$.

(i) Claim that the sequence $\{z_n\}$ is bounded in $\mathbb{R}^N$. On the contrary, assume that $| z_n | \to \infty$, then

$$\begin{aligned}
&\gamma_{\max} = I_{\max}(\overline{\nu}) < I_\infty(\overline{\nu}) \\
\leq &\liminf_{n\to\infty} \Big[\frac{1}{2} \| \overline{\nu_n} \|_H^2 - \frac{1}{P} \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon_n z + z_n)(\overline{\nu_n})_+^p \mathrm{d}z \Big] \\
= &\liminf_{n\to\infty} \Big[\frac{(s_{\max}^n)^2}{2} \| u_n \|_H^2 - \frac{(s_{\max}^n)^p}{p} \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon_n z)(u_n)_+^p \mathrm{d}z \Big] \\
= &\liminf_{n\to\infty} I_{\varepsilon_n}(s_{\max}^n u_n) \leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = \gamma_{\max},
\end{aligned}$$

that is a contradiction.

(ii) Claim that $z_0\in\mathbf{K}$. On the contrary, assume that $z_0\notin\mathbf{K}$, that is $a(z_0)<a_{\max}$. Then using the above argument to obtain that

$$\begin{aligned}\gamma_{\max}=I_{\max}(\overline{v})&<\frac{1}{2}\|\overline{v_n}\|_H^2-\frac{1}{P}\int_{\mathbb{R}^{\mathbb{N}}}a(z)(\overline{v_n})_+^p\mathrm{d}z\\&\leq\liminf\left[\frac{1}{2}\|\overline{v_n}\|_H^2-\frac{1}{P}\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon_n z+z_n)(\overline{v_n})_+^p\mathrm{d}z\right]\\&=\gamma_{\max},\end{aligned}$$

that is a contradiction. Since $v_n\rightharpoonup v\neq 0$ in $H^1(\mathbb{R}^{\mathbb{N}})$, we have that

$$Q_{\varepsilon_n}(u_n)=\frac{\int_{\mathbb{R}^{\mathbb{N}}}\chi(\varepsilon_n z)\,|\,v_n(z-\overline{z_n})\,|^p\,\mathrm{d}z}{\int_{\mathbb{R}^{\mathbb{N}}}|\,v_n(z-\overline{z_n})\,|^p\,\mathrm{d}z}=\frac{\int_{\mathbb{R}^{\mathbb{N}}}\chi(\varepsilon_n z+\varepsilon_n\overline{z_n})\,|\,v_n\,|^p\,\mathrm{d}z}{\int_{\mathbb{R}^{\mathbb{N}}}|\,v_n\,|^p\,\mathrm{d}z}\to z_0\subset\mathbf{K}_{\frac{æ_0}{2}}$$

as $n\to\infty$, that is a contradiction.

Hence, there is a number $\delta>0$ such that if $u\in\Omega$ and $I_{\max}(u)\leq\gamma_{\max}+\delta$. Then $Q_\varepsilon(u)\in\mathbf{K}_{\frac{æ_0}{2}}$ for every $c<\varepsilon^0$. Choosing $0<\delta_0<\delta$ such that

$$\gamma_{\max}+\delta_0<\gamma_{\max}-C_0\lambda^{\frac{2}{2-q}},\qquad\text{for every } 0<\varepsilon\leq\varepsilon^0. \tag{4.3}$$

This completes the proof. □

Lemma 4.6. *If* $u\in M^-_{\varepsilon,\lambda}$ *and* $J_{\varepsilon,\lambda}(u)\leq\gamma_{\max}+\frac{\delta_0}{2}$, *then there is a number* $\Lambda^*>0$ *so that* $Q_\varepsilon(u)\in\mathbf{K}_{\frac{æ_0}{2}}$ *for every* $0<\varepsilon<\Lambda^*$.

Proof. We apply the same computation in Lemma 2.11 to obtain that there is a unique positive number

$$s_\varepsilon^u=\left(\frac{\|u\|_H^2}{\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\,\mathrm{d}z}\right)^{\frac{1}{p-2}},$$

so that $s_\varepsilon^u u\in\Omega$ we want to show that $s_\varepsilon^u<C$ for some $C>0$ (independent of u). First, since $u\in M_{\varepsilon,\lambda}$

$$0<d_0\leq\alpha^-_{\varepsilon,\lambda}\leq J_{\varepsilon,\lambda}(u)\leq\gamma_{\max}+\frac{\delta_0}{2},$$

since $\langle J'_{\varepsilon,\lambda}(u),u\rangle=0$, then

$$\gamma_{\max}+\frac{\delta_0}{2}\geq J_{\varepsilon,\lambda}(u)=\left(\frac{1}{2}-\frac{1}{q}\right)\|u\|_H^2+\left(\frac{1}{q}-\frac{1}{p}\right)\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)\,\|u\|^p\,\mathrm{d}z\geq\frac{q-2}{2q}\|u\|_H^2,$$

that is

$$\|u\|_H^2\geq C_1=\frac{2q}{q-2}\left(\gamma_{\max}+\frac{\delta_0}{2}\right)$$

and

$$d_0 \le J_{\varepsilon,\lambda}(u) = \Big(\frac{1}{2}-\frac{1}{p}\Big) \| u \|_H^2 + \Big(\frac{1}{q}-\frac{1}{p}\Big) \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z) \| u \|^p \, \mathrm{d}z \ge \frac{p-2}{2p} \| u \|_H^2,$$

that is

$$\| u \|_H^2 \ge C_2 = \frac{2P}{P-2} d_0. \tag{4.4}$$

Moreover, we have that $J_{\varepsilon,\lambda}$ is coercive on $M_{\varepsilon,\lambda}$, then $0 < C_2 < \| u \|_H^2 < C_1$ for some C_1 and C_2 (independent of u). Next, we claim that $\| u \|_{L^p}^p > C_3 > 0$ for some C_3 (independent of u). On the contrary, there is a sequence $\{u_n\} \subset M_{\varepsilon,\lambda}^-$ so that $\| u_n \|_{L^p}^p = o_n(1)$ as $n \to \infty$. By (2.3)

$$\frac{2-q}{p-q} < \frac{\int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z) \| u_n \|_+^p \, \mathrm{d}z}{\| u \|_H^2} \le \frac{a_{\max} \| u \|_{L^p}^p}{C_2} = o_n(1),$$

that is a contradiction. Thus, $s_\varepsilon^u < C$ for some $C > 0$ (independent of u). Now, we get that

$$\begin{aligned}
\gamma_{\max} + \frac{\delta_0}{2} &\ge J_{\varepsilon,\lambda}(u) = \sup_{t \ge 0} J_{\varepsilon,\lambda}(tu) \ge J_{\varepsilon,\lambda}(s_\varepsilon^u u) \\
&= \frac{1}{2} \| s_\varepsilon^u u \|_H^2 - \frac{1}{p} \int_{\mathbb{R}^{\mathbb{N}}} a(\varepsilon z) \| s_\varepsilon^u u \|_+^p \, \mathrm{d}z - \frac{1}{q} \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z) (s_\varepsilon^u u)_+^q \, \mathrm{d}z \\
&\ge I_{\max}(s_\varepsilon^u u) - \frac{1}{q} \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z) (s_\varepsilon^u u)_+^q \, \mathrm{d}z.
\end{aligned}$$

Form the above inequality, we conclude that

$$\begin{aligned}
I_\varepsilon(s_\varepsilon^u u) &\le \gamma_{\max} + \frac{\delta_0}{2} + \frac{1}{q} \int_{\mathbb{R}^{\mathbb{N}}} \lambda h(\varepsilon z) (s_\varepsilon^u u)_+^q \, \mathrm{d}z \\
&\le \gamma_{\max} + \frac{\delta_0}{2} + \lambda \| h \|_\# S^q \| s_\varepsilon^u u \|_H^q \\
&< \gamma_{\max} + \frac{\delta_0}{2} + \lambda C^q (C_1)^{\frac{q}{2}} \| h \|_\# S^q.
\end{aligned}$$

Hence, there is $0 < \Lambda^* \le \varepsilon^0$ such that for $0 < \varepsilon \le \Lambda^*$

$$I_{\max}(s_\varepsilon^u u) \le \gamma_{\max} + \delta_0, \qquad \text{where } \; s_\varepsilon^u u \in \Omega.$$

By Lemma 4.6, we get

$$Q_\varepsilon(s_\varepsilon^u u) = \frac{\int_{\mathbb{R}^{\mathbb{N}}} \chi(\varepsilon z) \, | s_\varepsilon^u u(z) |^p \, \mathrm{d}z}{\int_{\mathbb{R}^{\mathbb{N}}} | s_\varepsilon^u u(z) |^p \, \mathrm{d}z} \in \mathbf{K}_{\frac{\boldsymbol{\ae}_0}{2}}, \qquad \text{for every } 0 < \varepsilon < \Lambda^*,$$

or $Q_\varepsilon \in \mathbf{K}_{\frac{\boldsymbol{\ae}_0}{2}}$.

Applying the above lemma, we get that

$$\overline{\beta^i_{\varepsilon,\lambda}} \geq \gamma_{\max} + \frac{\delta_0}{2}, \qquad \text{for every } 0<\varepsilon<\Lambda^*. \tag{4.5}$$

By Lemmas 4.3, 4.4, and Eq. (4.3), there every $0<\varepsilon^*<\Lambda^*$. So that

$$\beta^i_{\varepsilon,\lambda} \leq J_{\varepsilon,\lambda}\Big((t^i_\varepsilon)^-)w^i_\varepsilon\Big) \leq \gamma_{\max} + \frac{\delta_0}{3} < \gamma_{\max} - C_0\lambda^{\frac{2}{2-q}}. \tag{4.6}$$

This completes the proof. □

Lemma 4.7. *Given $u\in O^i_\varepsilon$, then there is an $\eta>0$ and differentiable functional $l: B(0;\eta)\subset H^1(\mathbb{R}^{\mathbb{N}})\to\mathbb{R}^+$ such that*

$$l(0)=1,\ l(\nu)(u-\nu)\in O^i_\varepsilon, \qquad \text{for every } \nu\in B(0;\eta),$$

and

$$\langle l'(\nu),\phi\rangle\,|_{(l,\nu)=(1,0)} = \frac{\langle \psi'_{\varepsilon,\lambda}(u),\phi\rangle}{\langle \psi'_{\varepsilon,\lambda}(u),u\rangle}, \qquad \text{for every } \phi\in C^\infty_c(\mathbb{R}^{\mathbb{N}}), \tag{4.7}$$

where $\psi_{\varepsilon,\lambda}(u)=\langle J'_{\varepsilon,\lambda}(u),u\rangle$.

Proof. See Cao and Zhou [8]. □

Lemma 4.8. *For each $1\leq i\leq k$, there is a $(PS)_{\beta^i_{\varepsilon,\lambda}}$-sequence $\{u_n\}\subset O^i_\varepsilon$ in $H^1(\mathbb{R}^{\mathbb{N}})$ for $J_{\varepsilon,\lambda}$.*

Proof. See [1, Lemma 4.7]. □

Theorem 4.1. *According to a_1, a_2, h_1, there is a positive number $(\varepsilon^*)^{-2}$ such that for $\lambda,\mu>(\varepsilon^*)^{-2}$, Eq. $(E_{\lambda,\mu})$ has $k+1$ positive solution in $\mathbb{R}^{\mathbb{N}}$.*

Proof. We know that there is a $(PS)_{\beta^i_{\varepsilon,\lambda}}$-sequence $\{u_n\}\subset M^-_{\varepsilon,\lambda}$ in $H^1(\mathbb{R}^{\mathbb{N}})$ for $J_{\varepsilon,\lambda}$ for every $1\leq i\leq k$, and (4.5). Since $J_{\varepsilon,\lambda}$ satisfy the $(PS)_\beta$-condition for $\beta\in(-\infty,\gamma_{\max}-C_0\lambda^{\frac{2}{2-q}})$, then $J_{\varepsilon,\lambda}$ has at least k critical points in $M^-_{\varepsilon,\lambda}$ for $0<\varepsilon\leq\varepsilon^*$. It follows that Eq. $(E_{\lambda,\mu})$ has k nonnegative solution in $\mathbb{R}^{\mathbb{N}}$. Applying the maximum principle and Theorem 3.4, Eq. $(E_{\varepsilon,\lambda})$ has $k+1$ positive solution in $\mathbb{R}^{\mathbb{N}}$. □

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