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BKM's Criterion of Weak Solutions for the 3D Boussinesq Equations

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Abstract. In this present paper, we investigate the Cauchy problem for 3D incompressible Boussinesq equations and establish the Beale-Kato-Majda regularity criterion of smooth solutions in terms of the velocity field in the homogeneous *BMO* space.

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1 Introduction

This paper is devoted to establish BKM's criterion of smooth solutions for the Cauchy problem for 3D Boussinesq equations with viscosity in $\mathcal{R}^3$

$$u_t+u\cdot\nabla u-\eta\triangle u+\nabla p=\theta e_3, \tag{1.1}$$

$$\theta_t+u\cdot\nabla\theta-\nu\triangle\theta=0, \tag{1.2}$$

$$\nabla\cdot u=0, \tag{1.3}$$

$$t=0:\ u=u_0(x),\ \ \theta=\theta_0(x), \tag{1.4}$$

here u is the velocity field, p is the pressure, θ is the small temperature deviations which depends on the density. $\eta\geq 0$ is the viscosity, $\nu\geq 0$ is called the molecular diffusivity and $e_3=(0,0,1)^T$. The above systems describe the evolution of the velocity field u for a three-dimensional incompressible fluid moving under the gravity and the earth rotation which

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come from atmospheric or oceanographic turbulence where rotation and stratification play an important role. When the initial density θ_0 is identically zero (or constant) and $\eta=0$, then (1.1)-(1.4) reduce to the classical incompressible Euler equation:

$$u_t+u\cdot\nabla u+\nabla p=0, \tag{1.5}$$

$$\nabla\cdot u=0, \tag{1.6}$$

$$u(x,t)|_{t=0}=u_0(x). \tag{1.7}$$

For the incompressible Euler equation and Navier-Stokes equation, a well-known criterion for the existence of global smooth solutions is the Beale-Kato-Majda criterion in [1] which states the control of the vorticity when $\omega=\text{curl }u$ in $L^1(0,T;L^\infty)$, this is sufficient to get the global well-posedness of solutions, i.e., any solution u is smooth up to time T under the assumption that $\int_0^T\|\nabla\times u(t)\|_{L^\infty}\mathrm{d}t<+\infty$. Kozono and Taniuchi [2] improved the Beale-Kato-Majda criterion under the assumption $\int_0^T\|\nabla\times u(t)\|_{\mathrm{BMO}}\mathrm{d}t<+\infty$. The regularity criteria for the Navier-Stokes equations, we can refer to Bahouri, Chemin and Danchin [3], Cao and Titi [4], Kato and Ponce [5], Kozono and Taniuchi [2], Zhou [6,7], Zhou and Lei [8], Zhang and Chen [9].

The global well-posedness for two-dimensional Boussinesq equations which has recently drawn a lot of attention. More precisely, the global well-posedness has been shown in various function spaces and for different viscosity, we can refer to [10–20]. When $\eta=\nu=0$, the Boussinesq system exhibits vorticity intensification and the global well-posedness issue remains an unsolved challenging open problem (if θ_0 is a constant) which may be formally compared to the similar problem for the three-dimensional axisymmetric Euler equations with swirl.

For the three-dimensional case, Hmidi and Rousset [16, 17] proved the global well-posedness for the 3D Navier-Stokes-Boussinesq equations and Euler-Boussinesq equations with axisymmetric initial data without swirl respectively. Danchin and Paicu [12] obtained the global existence and uniqueness result in Lorentz space for the Boussinesq equations with small data.

Our purpose of this paper is to obtain logarithmically improved regularity (BKM's) criterion of smooth solutions in terms of velocity field in BMO space.

Now we state our result as follows.

Theorem 1.1. *Assume that* $(u_0,\theta_0)\in H^m(\mathcal{R}^3)$ *holds with* $\mathrm{div}u_0=0$ *and* $m\geq 3$. *If* u *satisfies the condition*

$$\int_0^T\frac{\|\nabla\times u(t)\|_{\mathrm{BMO}}}{\sqrt{\ln(e+\|\nabla\times u(t)\|_{\mathrm{BMO}})}}\mathrm{d}t<+\infty, \tag{1.8}$$

then the solution (u,θ) *for the Cauchy problem* (1.1)-(1.4) *can be extended smoothly beyond* T.

The paper is organized as follows. We shall state some important inequalities in Section 2 and prove Theorem 1.1 in Section 3.

2 Preliminaries

Throughout this paper we use the following notations. $L^p(\mathcal{R}^3)$ denotes the generic Lebegue space, $H^m(\mathcal{R}^3)$ denotes the standard Sobolev space. BMO is the bounded mean oscillations space. $\dot{B}^0_{m,n}(\mathcal{R}^3)$ is the homogeneous Besov space, where $0\leq m,\ n\leq +\infty$. $\mathcal{S}(\mathcal{R}^n)$ be the Schwartz class of rapidly decreasing functions.

The Fourier transformation of $f\in\mathcal{S}(\mathcal{R}^n)$ is defined as

$$\mathcal{F}f=\hat{f}(\xi)=\int_{R^n} e^{-ix\cdot\xi}f(x)\mathrm{d}x \tag{2.1}$$

and the inverse Fourier transformation of $g\in\mathcal{S}(\mathcal{R}^n)$ is defined as

$$\mathcal{F}^{-1}g=\check{g}(x)=\int_{R^n} e^{ix\cdot\xi}g(\xi)\mathrm{d}\xi. \tag{2.2}$$

Next, we shall recall the Littlewood-Paley decomposition and define some functional spaces which can be found in [3,21,22].

Definition 2.1. *Denote $\mathcal{C}$ as the annulus of center on 0 with short radius $3/4$ and long radius $8/3$. Then there exist two positive functions $\varphi\in C_0^\infty(B(0,4/3))$ and $\chi\in C_0^\infty(\mathcal{C})$ such that*

$$\chi(\xi)+\sum_{q\geq 0}\varphi(2^{-q}\xi)=1, \tag{2.3}$$

$$|p-q|\geq 2 \Rightarrow \mathrm{supp}\varphi(2^{-q}\cdot)\cap\mathrm{supp}\varphi(2^{-p}\cdot)=\Phi, \tag{2.4}$$

$$q\geq 1 \Rightarrow \mathrm{supp}\chi\cap\mathrm{supp}\varphi(2^{-q}\cdot)=\Phi. \tag{2.5}$$

Remark 2.1. The frequency localization operator is defined as

$$\Delta_k u=\int_{\mathcal{R}^n}\check{\phi}(y)u(x-2^{-q}y)\mathrm{d}y. \tag{2.6}$$

Definition 2.2. *BMO denotes the homogeneous bounded mean oscillation space which equipped with the norm*

$$\|f\|_{\mathrm{BMO}}=\sup_{x\in\mathcal{R}^n,r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}\Big|f(y)-\frac{1}{B_r(y)}\int_{B_r(y)}f(z)dz\Big|\mathrm{d}y. \tag{2.7}$$

Definition 2.3. (The Triebel-Lizorkin space $\dot{F}^s_{p,q}$) *The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is defined as the set of tempered distributions u, i.e.,*

$$\|u\|_{\dot{F}^s_{p,q}}=\Big\|\Big(\sum_{k\in\mathcal{Z}}2^{sqk}|\Delta_k u|^q\Big)^{1/q}\Big\|_{L^p}<+\infty. \tag{2.8}$$

*Moreover, when $s=0$, $p=\infty$, $q=2$, $\dot{F}^0_{\infty,2}=$*BMO.

Lemma 2.1. (The Bernstein inequality) *Let $\mathcal{C}$ is a annulus of center on 0, B is a ball of center on zero, then there exists a constant $C>0$ such that for any integer $k\geq 0$ and function $u\in L^{\alpha}(\mathcal{R}^n)$, $b\geq a\geq 1$, we have*

$$\sup_{|\alpha|=k}\|\partial^{\alpha}u\|_{L^{\alpha}(\mathcal{R}^n)}\leq C^{k+1}\lambda^{k+n(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^{\alpha}(\mathcal{R}^n)}, \qquad \text{supp } \hat{u}\subset\lambda B, \tag{2.9}$$

$$C^{-(k+1)}\lambda^{k}\|u\|_{L^{\alpha}(\mathcal{R}^n)}\leq \sup_{|\alpha|=k}\|\partial^{\alpha}u\|_{L^{\alpha}(\mathcal{R}^n)}\leq C^{k+1}\lambda^{k}\|u\|_{L^{\alpha}(\mathcal{R}^n)}, \qquad \text{supp } \hat{u}\subset\lambda\mathcal{C}. \tag{2.10}$$

Proof. See, e.g., [3]. □

Lemma 2.2. (The Special Bernstein inequality) *For any integer $k\geq 0$, $1\leq p\leq q\leq +\infty$ and function $u\in L^p(\mathcal{R}^n)$, we have*

$$c2^{km}\|\Delta_k u\|_{L^p(\mathcal{R}^n)}\leq\|\nabla^m\Delta_k u\|_{L^p(\mathcal{R}^n)}\leq C2^{km}\|\Delta_k u\|_{L^p(\mathcal{R}^n)}, \tag{2.11}$$

$$\|\Delta_k u\|_{L^q(\mathcal{R}^n)}\leq C2^{n(\frac{1}{p}-\frac{1}{q})k}\|\Delta_k u\|_{L^p(\mathcal{R}^n)}, \qquad \text{supp } \hat{u}\subset 2\mathcal{C}, \tag{2.12}$$

where c and C are positive constants independent of u and k.

Proof. See, e.g., [3]. □

Lemma 2.3. (The Gagliardo-Nirenberg inequality)

$$\|\nabla^i f\|_{L^{2m/i}}\leq C\|f\|_{L^\infty}^{1-i/m}\|\nabla^m f\|_{L^2}^{i/m},\ i\in[0,m] \tag{2.13}$$

holds for all $u\in L^{\infty}(\mathcal{R}^n)\cap H^m(\mathcal{R}^n)$.

Lemma 2.4. (The Interpolation Inequalities) *The following inequalities hold in the three dimensional Lebesgue space*

$$\|\nabla u\|_{L^4}\leq C\|u\|_{L^4}^{1/5}\|\Delta u\|_{L^2}^{4/5}, \tag{2.14}$$

$$\|\nabla u\|_{L^2}\leq C\|u\|_{L^2}^{2/3}\|\nabla^3 u\|_{L^2}^{1/3}, \tag{2.15}$$

$$\|u\|_{L^\infty}\leq C\|u\|_{L^2}^{1/4}\|\nabla^2 u\|_{L^2}^{3/4}, \tag{2.16}$$

$$\|u\|_{L^4}\leq C\|u\|_{L^2}^{3/4}\|\nabla^3 u\|_{L^2}^{1/4}. \tag{2.17}$$

Lemma 2.5. *The following inequality holds:*

$$\|\nabla^m(u\cdot\nabla v)-u\cdot\nabla\nabla^m v\|_{L^2}\leq C(\|\nabla u\|_{L^\infty}\|\nabla^m v\|_{L^2}+\|\nabla v\|_{L^\infty}\|\nabla^m u\|_{L^2}). \tag{2.18}$$

Proof. See, e.g., [3]. □

Lemma 2.6. *There exists a uniform positive constant C such that*

$$\|\nabla u\|_{L^\infty}\leq C\left(1+\|u\|_{L^2}+\|\nabla\times u\|_{\mathrm{BMO}}\sqrt{\ln(e+\|u\|_{H^3})}\right) \tag{2.19}$$

holds for all $u\in H^3(\mathcal{R}^3)$ with $\nabla\cdot u=0$.

Proof. Using the similar technique as in [8], we can derive our result. From the Littlewood-Paley decomposition, we have

$$\nabla u=\sum_{k=-\infty}^{+\infty}\Delta_k\nabla u=\Big(\sum_{k=-\infty}^{0}+\sum_{k=1}^{A}+\sum_{k=A+1}^{+\infty}\Big)\Delta_k\nabla u. \tag{2.20}$$

Using the Bernstein inequality, setting $A=\frac{\ln(e+\|u\|_{H^3})}{(2-n/2)\ln 2}+1$, we deduce that

$$\begin{aligned}\|\nabla u\|_{L^\infty}\le&\sum_{k=-\infty}^{0}\|\Delta_k\nabla u\|_{L^\infty}+\|\sum_{k=1}^{A}\Delta_k\nabla u\|_{L^\infty}+\sum_{k=A+1}^{+\infty}\|\Delta_k\nabla u\|_{L^\infty}\\ \le&\sum_{k=-\infty}^{0}2^{k(1+n/2)}\|\Delta_k\nabla u\|_{L^2}+A^{1/2}\Big\|(\sum_{k=1}^{A}|\Delta_k\nabla u|^2)^{1/2}\Big\|_{L^\infty}\\ &+\sum_{k=A+1}^{+\infty}2^{-k(2-n/2)}\|\Delta_k\nabla^3u\|_{L^2}\\ \le&C(\|u\|_{L^2}+A^{1/2}\|\nabla u\|_{\mathrm{BMO}}+2^{-A(2-n/2)}\|\nabla^3u\|_{L^2}).\end{aligned} \tag{2.21}$$

Denote $R_j=(\partial/\partial x_j)(-\Delta)^{-1/2}$ $(R=(R_1,R_2,\cdots,R_n))$ be the Riesz transformation, from the Biot-Savard law, we see $u_{X_j}=R_j(R\times\nabla u)$ $(j=1,2,\cdots,n)$, Since R is a bounded operator in BMO, we derive that $\|\nabla u\|_{BMO}\le C\|\nabla\times u\|_{\mathrm{BMO}}$. Combining (2.20) and (2.21), we complete the proof of lemma. □

3 Proof of main theorem

Proof of Theorem 1.1: Multiplying (1.1) by u, using (1.3) and integrating by parts in $\mathcal{R}^3$, we derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2+\eta\|\nabla u\|_{L^2}^2=\int_{\mathcal{R}^3}\theta e_3\cdot u\mathrm{d}x\le\|\theta\|_{L^2}\|u\|_{L^2}\le\frac{1}{2}\|\theta\|_{L^2}^2+\frac{1}{2}\|u\|_{L^2}^2. \tag{3.1}$$

Multiplying (1.2) by θ, using (1.3) and integrating in $\mathcal{R}^3$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2+\nu\|\nabla\theta\|_{L^2}=0. \tag{3.2}$$

Combining (3.1) and (3.2), integrating with respect to t, using the Gronwall inequality, we conclude that

$$\begin{aligned}&\|u\|_{L^2}^2+\|\theta\|_{L^2}^2+2\eta\int_0^t\|\nabla u(s)\|_{L^2}^2\mathrm{d}s+2\nu\int_0^t\|\nabla\theta(s)\|_{L^2}\mathrm{d}s\\ &\quad=2\int_0^t\int_{\mathcal{R}^3}\theta(s)e_3\cdot u\mathrm{d}x\mathrm{d}s+\int_0^t\|\theta(s)\|_{L^2}^2\mathrm{d}s+\int_0^t\|u(s)\|_{L^2}^2\mathrm{d}s+\|u_0\|_{L^2}^2+\|\theta_0\|_{L^2}^2, \qquad (3.3)\\ &\|u\|_{L^\infty(0,T;L^2)}+\|u\|_{L^2(0,T;H^1)}\le C, \qquad (3.4)\\ &\|\theta\|_{L^\infty(0,T;L^2)}+\|\theta\|_{L^2(0,T;H^1)}\le C. \qquad (3.5)\end{aligned}$$

Applying ∇ to the both sides of (1.1), taking a L^2 inner product of the resulting equation with ∇u, integrating by parts, we derive

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2+\mu\|\nabla^2 u(t)\|_{L^2}^2=-\int_{\mathcal{R}^3}\nabla(u\cdot\nabla u)\nabla u\mathrm{d}x+\int_{\mathcal{R}^3}\nabla(\theta e_3)\nabla u\mathrm{d}x. \tag{3.6}$$

The similar steps to (1.2), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\theta(t)\|_{L^2}^2+\nu\|\nabla^2\theta(t)\|_{L^2}^2=-\int_{\mathcal{R}^3}\nabla(u\cdot\nabla\theta)\nabla\theta\mathrm{d}x. \tag{3.7}$$

From (3.6), (3.7) and $\nabla\cdot u=0$, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u(t)\|_{L^2}^2+\|\nabla\theta(t)\|_{L^2}^2)+\mu\|\nabla^2 u(t)\|_{L^2}^2+\nu\|\nabla^2\theta(t)\|_{L^2}^2\\&=-\int_{\mathcal{R}^3}\Big[\nabla(u\cdot\nabla u)-u\cdot\nabla\nabla u\Big]\nabla u\mathrm{d}x-\int_{\mathcal{R}^3}\Big[\nabla(u\cdot\nabla\theta)-u\cdot\nabla\nabla\theta\Big]\nabla\theta\mathrm{d}x+\int_{\mathcal{R}^3}\nabla(\theta e_3)\nabla u\mathrm{d}x\\&=I_1+I_2+I_3.\end{aligned} \tag{3.8}$$

By Lemmas 2.5 and 2.6, using the incompressible condition $\nabla\cdot u=0$, we give the estimate of $I_i(i=1,2,3)$.

$$I_1=-\int_{\mathcal{R}^3}\Big[\nabla(u\cdot\nabla u)-u\cdot\nabla\nabla u\Big]\nabla u\mathrm{d}x\le 2C\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}^2, \tag{3.9}$$

$$\begin{aligned}I_2=&-\int_{\mathcal{R}^3}\Big[\nabla(u\cdot\nabla\theta)-u\cdot\nabla\nabla\theta\Big]\nabla\theta\mathrm{d}x=-\int_{\mathcal{R}^3}\nabla u\cdot\nabla\theta\nabla\theta\mathrm{d}x\\&\le C\|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^2}^2,\end{aligned} \tag{3.10}$$

$$I_3=\int_{\mathcal{R}^3}\nabla(\theta e_3)\nabla u\mathrm{d}x\le\frac{1}{2}\|\nabla u\|_{L^2}^2+\frac{1}{2}\|\theta\|_{L^2}^2. \tag{3.11}$$

It follows from (3.8)-(3.11) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u(t)\|_{L^2}^2+\|\nabla\theta(t)\|_{L^2}^2)+\mu\|\nabla^2 u(t)\|_{L^2}^2+\nu\|\nabla^2\theta(t)\|_{L^2}^2\\&\le C(1+\|\nabla u\|_{L^\infty})(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)\\&\le C\left(1+\|u\|_{L^2}+\|\nabla\times u\|_{\mathrm{BMO}}\sqrt{\ln(e+\|u\|_{H^3})}\right)(\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2).\end{aligned} \tag{3.12}$$

By the Gronwall inequality, we arrive at

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2+\|\nabla\theta(t)\|_{L^2}^2+2\mu\int_{t_0}^t\|\nabla^2 u(s)\|_{L^2}^2\mathrm{d}s+2\nu\int_{t_0}^t\|\nabla^2\theta(s)\|_{L^2}^2\mathrm{d}s\\&\le(\|\nabla u(t_0)\|_{L^2}^2+\|\nabla\theta(t_0)\|_{L^2}^2)\\&\quad\times\exp\left\{\mathrm{C}\int_{\mathrm{t}_0}^{\mathrm{t}}\left(1+\|\mathrm{u(s)}\|_{\mathrm{L}^2}+\|\nabla\times\mathrm{u(s)}\|_{\mathrm{BMO}}\sqrt{\ln(\mathrm{e}+\|\mathrm{u(s)}\|_{\mathrm{H}^3})}\right)\mathrm{ds}\right\}.\end{aligned} \tag{3.13}$$

From (1.8), there exist an arbitrary small constant $\varepsilon > 0$ and $T_* < T$ such that

$$\int_{T_*}^{T} \frac{\|\nabla\times u(t)\|_{\rm BMO}}{\sqrt{\ln(e+\|\nabla\times u(t)\|_{\rm BMO})}}{\rm d}t \le \varepsilon. \tag{3.14}$$

Hence, combining (3.13) and (3.3)-(3.5), we conclude

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2 + |\nabla\theta(t)\|_{L^2}^2 + 2\mu\int_{t_0}^{t}\|\nabla^2 u(s)\|_{L^2}^2{\rm d}s + 2\nu\int_{t_0}^{t}\|\nabla^2\theta(s)\|_{L^2}^2{\rm d}s\\
\le& C_0\exp\Big\{{\rm C_1}\int_{\rm T_*}^{\rm t}\|\nabla\times{\rm u(s)}\|_{\rm BMO}\sqrt{\ln({\rm e}+\|{\rm u(s)}\|_{\rm H^3})}{\rm ds}\Big\}\\
\le& C_0\exp\{{\rm C_1}''\ln({\rm e}+{\rm A(t)})\} \le C_2(e+A(t))^{C_1\varepsilon},
\end{aligned} \tag{3.15}$$

where $A(t)=\sup_{T_*\le s\le t}(\|\nabla^3 u(s)\|_{L^2}^2+\|\nabla^3\theta(s)\|_{L^2}^2)$, $t\in[T_*,T]$, C_0 depends on $\|\nabla u(T_*)\|_{L^2}^2+\|\nabla\theta(T_*)\|_{L^2}^2$, $C_1>0$ is a uniform constant.

Applying ∇^m to (1.1) and (1.2), then taking L^2 inner product of the resulting equation with $\nabla^m u$ and $\nabla^m\theta$ respectively, integrating by parts, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\rm d}{{\rm d}t}\|\nabla^m u(t)\|_{L^2}^2 + \eta\|\nabla^m\nabla u(t)\|_{L^2}^2\\
&\qquad = -\int_{\mathcal{R}^3}\nabla^m(u\cdot\nabla u)\nabla^m u{\rm d}x + \int_{\mathcal{R}^3}\nabla^m(\theta e_3)\nabla^m u{\rm d}x,
\end{aligned} \tag{3.16}$$

$$\frac{1}{2}\frac{\rm d}{{\rm d}t}\|\nabla^m\theta(t)\|_{L^2}^2 + \nu\|\nabla^m\nabla\theta(t)\|_{L^2}^2 = -\int_{\mathcal{R}^3}\nabla^m(u\cdot\nabla\theta)\nabla^m\theta{\rm d}x. \tag{3.17}$$

It follows from (3.16), (3.17) and $\nabla\cdot u=0$ that

$$\begin{aligned}
&\frac{1}{2}\frac{\rm d}{{\rm d}t}(\|\nabla^m u\|_{L^2}^2+\|\nabla^m\theta\|_{L^2}^2) + \eta\|\nabla^m\nabla u(t)\|_{L^2}^2 + \nu\|\nabla^m\nabla\theta(t)\|_{L^2}^2\\
=& -\int_{\mathcal{R}^3}\nabla^m(u\cdot\nabla u)\nabla^m u{\rm d}x + \int_{\mathcal{R}^3}\nabla^m(\theta e_3)\nabla^m u{\rm d}x - \int_{\mathcal{R}^3}\nabla^m(u\cdot\nabla\theta)\nabla^m\theta{\rm d}t\\
=& -\int_{\mathcal{R}^3}[\nabla^m(u\cdot\nabla u) - u\cdot\nabla\nabla^m u]\nabla^m u{\rm d}x + \int_{\mathcal{R}^3}\nabla^m(\theta e_3)\nabla^m u{\rm d}x\\
&\quad - \int_{\mathcal{R}^3}[\nabla^m(u\cdot\nabla\theta) - u\cdot\nabla\nabla^m\theta]\nabla^m\theta{\rm d}x\\
=& I_4 + I_5 + I_6.
\end{aligned} \tag{3.18}$$

Since the proof for the case $m>3$ is similar to $m=3$, here we only need to prove the case $m=3$. By the Hölder inequality, the Cauchy inequality and Lemma 2.6, we get

$$\begin{aligned}
I_4 =& \Big|-\int_{\mathcal{R}^3}\big[\nabla^3(u\cdot\nabla u) - u\cdot\nabla\nabla^3 u\big]\,\nabla^3 u{\rm d}x\Big| \le \|\nabla^3(u\cdot\nabla u) - u\cdot\nabla^3 u\|_{L^2}\|\nabla^3 u\|_{L^2}\\
\le& C\|\nabla u\|_{L^\infty}\|\nabla^3 u\|_{L^2}^2,
\end{aligned} \tag{3.19}$$

$$
\begin{aligned}
I_5 &= \int_{\mathcal{R}^3} \nabla^3(\theta e_3)\nabla^3 u \mathrm{d}x \leq \frac{1}{2}\left(\|\nabla^3\theta\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2\right), && (3.20)\\
I_6 &= -\int_{\mathcal{R}^3} \left[\nabla^3(u\cdot\nabla\theta) - u\cdot\nabla\nabla^3\theta\right]\nabla^3\theta \mathrm{d}x\\
&= -3\int_{\mathcal{R}^3} \nabla u\cdot\nabla\nabla^2\theta\nabla^3\theta\mathrm{d}x - 3\int_{\mathcal{R}^3}\nabla^2 u\cdot\nabla\nabla\theta\nabla^3\theta\mathrm{d}x - \int_{\mathcal{R}^3}\nabla^3 u\cdot\nabla\theta\nabla^3\theta\mathrm{d}x\\
&= -3\int_{\mathcal{R}^3} \nabla u\cdot\nabla\nabla^2\theta\nabla^3\theta\mathrm{d}x - 3\int_{\mathcal{R}^3}\nabla^2 u\cdot\nabla\nabla\theta\nabla^3\theta\mathrm{d}x + \int_{\mathcal{R}^3}\nabla^3 u\cdot\theta\nabla^4\theta\mathrm{d}x\\
&\leq C\|\nabla u\|_{L^\infty}\|\nabla^3\theta\|_{L^2}^2 + C\|\nabla^2 u\|_{L^4}\|\nabla^2\theta\|_{L^4}\|\nabla^3\theta\|_{L^2} + \|\theta\|_{L^\infty}\|\nabla^3 u\|_{L^2}\|\nabla^4\theta\|_{L^2}\\
&= I_7 + I_8 + I_9. && (3.21)
\end{aligned}
$$

Using Lemma 2.4 and $\|\theta\|_{L^\infty}\leq\|\theta_0\|_{L^\infty}$, we obtain

$$
\begin{aligned}
I_8 &= 3\|\nabla^2 u\|_{L^4}\|\nabla^2\theta\|_{L^4}\|\nabla^3\theta\|_{L^2}\\
&\leq C\|\nabla u\|_{L^\infty}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\|\theta\|_{L^\infty}^{\frac{1}{2}}\|\nabla^4\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^4\theta\|_{L^2}^{\frac{1}{2}}\\
&\leq C\|\nabla u\|_{L^\infty}^{\frac{1}{2}}\|\nabla^3 u\|_{L^2}^{\frac{1}{2}}\|\theta_0\|_{L^\infty}^{\frac{1}{2}}\|\nabla^4\theta\|_{L^2}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}}\\
&\leq \frac{\nu}{4}\|\nabla^4\theta\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\|\nabla^3 u\|_{L^2}\|\nabla^2\theta\|_{L^2}\\
&\leq \frac{\nu}{4}\|\nabla^4\theta\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\left(1+\|\nabla^3 u\|_{L^2}^2+\|\nabla^3\theta\|_{L^2}^2\right)\\
&\leq \frac{\nu}{4}\|\nabla^4\theta\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}(e+A(t)). && (3.22)
\end{aligned}
$$

Noting that $\|\theta\|_{L^\infty}\leq\|\theta_0\|_{L^\infty}$ and using the Cauchy inequality, we deduce

$$
\begin{aligned}
I_9 &= \|\theta\|_{L^\infty}\|\nabla^3 u\|_{L^2}\|\nabla^4\theta\|_{L^2} \leq \frac{\nu}{4}\|\nabla^4\theta\|_{L^2}^2 + C\|\theta\|_{L^\infty}^2\|\nabla^3 u\|_{L^2}^2\\
&\leq \frac{\nu}{4}\|\nabla^4\theta\|_{L^2}^2 + C\|\nabla^3 u\|_{L^2}^2 \leq \frac{\nu}{4}\|\nabla^4\theta\|_{L^2}^2 + C(e+A(t)). && (3.23)
\end{aligned}
$$

From direct computation, we have

$$
I_7 \leq C\|\nabla u\|_{L^\infty}(e+A(t)). \tag{3.24}
$$

Combining (3.18)-(3.24) and (3.4), we conclude

$$
-\int_{\mathcal{R}^3}\left[\nabla^3(u\cdot\nabla\theta)-u\cdot\nabla\nabla^3\theta\right]\nabla^3\theta\mathrm{d}x \leq \frac{\nu}{2}\|\nabla^4\theta\|_{L^2}^2 + C\Big(\|\nabla u\|_{L^\infty}+1\Big)\Big(e+A(t)\Big). \tag{3.25}
$$

Thus, it follows from (3.18)-(3.25) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla^3 u(t)\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2\Big) + \nu\|\nabla^4\theta\|_{L^2}^2 \leq C\Big(\|\nabla u\|_{L^\infty}+1\Big)\Big(e+A(t)\Big), \tag{3.26}
$$

holds for all $T_*\leq t<T$.

Integrating (3.26) over $[T_*,s]$ with respect to t, using Lemma 2.5, we arrive at

$$
\begin{aligned}
&\|\nabla^3 u(s)\|_{L^2}^2+\|\nabla^3\theta(s)\|_{L^2}^2+\nu\int_{T_*}^{s}\|\nabla^4\theta(t)\|_{L^2}^2\mathrm{d}t\\
\le&\|\nabla^3 u(T_*)\|_{L^2}^2+\|\nabla^3\theta(T_*)\|_{L^2}^2\\
&\quad+C\int_{T_*}^{s}\Big(1+\|u(t)\|_{L^2}+\|\nabla\times u(t)\|_{\mathrm{BMO}}\sqrt{\ln(e+\|u(t)\|_{H^3})}\Big)(e+A(t))\mathrm{d}t\\
\le&\|\nabla^3 u(T_*)\|_{L^2}^2+\|\nabla^3\theta(T_*)\|_{L^2}^2\\
&\quad+C\int_{T_*}^{s}\Big(1+\|u(t)\|_{L^2}+\|\nabla\times u(t)\|_{\mathrm{BMO}}\sqrt{\ln(e+A(t))}\Big)(e+A(t))\mathrm{d}t, \qquad (3.27)
\end{aligned}
$$

which implies

$$
\begin{aligned}
e+A(t)\le& e+\|\nabla^3 u(T_*)\|_{L^2}^2+\|\nabla^3\theta(T_*)\|_{L^2}^2\\
&+C\int_{T_*}^{t}\Big(1+\|u(s)\|_{L^2}+\|\nabla\times u(s)\|_{\mathrm{BMO}}\sqrt{\ln(e+\|u(s)\|_{H^3})}\Big)(e+A(s))\mathrm{d}s. \qquad (3.28)
\end{aligned}
$$

For all $T_*\le t<T$, then using the Gronwall inequality and (3.28), we deduce that $e+A(t)$ is bounded, i.e.,

$$
\|\nabla^3 u(t)\|_{L^2}^2+\|\nabla^3\theta(t)\|_{L^2}^2\le C, \qquad (3.29)
$$

where C is dependent on $\|\nabla^3 u(T_*)\|_{L^2}^2+\|\nabla^3\theta(T_*)\|_{L^2}^2$. Thus, we complete the proof of Theorem 1.1. □

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