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Classification of Solutions to a Critically Nonlinear System of Elliptic Equations on Euclidean Half-Space

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Abstract. For $N\geq 3$ and non-negative real numbers a_{ij} and b_{ij} $(i,j=1,\cdots,m)$, the semi-linear elliptic system

$$\begin{cases}\Delta u_i+\prod\limits_{j=1}^{m}u_j^{a_{ij}}=0, & \text{in } \mathbb{R}^N_+,\\ \dfrac{\partial u_i}{\partial y_N}=c_i\prod\limits_{j=1}^{m}u_j^{b_{ij}}, & \text{on } \partial\mathbb{R}^N_+,\end{cases}\qquad i=1,\cdots,m,$$

is considered, where $\mathbb{R}^N_+$ is the upper half of N-dimensional Euclidean space. Under suitable assumptions on the exponents a_{ij} and b_{ij}, a classification theorem for the positive $C^2(\mathbb{R}^N_+)\cap C^1(\overline{R^N_+})$-solutions of this system is proven.

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1 Introduction

Let $N\geq 3$ be a positive integer and let $\mathbb{R}^N_+=\{(y_1,\cdots,y_N)\in\mathbb{R}^N: y_N>0\}$ denote the upper half of N-dimensional Euclidean space. Fix a positive integer m and set $J=\{1,\cdots,m\}$. Let $\mathcal{A}=[a_{ij}]$ be an $m\times m$ matrix with nonnegative entries. We are concerned with the classical solutions of the semi-linear elliptic system

$$\Delta u_i+\prod_{j=1}^{m}u_j^{a_{ij}}=0,\qquad \text{in } \Omega\subset\mathbb{R}^N \text{ for all } i\in J. \tag{1.1}$$

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This system and its variants have been studied extensively in numerous contexts. For example, (1.1) arises as the system of equations for a steady-state solution to the corresponding parabolic reaction-diffusion system. In particular, when $m=2$ the system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1 + u_1^{a_{11}} u_2^{a_{12}}, & \text{for } y\in\Omega, t>0, \\ \dfrac{\partial u_2}{\partial t} = \Delta u_2 + u_1^{a_{21}} u_2^{a_{22}}, & \text{for } y\in\Omega, t>0, \end{cases} \tag{1.2}$$

has received much attention. For example, when $a_{11}=a_{22}=0$, (1.2) gives a simple model for heat propagation in a two-component combustible mixture [1]. Variants of (1.2) have also been used to model the diffusing densities of two biological species when each specie finds its subsidence from the activity of the other specie [2]. It is well-known that a thorough understanding of (1.1) is highly beneficial to obtaining an understanding of (1.2). For example, under appropriate assumptions on $\mathcal{A}$, in [3] and [4] Mitidieri proved nonexistence results for (1.1) when $\Omega=\mathbb{R}^N$ and $m=2$. These results were refined by Zheng in [5] and then used to derive blow-up (in time) estimates for solutions of (1.2) that satisfy suitable initial and boundary conditions. For more results concerning these parabolic systems and their variants the reader is referred to [6, 7] and the references therein.

An interesting case of (1.1) arises when $\mathcal{A}$ satisfies

$$\begin{cases} a_{ij}\geq 0, & \text{for all } (i,j)\in J\times J, \\ \mathcal{A} \text{ is irreducible }, & \\ \sum_{j=1}^{m} a_{ij} = \frac{N+2}{N-2}, & \text{for all } i\in J. \end{cases} \tag{1.3}$$

Recall that an $m\times m$-matrix $\mathcal{A}$ is called *irreducible* if there is no partition $J=I_1\cup I_2$ such that $a_{ij}=0$ for all $i\in I_1$, and $j\in I_2$. When $m=1$ equations (1.1) reduce to

$$\Delta u + K u^{(N+2)/(N-2)} = 0, \tag{1.4}$$

with $K=1$. Eq. (1.4) has been studied extensively as it arises in relation to the famous Yamabe problem. The Yamabe problem asks whether it is always possible to conformally deform the metric g of a given smooth compact Riemannian manifold to a metric $\hat{g}=u^{4/(N-2)}g$ whose scalar curvature is constant. Through the works of Trudinger [8], Aubin [9] and Schoen [10], the Yamabe problem was proven affirmative. See [11] and the references therein for results regarding the Yamabe problem. For $\mathcal{A}$ satisfying (1.3) and $\Omega=\mathbb{R}^N$, the classical solutions of (1.1) were classified by Chipot, Shafrir and Wolansky in [12] (see also [13]). Their result is the following.

Theorem 1.1 (Chipot, Shafrir and Wolansky [12])**.** *Suppose $\mathcal{A}$ satisfies* (1.3). *If $u_1,\cdots,u_m$ are positive $C^2(\mathbb{R}^N)$-solutions of* (1.1) *with $\Omega=\mathbb{R}^N$ then*

$$u_i(y) = \frac{\beta_i}{\left(\sigma^2 + |y-y^0|^2\right)^{(N-2)/2}}, \qquad \text{for all } i\in J, \tag{1.5}$$

for some $y^0\in\mathbb{R}^N$ *and some positive constants* σ^2 *and* $\beta_1,\cdots,\beta_m$ *satisfying*

$$\log\beta_i=\sum_{j=1}^m a_{ij}\log\beta_j-\log\left(\sigma^2N(N-2)\right),\qquad \text{for all } i\in J. \tag{1.6}$$

This theorem is the system-generalization of the classification of entire solutions to (1.4) given in [14].

Many interesting questions involving variants of (1.4) have been considered. For example, for real numbers K and c the equations

$$\begin{cases}\Delta u+Ku^{(N+2)/(N-2)}=0, & \text{in } \mathbb{R}^N_+,\\ \dfrac{\partial u}{\partial y_N}=cu^{N/(N-2)}, & \text{on } \partial\mathbb{R}^N_+,\end{cases} \tag{1.7}$$

arise in relation to the boundary-Yamabe problem which seeks to determine whether the metric g of smooth compact Riemannian manifold M with boundary can be conformally deformed into a metric $\hat{g}$ such that both the scalar curvature and the boundary mean curvature of $\hat{g}$ are constant. The boundary-Yamabe problem is still open. For a detailed discussion on the boundary-Yamabe problem, the reader is referred to Escobar [15, 16], Han-Li [17, 18], Marques [19] and the references therein. The solutions of equations (1.7) were classified separately by Li and Zhu in [20] and Chipot, Shafrir and Fila in [21]. Later in [22], the solutions of (1.7) with more general nonlinearities were classified. The result is as follows

Theorem 1.2 (Li-Zhu [20], Chipot-Shafrir-Fila [21] and Li-Zhang [22]). *If* u *is a non-negative* $C^2(\mathbb{R}^N_+)\cap C^1(\overline{\mathbb{R}^N_+})$*-solution of* (1.7) *with* $K=N(N-2)$*, then either* $u\equiv 0$ *or there exists* $\sigma>0$ *and* $(y^0_1,\cdots,y^0_{N-1})\in\partial\mathbb{R}^N_+$ *such that*

$$u(y)=\left(\frac{\sigma}{\sigma^2+|y-y^0|^2}\right)^{(N-2)/2},\qquad \text{for all } y\in\mathbb{R}^N_+,$$

where $y^0=(y^0_1,\cdots,y^0_{N-1},y^0_N)$ *and* $y^0_N=\sigma c/(N-2)$.

In this paper, an analogue of Theorem 1.2 is proven for the generalization of (1.7) to a system of equations. To generalize the boundary nonlinearity in (1.7) let $c_1,\cdots,c_m$ be real numbers and let $\mathcal{B}=[b_{ij}]$ be an $m\times m$ matrix satisfying

$$\begin{cases} b_{ij}\geq 0, & \text{for all } (i,j)\in J\times J,\\ \sum\limits_{j=1}^m b_{ij}=\frac{N}{N-2}, & \text{for all } i\in J,\\ b_{ij}=\frac{N}{N-2}\delta_{ij}, & \text{for all } i\in J \text{ such that } c_i\geq 0,\end{cases} \tag{1.8}$$

and consider the system

$$\begin{cases}\Delta u_i+\prod\limits_{j=1}^m u_j^{a_{ij}}=0, & \text{in } \mathbb{R}^N_+,\\ \dfrac{\partial u_i}{\partial y_N}=c_i\prod\limits_{j=1}^m u_j^{b_{ij}}, & \text{on } \partial\mathbb{R}^N_+,\\ u_i>0, & \text{on } \overline{\mathbb{R}^N_+},\end{cases}\qquad \text{for all } i\in J. \tag{1.9}$$

Our main theorem is as follows.

Theorem 1.3. *Suppose $\mathcal{A}$ satisfies* (1.3) *and $\mathcal{B}$ satisfies* (1.8). *If $(u_1,\cdots,u_m)$ is a $C^2(\mathbb{R}^N_+)\cap C^1(\overline{\mathbb{R}^N_+})$-solution of* (1.9) *then there exist positive constants $\sigma,\beta_1,\cdots,\beta_m$ satisfying* (1.6) *and $(y_1^0,\cdots,y_{N-1}^0)\in\partial\mathbb{R}^N_+$ such that u_i is given by* (1.5) *with $y^0=(y_1^0,\cdots,y_{N-1}^0,y_N^0)$, where*

$$y_N^0=\sigma^2 Nc_i\prod_{j=1}^m \beta_j^{b_{ij}-a_{ij}},\qquad \text{for all } i\in J. \tag{1.10}$$

In particular, $\sigma^2 Nc_i\prod_{j=1}^m\beta_j^{b_{ij}-a_{ij}}$ is independent of i.

Remark 1.1. The third item of (1.8) says that if $i\in J$ is an index for which $c_i\geq 0$, then the boundary equation for u_i is

$$\frac{\partial u_i}{\partial y_N}=c_i u_i^{N/(N-2)},\qquad \text{on } \partial\mathbb{R}^N_+.$$

This assumption is made for convenience as it makes some of the proofs simpler. See, for example the proof of Claim 3.2.

The proof of Theorem 1.3 is via the method of moving spheres and is inspired by the proofs of Theorems 1.2 and 1.1 given in [22] and [12] respectively. The organization of this paper is as follows. In Section 2 we show that the moving sphere process can start. In Section 3 we obtain a symmetry relation between u_i and its "critical" Kelvin transformations. In Section 4 we first use a calculus lemma to deduce the form of the restriction of u_i to $\partial\mathbb{R}^N_+$. Next we transform the problem defined on $\mathbb{R}^N_+$ to a new problem defined on a ball. After determining that the solutions of the transformed problem must be radial, a system of ODE is obtained and the solution to this system is determined. The conclusion of Theorem 1.3 will follow after returning to the original problem.

Throughout, C will be used to denote a positive constant depending only on N. The value of C may change from line to line. The Euclidean ball of radius r and center x will be denoted $B_r(x)$. When $x=0$ the notation B_r will be used.

2 The moving sphere process can start

Let $u_1,\cdots,u_m$ be as in the hypotheses of Theorem 1.3. As the proof of Theorem 1.3 is via the method of moving spheres, we wish to consider the following $\partial\mathbb{R}^N_+\times(0,\infty)$-indexed family of Kelvin inversions of u_i. For $x\in\partial\mathbb{R}^N_+$ and $\lambda>0$ let

$$\Sigma_{x,\lambda}=\mathbb{R}^N_+\setminus\overline{B}_\lambda(x)$$

and define

$$u_{i,x,\lambda}(y)=\left(\frac{\lambda}{|y-x|}\right)^{N-2}u_i\left(x+\frac{\lambda^2(y-x)}{|y-x|^2}\right),\qquad \text{for } y\in\overline{\mathbb{R}^N_+}\setminus\{x\} \text{ and } i\in J.$$

By using (1.3), (1.8) and (1.9) and computing directly, one may verify that $u_{1,x,\lambda},\cdots,u_{m,x,\lambda}$ satisfy

$$\begin{cases} \Delta u_{i,x,\lambda}+\prod_{j=1}^{m} u_{j,x,\lambda}^{a_{ij}}=0, & \text{in } \mathbb{R}_+^N, \\ \dfrac{\partial u_{i,x,\lambda}}{\partial y_N}=c_i\prod_{j=1}^{m} u_{j,x,\lambda}^{b_{ij}}, & \text{on } \partial\mathbb{R}_+^N\setminus\{x\}, \\ u_{i,x,\lambda}>0, & \text{in } \overline{\mathbb{R}_+^N}\setminus\{x\}, \end{cases} \qquad \text{for all } i\in J. \tag{2.1}$$

Since we want to compare u_i to $u_{i,x,\lambda}$, we define the differences

$$w_{i,x,\lambda}(y)=u_i(y)-u_{i,x,\lambda}(y), \qquad \text{for } y\in\overline{\mathbb{R}_+^N}\setminus\{x\} \text{ and } i\in J.$$

Using (1.9) and (2.1) one can verify that $w_{i,x,\lambda}$ satisfies

$$\begin{cases} -\Delta w_{i,x,\lambda}=\prod_{j=1}^{m} u_j^{a_{ij}}-\prod_{j=1}^{m} u_{j,x,\lambda}^{a_{ij}}, & \text{in } \Sigma_{x,\lambda}, \\ \dfrac{\partial w_{i,x,\lambda}}{\partial y_N}=c_i\left(\prod_{j=1}^{m} u_j^{b_{ij}}-\prod_{j=1}^{m} u_{j,x,\lambda}^{b_{ij}}\right), & \text{on } \partial\Sigma_{x,\lambda}\cap\partial\mathbb{R}_+^N, \end{cases} \qquad \text{for all } i\in J. \tag{2.2}$$

Moreover,

$$w_{i,x,\lambda}=0, \qquad \text{on } \partial\Sigma_{x,\lambda}\cap\partial B_\lambda(x), \text{ for all } i\in J. \tag{2.3}$$

As the proofs of many of the propositions given will be similar for $x=0$ and for general $x\in\partial\mathbb{R}_+^N$, when considering $x=0$ we will use the following simplified notation

$$\Sigma_{0,\lambda}=\Sigma_\lambda, \qquad u_{i,0,\lambda}=u_{i,\lambda} \qquad \text{and} \qquad w_{i,0,\lambda}=w_{i,\lambda}. \tag{2.4}$$

Proposition 2.1. *For each $x\in\partial\mathbb{R}_+^N$, there exists $\lambda_0(x)>0$ such that for all $\lambda\in(0,\lambda_0(x))$,*

$$w_{i,x,\lambda}\geq 0, \qquad \Sigma_{x,\lambda} \text{ for all } i\in J.$$

According to Proposition 2.1, for $x\in\partial\mathbb{R}_+^N$, we may define

$$\overline{\lambda}(x)=\sup\{\lambda>0: w_{i,x,\mu}\geq 0 \text{ in } \Sigma_{x,\mu} \text{ for all } \mu\in(0,\lambda) \text{ and all } i\in J\}.$$

For convenience, the proof of Proposition 2.1 will only be given for $x=0$ and the notation in (2.4) will be used. The proof for general $x\in\partial\mathbb{R}_+^N$ is similar to the proof for $x=0$. We begin by establishing three lemmas.

Lemma 2.1. *There exists $r_0>0$ such that for all $i\in J$ and all $\lambda\in(0,r_0)$,*

$$w_{i,\lambda}(y)>0, \qquad \textit{for all } y\in\overline{B_{r_0}^+}\setminus\overline{B}_\lambda.$$

Proof. For $(r,\theta)\in[0,\infty)\times\overline{\mathbb{S}^{N-1}_+}$ and $i\in J$ set $g_i(r,\theta)=r^{(N-2)/2}u_i(r,\theta)$, where $\overline{\mathbb{S}^{N-1}_+}$ is the closed, $(N-1)$-dimensional upper half sphere. Set

$$r_0=\min\left\{1,\frac{N-2}{4}\left(\min_{j\in J}\min_{\overline{B_1^+}}u_j\right)\left(\max_{j\in J}\|Du_j\|_{C^0(\overline{B_1^+})}\right)^{-1}\right\}.$$

For all $0<r\le r_0$ and for all $i\in J$, we have

$$\frac{\partial g_i}{\partial r}(r,\theta)\ge r^{(N-4)/2}\left(\frac{N-2}{2}\min_{\overline{B_1^+}}u_i-r\|Du_i\|_{C^0(\overline{B_1^+})}\right)>0.$$

In particular, if $0<\lambda\le r_0$ then with $\theta=y/|y|$,

$$w_{i,\lambda}(y)=|y|^{(2-N)/2}\left(g_i(|y|,\theta)-g_i\left(\frac{\lambda^2}{|y|},\theta\right)\right)>0,\qquad \text{for all } y\in\overline{B^+_{r_0}}\setminus\overline{B}_\lambda \text{ and all } i\in J.$$

□

Lemma 2.2. *If i is an index for which $c_i<0$, then* $\liminf_{|y|\to\infty}|y|^{N-2}u_i(y)>0$.

Proof. If $c_i\ge 0$ for all $i\in J$, there is nothing to prove. Otherwise, fix $R>0$ and fix $i\in J$ for which $c_i<0$. By (2.1) the hypotheses of Lemma 5.1 are satisfied by $u_{i,R}$. Therefore, for each $z\in\overline{B_R^+}\setminus\{0\}$

$$\left(\frac{R}{|z|}\right)^{N-2}u_i\left(\frac{R^2z}{|z|^2}\right)=u_{i,R}(z)\ge\min_{\partial B_R\cap\overline{\mathbb{R}^N_+}}u_{i,R}=\min_{\partial B_R\cap\overline{\mathbb{R}^N_+}}u_i.$$

Now, if $y\in\overline{\mathbb{R}^N_+}\setminus B_R$, set $z=R^2y/|y|^2$. Then $z\in\overline{B_R^+}\setminus\{0\}$, $y=R^2z/|z|^2$, and the above inequalities give

$$u_i(y)\ge\left(\min_{\partial B_R\cap\overline{\mathbb{R}^N_+}}u_i\right)R^{N-2}|y|^{2-N}.$$

Lemma 2.2 follows immediately. □

Lemma 2.3. *If i is an index for which $c_i\ge 0$, then* $\liminf_{|y|\to\infty}|y|^{N-2}u_i(y)>0$.

Proof. If $c_i<0$ for all $i\in J$ there is nothing to prove. Otherwise, fix an index i for which $c_i\ge 0$ and let

$$\mathcal{O}_i=\{y\in\mathbb{R}^N_+:u_i(y)<|y|^{2-N}\}.$$

Clearly, to prove Lemma 2.3 it suffices to show $\liminf_{|y|\to\infty;y\in\overline{\mathcal{O}_i}}|y|^{N-2}u_i(y)>0$. For $y\in\overline{\mathcal{O}_i}$ we have $u_i(y)^{N/(N-2)}\le|y|^{-2}u_i(y)$, so u_i satisfies

$$\begin{cases}-\Delta u_i>0, & \text{in } \mathcal{O}_i,\\ \dfrac{\partial u_i}{\partial y_N}-C_1|y|^{-2}u_i<0, & \text{on } \partial\mathbb{R}^N_+\cap\overline{\mathcal{O}}_i,\end{cases}$$

for some constant $C_1=C_1(\max_j|c_j|)>0$. For $A\gg 1$ fixed and to be determined, define

$$\xi(y)=|y-Ae_N|^{2-N}+|y|^{1-N},\qquad \text{for } |y|\geq 2A. \tag{2.5}$$

By direct computation, one may verify that ξ satisfies

$$\begin{cases} \Delta\xi>0, & \text{in } \mathbb{R}^N_+\setminus\overline{B}_{2A},\\ |y|^{-2}\xi(y)\leq C|y|^{-N}, & \text{in } \overline{\mathbb{R}^N_+}\setminus B_{2A},\\ \dfrac{\partial\xi}{\partial y_N}(y)=A(N-2)|y-Ae_N|^{-N}, & \text{on } \partial\mathbb{R}^N_+\setminus B_{2A}, \end{cases} \tag{2.6}$$

where C depends only on N. Therefore, we may choose $A=A(N,\max_j|c_j|)$ sufficiently large so that

$$\left(\frac{\partial}{\partial y_N}-C_1|y|^{-2}\right)\xi(y)>0,\qquad \text{on } \partial\mathbb{R}^N_+\setminus B_{2A}.$$

Fixing such an A and choosing $\epsilon>0$ small enough to achieve $u_i(y)>\epsilon\xi(y)$ on $(\partial B_{2A}\cap\overline{\mathbb{R}^N_+})\cup(\partial\mathcal{O}_i\cap\mathbb{R}^N_+)$, we obtain

$$\begin{cases} -\Delta(u_i-\epsilon\xi)>0, & \text{in } \mathcal{O}_i\setminus\overline{B}_{2A},\\ \left(\dfrac{\partial}{\partial y_N}-C_1|y|^{-2}\right)(u_i-\epsilon\xi)<0, & \text{on } (\partial\mathbb{R}^N_+\cap\overline{\mathcal{O}}_i)\setminus B_{2A},\\ (u_i-\epsilon\xi)(y)\geq 0, & \text{on } (\partial B_{2A}\cap\overline{\mathbb{R}^N_+})\cup[(\partial\mathcal{O}_i\cap\mathbb{R}^N_+)\setminus B_{2A}]. \end{cases} \tag{2.7}$$

Moreover, $\liminf_{|y|\to\infty}(u_i-\epsilon\xi)\geq 0$, so if $u_i-\epsilon\xi$ is negative at some point of $\overline{\mathcal{O}}_i\setminus B_{2A}$, then $u_i-\epsilon\xi$ must achieve a negative minimum at some point $\tilde{y}\in\overline{\mathcal{O}}_i\setminus B_{2A}$. By the maximum principle, we may assume $\tilde{y}\in\partial(\mathcal{O}_i\setminus B_{2A})$. The third item of (2.7) imposes $\tilde{y}\in(\partial\mathbb{R}^N_+\cap\overline{\mathcal{O}}_i)\setminus B_{2A}$. On the other hand, $(u_i-\epsilon\xi)(\tilde{y})<0$ and $\frac{\partial}{\partial y_N}(u_i-\epsilon\xi)(\tilde{y})\geq 0$, so the second item of (2.7) is violated. We conclude that $u_i-\epsilon\xi\geq 0$ in $\overline{\mathcal{O}}_i\setminus B_{2A}$. Consequently,

$$\liminf_{|y|\to\infty;y\in\overline{\mathcal{O}}_i}|y|^{N-2}u_i(y)\geq\epsilon\liminf_{|y|\to\infty}|y|^{N-2}\xi(y)>0.$$

Lemma 2.3 is established. □

Proof of Proposition 2.1. Let r_0 be as in Lemma 2.1. By Lemmas 2.2 and 2.3 we may first choose $c_0\in(0,1]$ such that

$$u_i(y)\geq c_0|y|^{2-N},\qquad \text{for all } y\in\overline{\mathbb{R}^N_+}\setminus B_{r_0},\quad \text{and all } i\in J,$$

and then choose $\lambda_0\in(0,r_0)$ such that

$$\lambda_0^{N-2}\left(\max_j\max_{\overline{B^+_{r_0}}}u_j\right)\leq c_0.$$

For such λ_0, if $0<\lambda\le\lambda_0$ then

$$u_{i,\lambda}(y)\le\lambda_0^{N-2}\left(\max_j\max_{\overline{B_{r_0}^+}}u_j\right)|y|^{2-N}\le c_0|y|^{2-N}\le u_i(y),\qquad \text{for all } y\in\overline{\mathbb{R}^N_+}\backslash B_{r_0} \text{ and all } i\in J.$$

Combining this with Lemma 2.1 establishes Proposition 2.1. □

3 A symmetry relation for $u_1,\cdots,u_m$

In this section we prove the following proposition.

Proposition 3.1. *For each* $x\in\partial\mathbb{R}^N_+$, $\overline{\lambda}(x)<\infty$ *and*

$$w_{i,x,\overline{\lambda}(x)}(y)\equiv 0,\qquad \text{for all } y\in\overline{\mathbb{R}^N_+}\setminus\{x\}\text{ and all } i\in J.$$

For convenience Proposition 3.1 will be proven for $x=0$ only. Proposition 3.1 will be established with the aid of some lemmas.

Lemma 3.1. *Let* $\mathcal{A}$ *be a matrix satisfying* (1.3) *and let* $x_0\in\partial\mathbb{R}^N_+$. *For* $\lambda\in(0,\overline{\lambda}(x_0)]$, *if there exists* $i_0\in J$ *for which* $w_{i_0,x_0,\lambda}\equiv 0$ *in* $\Sigma_{x_0,\lambda}$, *then*

$$w_{i,x_0,\lambda}\equiv 0,\qquad \text{in } \overline{\mathbb{R}^N_+}\setminus\{x_0\}\text{ for all } i\in J. \tag{3.1}$$

Proof. Clearly, it suffices to show that the equality in (3.1) holds for all $y\in\Sigma_{x_0,\lambda}$. The proof is given for $x_0=0$ only. The proof for general $x_0\in\partial\mathbb{R}^N_+$ is similar. Fix $0<\lambda\le\overline{\lambda}$. According to (2.2), the interior equation for $w_{i,\lambda}$ may be written

$$-\Delta w_{i,\lambda}=\sum_{j=1}^m\phi_{ij}(u_j^{a_{ij}}-u_{j,\lambda}^{a_{ij}}),\qquad \text{in } \Sigma_\lambda \text{ for all } i\in J, \tag{3.2}$$

where

$$\phi_{ij}=\left(\prod_{k=1}^{j-1}u_{k,\lambda}^{a_{ik}}\right)\left(\prod_{\ell=j+1}^{m}u_\ell^{a_{i\ell}}\right)>0.$$

Here the notational conventions $\prod_{k=1}^0 u_{k,\lambda}^{a_{ik}}=1$ and $\prod_{\ell=m+1}^m u_\ell^{a_{i\ell}}=1$ are used. Let i_0 be as in the hypotheses of the lemma and consider fixed but arbitrary $j_0\in J$. By irreducibility of $\mathcal{A}$ and non negativity of the entries of $\mathcal{A}$, there exists $k\le m-1$ and a sequence $i_0,i_1,\cdots,i_k=j_0$ of distinct elements of J such that

$$a_{i_\alpha i_{\alpha+1}}>0,\qquad \text{for all } \alpha\in\{0,1,\cdots,k-1\}.$$

Since $w_{i_0,\lambda}\equiv 0$ and by Eq. (3.2) with $i=i_0$ we have

$$0=\sum_{j=1}^m\phi_{i_0j}\left(u_j^{a_{i_0j}}-u_{j,\lambda}^{a_{i_0j}}\right),\qquad \text{in } \Sigma_\lambda.$$

By positivity of ϕ_{ij} and since $a_{i_0 i_1} > 0$ this equation ensures that $w_{i_1,\lambda} \equiv 0$ in Σ_λ. Similarly, using $a_{i_1 i_2} > 0$ and Eq. (3.2) with $i = i_1$ we deduce that $w_{i_2,\lambda} \equiv 0$ in Σ_λ. Repeating this argument a total of k times shows that $w_{i_\alpha,\lambda} \equiv 0$ in Σ_λ for all $\alpha \in \{1,2,\cdots,k\}$. In particular, $w_{j_0,\lambda} \equiv 0$ in Σ_λ. $\square$

Lemma 3.2. *If $x_0 \in \partial\mathbb{R}^N_+$ with $\overline{\lambda}(x_0) < \infty$, then $w_{i,x_0,\overline{\lambda}(x_0)} \equiv 0$ in $\overline{\mathbb{R}^N_+} \setminus \{x_0\}$ for all $i \in J$.*

Proof. For simplicity, we assume $x_0 = 0$. By Lemma 3.1, it suffices to show that there exists $i \in J$ such that $w_{i,\overline{\lambda}} \equiv 0$ in $\overline{\mathbb{R}^N_+} \setminus \{0\}$. In fact, we only need to show this equality holds in $\Sigma_{\overline{\lambda}}$ for some $i \in J$. For the sake of obtaining a contradiction, suppose that for all $i \in J$, there is some point of $\Sigma_{\overline{\lambda}}$ at which $w_{i,\overline{\lambda}}$ is positive. By the maximum principle we have

$$w_{i,\overline{\lambda}}(y) > 0, \qquad \text{for all } y \in \Sigma_{\overline{\lambda}} \text{ and all } i \in J. \tag{3.3}$$

Moreover,

$$w_{i,\overline{\lambda}}(y) > 0, \qquad \text{for all } y \in \partial\Sigma_{\overline{\lambda}} \setminus \partial B_{\overline{\lambda}} \text{ and all } i \in J. \tag{3.4}$$

Indeed, if $\tilde{y} \in \partial\Sigma_{\overline{\lambda}} \setminus \partial B_{\overline{\lambda}}$ and $i_0 \in J$ are such that with $w_{i_0,\overline{\lambda}}(\tilde{y}) = 0$, then apply Hopf's Lemma to $w_{i_0,\overline{\lambda}}$ on any ball $B \subset \Sigma_{\overline{\lambda}}$ such that $\partial B \cap \partial\Sigma_{\overline{\lambda}} = \{\tilde{y}\}$ to deduce

$$\frac{\partial w_{i_0,\overline{\lambda}}}{\partial y_N}(\tilde{y}) > 0. \tag{3.5}$$

On the other hand, if $c_{i_0} < 0$ then

$$\frac{\partial w_{i_0,\overline{\lambda}}}{\partial y_N}(\tilde{y}) = c_{i_0}\left(\prod_{j=1}^m u_j(\tilde{y})^{b_{i_0 j}} - \prod_{j=1}^m u_{j,\overline{\lambda}}(\tilde{y})^{b_{i_0 j}}\right) \leq 0.$$

If $c_{i_0} \geq 0$, then

$$\frac{\partial w_{i_0,\overline{\lambda}}}{\partial y_N}(\tilde{y}) = c_{i_0}\left(u_{i_0}(\tilde{y})^{N/(N-2)} - u_{i_0,\overline{\lambda}}(\tilde{y})^{N/(N-2)}\right) = 0.$$

In either case, (3.5) is violated, so (3.4) holds.

Now, for $y \in \partial B_{\overline{\lambda}} \cap \partial\Sigma_{\overline{\lambda}}$, let $\nu = \nu(y)$ denote the unit outer normal vector to $B_{\overline{\lambda}}$ (pointing into $\overline{\Sigma}_{\overline{\lambda}}$).

Claim 3.2. There exists $\epsilon > 0$ such that

$$\frac{\partial w_{i,\overline{\lambda}}}{\partial \nu}(y) \geq \epsilon, \qquad \text{for all } y \in \partial\Sigma_{\overline{\lambda}} \cap \partial B_{\overline{\lambda}} \text{ and all } i \in J.$$

Proof of Claim 3.2. In view of (3.3) and (2.3), a routine application of Hopf's Lemma yields the positivity of $\partial w_{i,\overline{\lambda}}/\partial\nu(y)$ for all $y \in \partial\Sigma_{\overline{\lambda}} \setminus \partial\mathbb{R}^N_+$ and all $i \in J$. Since $\partial\Sigma_{\overline{\lambda}} \cap \partial B_{\overline{\lambda}}$ is compact, Claim 3.2 will be established once we show

$$\frac{\partial w_{i,\overline{\lambda}}}{\partial \nu}(y) > 0, \qquad \text{for all } y \in \partial B_{\overline{\lambda}} \cap \partial\mathbb{R}^N_+ \text{ and all } i \in J. \tag{3.6}$$

To show this, define

$$\Omega=\{y\in\Sigma_{\overline{\lambda}}:\mathrm{dist}(y,\partial B_{\overline{\lambda}}\cap\partial\mathbb{R}^N_+)<\frac{\overline{\lambda}}{2}\}$$

and

$$\phi(y)=\delta e^{\alpha y_N}(|y|^2-\overline{\lambda}^2),$$

where $\delta>0$ (small) and $\alpha>0$ (large) are positive constants which are to be determined. Elementary computations yield

$$\begin{cases}\Delta\phi>0, & \text{in } \Sigma_{\overline{\lambda}},\\ \phi\equiv 0, & \text{on } \partial B_{\overline{\lambda}},\\ \dfrac{\partial\phi}{\partial y_N}=\alpha\phi, & \text{on } \partial\mathbb{R}^N_+,\\ \dfrac{\partial\phi}{\partial\nu}=2\delta\overline{\lambda}e^{\alpha y_N}, & \text{on } \partial B_{\overline{\lambda}}.\end{cases} \tag{3.7}$$

Moreover, if i is an index for which $c_i<0$, then by using each of the second item of (2.2), (3.4) and the third item of (3.7) one may verify that for any choice of $\alpha>0$

$$\frac{\partial}{\partial y_N}(w_{i,\overline{\lambda}}-\phi)\le -\alpha\phi\le\frac{\alpha}{2}(w_{i,\lambda}-\phi), \qquad \text{on } \partial\Omega\cap\partial\mathbb{R}^N_+. \tag{3.8}$$

If i is an index for which $c_i\ge 0$, then by Mean-Value Theorem, there is $\psi_i(y)\in[u_{i,\overline{\lambda}}(y),u_i(y)]$ such that

$$\begin{aligned}\frac{\partial}{\partial y_N}(w_{i,\overline{\lambda}}-\phi)&=c_i\left(u_i^{N/(N-2)}-u_{i,\overline{\lambda}}^{N/(N-2)}\right)-\alpha\phi\\ &=\frac{N}{N-2}c_i\psi_i^{2/(N-2)}w_{i,\overline{\lambda}}-\alpha\phi\\ &\le\frac{N}{N-2}\left(\max_j|c_j|\right)\left(\max_j\max_{\overline{\Omega}}u_j\right)^{2/(N-2)}w_{i,\overline{\lambda}}-\alpha\phi.\end{aligned}$$

Therefore, by choosing $\alpha=\alpha(N,\max_j|c_j|,\max_j\max_{\overline{\Omega}}u_j)$ sufficiently large, we obtain

$$\frac{\partial}{\partial y_N}(w_{i,\overline{\lambda}}-\phi)\le\frac{\alpha}{2}(w_{i,\overline{\lambda}}-\phi), \qquad \text{on } \partial\Omega\cap\partial\mathbb{R}^N_+. \tag{3.9}$$

Combining (3.8) and (3.9) we see that there is a constant $C_1>0$ for which

$$\frac{\partial}{\partial y_N}(w_{i,\overline{\lambda}}-\phi)\le C_1(w_{i,\overline{\lambda}}-\phi), \qquad \text{on } \partial\Omega\cap\partial\mathbb{R}^N_+ \text{ for all } i\in J. \tag{3.10}$$

Fix any such C_1. After choosing δ sufficiently small $w_{i,\overline{\lambda}}-\phi$ is seen to satisfy

$$\begin{cases}-\Delta(w_{i,\overline{\lambda}}-\phi)>0, & \text{in } \Omega,\\ w_{i,\overline{\lambda}}-\phi\equiv 0, & \text{on } \partial\Omega\cap\partial B_{\overline{\lambda}},\\ w_{i,\overline{\lambda}}-\phi>0, & \text{on } \partial\Omega\setminus\partial\Sigma_{\overline{\lambda}},\end{cases} \qquad \text{for all } i\in J. \tag{3.11}$$

By the maximum principle, if there exists $i_0\in J$ such that $w_{i_0,\overline{\lambda}}-\phi$ is negative at some point of $\overline{\Omega}$ then $w_{i_0,\overline{\lambda}}-\phi$ achieves a negative minimum value over $\overline{\Omega}$ at some point $\tilde{y}\in\partial\Omega$. By the second and third items of (3.11), we may assume $\tilde{y}\in\partial\mathbb{R}^N_+\cap\{y:\overline{\lambda}<|y|\le 3\overline{\lambda}/2\}$. Since $\tilde{y}$ is a minimizer of $w_{i_0,\overline{\lambda}}-\phi$ and by (3.10), we have

$$0\le\frac{\partial}{\partial y_N}(w_{i_0,\overline{\lambda}}-\phi)(\tilde{y})\le C_1(w_{i_0,\overline{\lambda}}-\phi)(\tilde{y})<0,$$

a contradiction. We conclude that $w_{i,\overline{\lambda}}\ge\phi$ in $\overline{\Omega}$ for all $i\in J$. In particular, $\frac{\partial w_{i,\overline{\lambda}}}{\partial\nu}\ge\frac{\partial\phi}{\partial\nu}$ on $\partial B_{\overline{\lambda}}\cap\partial\mathbb{R}^N_+$ for all $i\in J$. Combining this with the last item of (3.7), we obtain inequality (3.6). Claim 3.2 follows. $\square$

In view of Claim 3.2 and the continuity of $\lambda\mapsto w_{i,\lambda}$, we may choose $R_0>\bar{\lambda}$ such that

$$\frac{\partial w_{i,\lambda}}{\partial r}(y)\ge\frac{\epsilon}{2},\qquad\text{for all } y\in\overline{B^+_{R_0}}\setminus B_\lambda,\ \text{all } \lambda\in[\overline{\lambda},R_0] \text{ and all } i\in J.$$

Therefore,

$$w_{i,\lambda}(y)>0,\qquad \text{in } \overline{B^+_{R_0}}\setminus\overline{B}_\lambda \text{ for all } \lambda\in[\bar{\lambda},R_0] \text{ and all } i\in J. \tag{3.12}$$

Claim 3.3. If i is an index for which $c_i<0$, then $\liminf_{|y|\to\infty}|y|^{N-2}w_{i,\overline{\lambda}}(y)>0$.

Proof of Claim 3.3. If $c_i\ge 0$ for all $i\in J$, there is nothing to prove. Otherwise, let i be an index for which $c_i<0$ and define

$$h_i(y)=\left(\min_{\partial B_{R_0}\cap\overline{\mathbb{R}^N_+}}w_{i,\overline{\lambda}}\right)R_0^{N-2}|y|^{2-N},\qquad \text{for } |y|\ge R_0.$$

By performing elementary computations using (3.3), (3.4) and the negativity of c_i, one may verify that $w_{i,\overline{\lambda}}-h_i$ satisfies

$$\begin{cases}-\Delta(w_{i,\overline{\lambda}}-h_i)\ge 0, & \text{in } \mathbb{R}^N_+\setminus\overline{B}_{R_0},\\ w_{i,\overline{\lambda}}-h_i\ge 0, & \text{on } \partial B_{R_0}\cap\overline{\mathbb{R}^N_+},\\ \dfrac{\partial(w_{i,\overline{\lambda}}-h_i)}{\partial y_N}=c_i\left(\prod_{j=1}^m u_j^{a_{ij}}-\prod_{j=1}^m u_{j,\overline{\lambda}}^{a_{ij}}\right)<0, & \text{on } \partial\mathbb{R}^N_+\setminus B_{R_0}.\end{cases} \tag{3.13}$$

Moreover, using (3.3) once again we have

$$\liminf_{|y|\to\infty}(w_{i,\overline{\lambda}}-h_i)(y)\ge 0. \tag{3.14}$$

Consequently, if $w_{i,\overline{\lambda}}-h_i$ is negative at some point of $\overline{\mathbb{R}^N_+}\setminus B_{R_0}$, then $w_{i,\overline{\lambda}}-h_i$ attains a negative minimum value over $\overline{\mathbb{R}^N_+}\setminus B_{R_0}$ at some point $\tilde{y}\in\overline{\mathbb{R}^N_+}\setminus B_{R_0}$. By the maximum principle, we may assume $\tilde{y}\in\partial(\mathbb{R}^N_+\setminus B_{R_0})$. By the second item of (3.13) we must have $\tilde{y}\in\partial\mathbb{R}^N_+\setminus\overline{B}_{R_0}$. On the other hand, since $\tilde{y}$ minimizes $w_{i,\overline{\lambda}}-h_i$ and by the third item of (3.13) we have

$$0\leq\frac{\partial}{\partial y_N}(w_{i,\overline{\lambda}}-h_i)(\tilde{y})<0,$$

a contradiction. We conclude that $w_{i,\overline{\lambda}}\geq h_i$ in $\mathbb{R}^N_+\setminus B_{R_0}$. Claim 3.3 follows immediately. □

Claim 3.4. If i is an index for which $c_i\geq 0$, then $\liminf_{|y|\to\infty}|y|^{N-2}w_{i,\overline{\lambda}}(y)>0$.

Proof of Claim 3.4. The proof is similar to the proof of Lemma 2.3. Suppose i is an index for which $c_i\geq 0$ and set

$$\mathcal{O}_i=\{y\in\Sigma_{\overline{\lambda}}:w_{i,\overline{\lambda}}(y)<u_{i,\overline{\lambda}}(y)\}.$$

To prove Claim 3.4, it suffices to show that

$$\liminf_{|y|\to\infty;y\in\overline{\mathcal{O}}_i}|y|^{N-2}w_{i,\overline{\lambda}}(y)>0.$$

We have

$$u_i(y)\leq 2\overline{\lambda}^{N-2}\left(\max_j\max_{\overline{B^+_{\overline{\lambda}}}}u_j\right)|y|^{2-N},\qquad\text{for all }y\in\mathcal{O}_i. \tag{3.15}$$

According to the Mean-Value Theorem, there is $\psi_i(y)\in[u_{i,\overline{\lambda}}(y),u_i(y)]$ such that for all $y\in\partial\Sigma_{\overline{\lambda}}\cap\partial\mathbb{R}^N_+$,

$$u_i(y)^{N/(N-2)}-u_{i,\overline{\lambda}}(y)^{N/(N-2)}=\frac{N}{N-2}\psi_i(y)^{2/(N-2)}w_{i,\overline{\lambda}}(y)\leq\frac{N}{N-2}u_i(y)^{2/(N-2)}w_{i,\overline{\lambda}}(y).$$

Therefore, using the boundary equation for $w_{i,\overline{\lambda}}$ in (2.2) corresponding to $c_i\geq 0$ and using inequality (3.15), there is a constant $C_1=C_1(N,\overline{\lambda},\max_j|c_j|,\max_j\max_{\overline{B^+_{\overline{\lambda}}}}u_j)>0$ such that

$$\left(\frac{\partial}{\partial y_N}-C_1|y|^{-2}\right)w_{i,\overline{\lambda}}\leq 0,\qquad\text{for all }y\in\overline{\mathcal{O}}_i\cap\partial\mathbb{R}^N_+.$$

For $A\gg 1$ large and to be determined, let $\xi(y)$ be as in (2.5). Then ξ still satisfies (2.6) and by choosing A sufficiently large (and depending on C_1) we may achieve

$$\left(\frac{\partial}{\partial y_N}-C_1|y|^{-2}\right)\xi(y)>0,\qquad\text{on }\partial\mathbb{R}^N_+\setminus B_{2A}.$$

Fix any such A and choose $\epsilon>0$ sufficiently small so that

$$(w_{i,\overline{\lambda}}-\epsilon\xi)(y)>0,\qquad\text{on }(\partial B_{2A}\cap\overline{\mathbb{R}^N_+})\cup(\partial\mathcal{O}_i\cap\mathbb{R}^N_+).$$

Then

$$\begin{cases} -\Delta(w_{i,\overline{\lambda}}-\epsilon\xi)>0, & \text{in } \mathcal{O}_i\setminus\overline{B}_{2A}, \\ (w_{i,\overline{\lambda}}-\epsilon\xi)>0, & \text{on } \partial(\mathcal{O}_i\setminus B_{2A})\setminus\partial\mathbb{R}^N_+, \\ \left(\dfrac{\partial}{\partial y_N}-C_1|y|^{-2}\right)(w_{i,\overline{\lambda}}-\epsilon\xi)<0, & \text{on } (\overline{\mathcal{O}}_i\setminus B_{2A})\cap\partial\mathbb{R}^N_+. \end{cases} \tag{3.16}$$

Moreover, $\liminf_{|y|\to\infty}(w_{i,\overline{\lambda}}-\epsilon\xi)(y)\geq 0$. Claim 3.4 now follows by the argument in the proof of Lemma 2.3. □

In view of Claims 3.3 and 3.4 and with R_0 as in (3.12) we may choose $c_0>0$ such that

$$w_{i,\overline{\lambda}}(y)\geq c_0|y|^{2-N}, \qquad \text{for all } y\in\overline{\mathbb{R}^N_+}\setminus B_{R_0} \text{ and all } i\in J.$$

Therefore, for any $\lambda>0$ and any $i\in J$ we have

$$\begin{aligned} w_{i,\lambda}(y)=&w_{i,\overline{\lambda}}(y)+w_{i,\lambda}(y)-w_{i,\overline{\lambda}}(y) \\ \geq& c_0|y|^{2-N}+\left(\overline{\lambda}^{N-2}u_i\left(\frac{\overline{\lambda}^2y}{|y|^2}\right)-\lambda^{N-2}u_i\left(\frac{\lambda^2y}{|y|^2}\right)\right)|y|^{2-N}, \end{aligned} \tag{3.17}$$

for all $y\in\overline{\mathbb{R}^N_+}\setminus B_{R_0}$. By uniform continuity of u_i on $\overline{B}^+_{R_0}$, there exists $\epsilon_0\in(0,R_0-\overline{\lambda})$ such that

$$\left|\overline{\lambda}^{N-2}u_i\left(\frac{\overline{\lambda}^2y}{|y|^2}\right)-\lambda^{N-2}u_i\left(\frac{\lambda^2y}{|y|^2}\right)\right|<\frac{c_0}{2}, \quad \text{for all } y\in\overline{\mathbb{R}^N_+}\setminus B_{R_0}, \text{ all } \lambda\in[\overline{\lambda},\overline{\lambda}+\epsilon_0] \text{ and all } i\in J.$$

Using this estimate in inequality (3.17), we conclude that

$$w_{i,\lambda}(y)>\frac{c_0}{2}|y|^{2-N}, \qquad \text{for all } y\in\overline{\mathbb{R}^N_+}\setminus B_{R_0}, \text{ all } \lambda\in[\overline{\lambda},\overline{\lambda}+\epsilon_0] \text{ and all } i\in J.$$

Combining this estimate with (3.12), we conclude that $w_{i,\lambda}(y)\geq 0$ in $\overline{\mathbb{R}^N_+}\setminus B_\lambda$ for all $\lambda\in[\overline{\lambda},\overline{\lambda}+\epsilon_0]$ and all $i\in J$. This contradicts the definition of $\overline{\lambda}$. Lemma 3.2 is established. □

Lemma 3.3. *If there exists $x_0\in\partial\mathbb{R}^N_+$ for which $\overline{\lambda}(x_0)=\infty$, then $\overline{\lambda}(x)=\infty$ for all $x\in\partial\mathbb{R}^N_+$.*

Proof. Suppose $x_0\in\partial\mathbb{R}^N_+$ is such that $\overline{\lambda}(x_0)=\infty$. By definition of $\overline{\lambda}(x_0)$, for all $\lambda>0$ we have

$$u_i(y)\geq\left(\frac{\lambda}{|y-x_0|}\right)^{N-2}u_i\left(x_0+\frac{\lambda^2(y-x_0)}{|y-x_0|^2}\right), \qquad \text{in } \Sigma_{x_0,\lambda} \text{ for all } i\in J.$$

Consequently, $|y|^{N-2}u_i(y)\to\infty$ as $|y|\to\infty$ for all $i\in J$. Now suppose $x\in\partial\mathbb{R}^N_+$ is such that $\overline{\lambda}(x)<\infty$. By Lemma 3.2, $u_i=u_{i,x,\bar{\lambda}(x)}$ on $\overline{\mathbb{R}^N_+}\setminus\{x\}$ for all $i\in J$. Multiplying this equality by $|y|^{N-2}$ and letting $|y|\to\infty$ we obtain

$$|y|^{N-2}u_i(y)\to\overline{\lambda}(x)^{N-2}u_i(x)<\infty, \qquad \text{for all } i\in J,$$

which is a contradiction. □

Lemma 3.4. *For each* $x\in\partial\mathbb{R}^N_+$, $\overline{\lambda}(x)<\infty$.

Proof. If Lemma 3.4 fails, then by Lemma 3.3, we have $\overline{\lambda}(x)=\infty$ for all $x\in\partial\mathbb{R}^N_+$. By Lemma 5.2, we see that for all $i\in J$, $u_i(y)$ depends only on y_N. In this case, (1.9) becomes

$$\begin{cases} u_i''(t)=-\prod_{j=1}^m u_j(t)^{a_{ij}}, & \text{in } (0,\infty), \\ u_i'(0)=c_i\prod_{j=1}^m u_j(0)^{b_{ij}}, & \\ u_i(t)>0, & \text{on } [0,\infty), \end{cases} \qquad \text{for all } i\in J. \tag{3.18}$$

Combining the first and third items of (3.18), we see that u_i' is strictly decreasing in $(0,\infty)$ for all $i\in J$.

Now, observe that there is no index $i_0\in J$ for which $u_{i_0}'(0)=0$. Indeed, if such an i_0 were to exist then since u_{i_0}' is strictly decreasing, we would have $u_{i_0}'(1)<0$. By choosing t sufficiently large we could achieve

$$u_{i_0}(t)=u_{i_0}(1)+\int_1^t u_{i_0}'(s)\,\mathrm{d}s\le u_{i_0}(1)+u_{i_0}'(1)(t-1)<0,$$

which contradicts the third item of (3.18). By a similar argument, we see that there is no index $i_0\in J$ for which $u_{i_0}'(0)<0$. Therefore, we must have $u_i'(0)>0$ for all $i\in J$. Moreover, by an argument similar to the above, we see that

$$u_i'(t)>0, \qquad \text{for all } t\in[0,\infty) \text{ and all } i\in J.$$

In particular, u_i' is decreasing and bounded below by zero, so

$$\ell_i=\lim_{t\to\infty}u_i'(t),$$

exists and is non-negative for all $i\in J$. Since both $u_i(0)>0$ and $u_i'(t)>0$ in $[0,\infty)$ for all $i\in J$, there exists $\epsilon>0$ such that

$$u_i(t)\ge\epsilon, \qquad \text{for all } t\in[0,\infty) \text{ and all } i\in J. \tag{3.19}$$

In particular, this estimate implies $\prod_{j=1}^m u_j(t)^{a_{ij}}\ge\epsilon^{(N+2)/(N-2)}$, from which we deduce

$$\prod_{j=1}^m u_j(t)^{a_{ij}}\notin L^1(0,\infty). \tag{3.20}$$

On the other hand, by the first equality of (3.18), we have

$$u_i'(t)-u_i'(0)=-\int_0^t\prod_{j=1}^m u_j(s)^{a_{ij}}\,\mathrm{d}s.$$

Letting $t\to\infty$ in this equation we obtain

$$u_i'(0)-\ell_i=\int_0^\infty\prod_{j=1}^m u_j(s)^{a_{ij}}\,\mathrm{d}s,$$

so that $\prod_{j=1}^m u_j^{a_{ij}}\in L^1(0,\infty)$. This contradicts (3.20). Lemma 3.4 is established. □

Proof of Proposition 3.1. Combine the results of Lemmas 3.2 and 3.4. □

4 Completion of the proof of Theorem 1.3

By Proposition 3.1, for all $x\in\partial\mathbb{R}^N_+$, we have both $\overline{\lambda}(x)<\infty$ and

$$u_i(y)=\left(\frac{\overline{\lambda}(x)}{|y-x|}\right)^{N-2}u_i\left(x+\frac{\overline{\lambda}(x)^2(y-x)}{|y-x|^2}\right),\qquad \text{in } \overline{\mathbb{R}^N_+}\setminus\{x\}\ \text{ for all } i\in J. \tag{4.1}$$

Restricting this equality to $\mathbb{R}^{N-1}=\partial\mathbb{R}^N_+$, writing $y=y'+y_Ne_N$ with $y'\in\partial\mathbb{R}^N_+$ and applying Lemma 5.3 on $\mathbb{R}^{N-1}$, for each $i\in J$ we obtain $A_i\geq 0$, $d_i>0$ and $\bar{x}_i\in\partial\mathbb{R}^N_+$ such that

$$u_i(y')=\frac{A_i}{\left(d_i^2+|y'-\bar{x}_i|^2\right)^{(N-2)/2}},\qquad \text{for all } y'\in\partial\mathbb{R}^N_+. \tag{4.2}$$

By this expression and by (4.1), it is easy to see that

$$A_i=\lim_{|y'|\to\infty}|y'|^{N-2}u_i(y')=\overline{\lambda}(x)^{N-2}u_i(x)>0,\qquad \text{for all } x\in\partial\mathbb{R}^N_+. \tag{4.3}$$

Next, observe that

$$d_i=d_j \quad\text{and}\quad \bar{x}_i=\bar{x}_j,\qquad \text{for all } (i,j)\in J\times J. \tag{4.4}$$

Indeed, by (4.3) we have

$$\frac{u_i(x)}{A_i}=\frac{u_j(x)}{A_j},\qquad \text{for all } x\in\partial\mathbb{R}^N_+ \text{ and all } (i,j)\in J\times J.$$

In view of (4.2), the above equality yields

$$d_i^2+|x-\bar{x}_i|^2=d_j^2+|x-\bar{x}_j|^2,\qquad \text{for all } x\in\partial\mathbb{R}^N_+ \text{ and all } (i,j)\in J\times J.$$

The equalities in (4.4) follow immediately.

Returning to (4.2) with (4.4), and using d to denote the common value of d_i and $\bar{x}$ to denote the common value of $\bar{x}_i$, we obtain

$$u_i(x) = \frac{A_i}{\left(d^2 + |x-\bar{x}|^2\right)^{(N-2)/2}}, \qquad \text{for all } x \in \partial\mathbb{R}^N_+ \text{ and all } i \in J. \tag{4.5}$$

Now that we know the form of the restriction of u_i to $\partial\mathbb{R}^N_+$, we wish to deduce the form of u_i. To achieve this we follow the arguments of [21–23]. Using (4.3) to replace A_i in (4.5), we see that

$$\overline{\lambda}(x)^2 = d^2 + |x-\bar{x}|^2, \qquad \text{for all } x \in \partial\mathbb{R}^N_+. \tag{4.6}$$

Setting $Q = \bar{x} + de_N$ and $P = \bar{x} - de_N$, Eq. (4.6) says that for each $x \in \partial\mathbb{R}^N_+$, $\partial B(x,\overline{\lambda}(x))$ contains both P and Q.

Next, for $y \in \mathbb{R}^N$ consider

$$Ty = P + \frac{4d^2(y-P)}{|y-P|^2}, \tag{4.7}$$

the conformal inversion of y about $\partial B(P,2d)$. By performing elementary computations, one may verify that T enjoys the following properties.

(i) $T = T^{-1}$ on $\mathbb{R}^N \cup \{\infty\}$,

(ii) $T(\mathbb{R}^N_+) = B(Q,2d)$,

(iii) For each $x \in \partial\mathbb{R}^N_+$, the image of $\partial B(x,\overline{\lambda}(x))$ under T is the hyperplane $\mathcal{H}(x)$ through Q that is orthogonal to $x-P$.

(iv) If z and $\tilde{z}$ are symmetric about $\mathcal{H}(x)$, then Tz and $T\tilde{z}$ are symmetric about $\partial B(x,\bar{\lambda}(x))$ in the sense that

$$T\tilde{z} = x + \frac{\bar{\lambda}(x)^2(Tz-x)}{|Tz-x|^2}. \tag{4.8}$$

See Fig. 4.1 for a visual representation of the mapping properties of T. For $z \in B(Q,2d)$

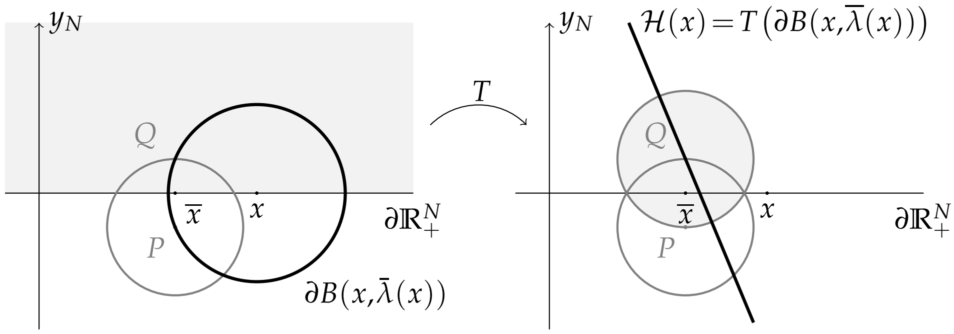

Figure 4.1: Visual representation of the properties of T

and $i\in J$, define

$$v_i(z)=\left(\frac{2d}{|z-P|}\right)^{N-2}u_i(Tz). \tag{4.9}$$

If $x\in\partial\mathbb{R}^N_+$, since u_i is symmetric about $\partial B(x,\overline{\lambda}(x))$ in the sense of Eq. (4.1), v_i is symmetric about $\mathcal{H}(x)$ in $B(Q,2d)$. Indeed, fix $x\in\partial\mathbb{R}^N_+$ and suppose $z,\tilde{z}\in B(Q,2d)$ are symmetric about $\mathcal{H}(x)$. By performing elementary computations using equations (4.1) and (4.8) we obtain

$$v_i(z)=\left(\frac{2d}{|z-P|}\right)^{N-2}\left(\frac{\bar{\lambda}(x)}{|Tz-x|}\right)^{N-2}u_i(T\tilde{z})=v_i(\tilde{z}).$$

Since this holds for all $x\in\partial\mathbb{R}^N_+$, v_i is radially symmetric about Q in $B(Q,2d)$.

Next, observe that the definition of v_i may be extended to P such that the resulting extension is continuous. Indeed, writing $y=Tz$ for $z\in B(Q,2d)$ and using (4.1) with $x=\overline{x}$ we have

$$\begin{aligned} v_i(z)&=\left(\frac{|y-P|}{2d}\right)^{N-2}u_i(y)\\ &=\left(\frac{|y-P|}{2d}\right)^{N-2}\left(\frac{\bar{\lambda}(\overline{x})}{|y-\overline{x}|}\right)^{N-2}u_i\left(\overline{x}+\frac{\bar{\lambda}(\overline{x})^2(y-\overline{x})}{|y-\overline{x}|^2}\right). \end{aligned}$$

Letting $z\to P$ from within $\overline{B}(Q,2d)\setminus\{P\}$ (so that $y\to\infty$ from within $\overline{\mathbb{R}^N_+}$) in this equality and using $\overline{\lambda}(\bar{x})=d$ gives

$$\lim_{z\to P;z\in\overline{B}(Q,2d)\setminus\{P\}}v_i(z)=\left(\frac{1}{2}\right)^{N-2}u_i(\bar{x})>0. \tag{4.10}$$

From now on, we identify v_i with its extension to P.

By an elementary computation, v_i is seen to satisfy

$$\begin{cases}\Delta v_i+\prod\limits_{j=1}^m v_j^{a_{ij}}=0, & \text{in } B(Q,2d),\\ \dfrac{\partial v_i}{\partial \nu}(z)+\dfrac{N-2}{4d}v_i(z)=-c_i\prod\limits_{j=1}^m v_j(z)^{b_{ij}}, & \text{on } \partial B(Q,2d),\\ v_i(z)>0, & \text{in } \overline{B}(Q,2d),\end{cases}\qquad \text{for all } i\in J, \tag{4.11}$$

where ν is the outward unit normal vector on the boundary of $B(Q,2d)$. Combining the first and third items of (4.11) implies that v_i is non-constant in $B(Q,2d)$ for all $i\in J$. By a simple maximum-principle argument and since v_i is radial about Q we see that v_i is strictly decreasing about Q in $B(Q,2d)$. Setting $r=|z-Q|$ we have $v_i(z)=\psi_i(r)$ for some smooth decreasing functions $\psi_i:[0,2d)\to(0,\infty)$. Using (4.10) and (4.11), these functions are seen to satisfy

$$\begin{cases} \psi_i''(r) + \dfrac{N-1}{r}\psi_i'(r) + \prod_{j=1}^{m}\psi_j(r)^{a_{ij}} = 0, & \text{for } 0<r<2d, \\ \psi_i'(2d) + \dfrac{N-2}{4d}\psi_i(2d) = -c_i\prod_{j=1}^{m}\psi_j(2d)^{b_{ij}}, & \text{for all } i\in J. \\ \psi_i(2d) = 2^{2-N}u_i(\bar{x}). \end{cases} \tag{4.12}$$

By the uniqueness of solutions to this system, there are positive constants $\alpha_1,\cdots,\alpha_m$ and μ satisfying

$$\log\alpha_i = \sum_{j=1}^{m} a_{ij}\log\alpha_j - \log\left(\mu^2 N(N-2)\right), \qquad \text{for all } i\in J \tag{4.13}$$

such that

$$\psi_i(r) = \frac{\alpha_i}{(\mu^2+r^2)^{(N-2)/2}}, \qquad \text{for all } i\in J.$$

Using this in Eq. (4.9) with $z=Ty$, we have

$$u_i(y) = \left(\frac{|Ty-P|}{2d}\right)^{N-2} \frac{\alpha_i}{\left(\mu^2+|Ty-Q|^2\right)^{(N-2)/2}} = \frac{\beta_i}{\left(\sigma^2+|y-y^0|^2\right)^{(N-2)/2}}, \tag{4.14}$$

for all $y\in\overline{\mathbb{R}^N_+}$ and all $i\in J$, where

$$\beta_i = \left(\frac{4d^2}{\mu^2+4d^2}\right)^{(N-2)/2}\alpha_i, \qquad \sigma^2 = \mu^2\left(\frac{4d^2}{\mu^2+4d^2}\right)^2 \qquad \text{and} \qquad y^0 = \overline{x} - d\frac{\mu^2-4d^2}{\mu^2+4d^2}e_N.$$

In particular, y^0 is independent of i. For convenience, the details of the computation that yields the second equality in (4.14) are provided in Lemma 5.4 of the appendix. By (4.13) and the expressions of σ^2 and β_i, it is routine to verify that σ^2 and $\beta_1,\cdots,\beta_m$ satisfy (1.6). Moreover, by using both the second item of (4.12) and (4.13) one may verify that (1.10) is satisfied.

5 Appendix

Lemma 5.1. *Let $R>0$ and suppose v is a solution of*

$$\begin{cases} -\Delta v \geq 0, & \text{in } B_R^+, \\ \dfrac{\partial v}{\partial y_N} < 0, & \text{on } (\partial B_R^+ \cap \partial\mathbb{R}^N_+)\setminus\{0\}, \\ v>0, & \text{on } \overline{B_R^+}\setminus\{0\}. \end{cases}$$

Then $v(y) \geq \min_{\partial B_R\cap\overline{\mathbb{R}^N_+}} v$ for all $y\in\overline{B_R^+}\setminus\{0\}$.

Proof. Set $m_R=\min_{\partial B_R\cap\overline{\mathbb{R}^N_+}} v$ and fix $0<\epsilon<R$. Define

$$\phi(y)=m_R\frac{\epsilon^{2-N}-|y|^{2-N}}{\epsilon^{2-N}-R^{2-N}},\qquad \text{for } \epsilon\le|y|\le R.$$

One may easily verify that $v-\phi$ satisfies

$$\begin{cases}-\Delta(v-\phi)\ge 0, & \text{in } B_R^+\setminus B_\epsilon,\\ \dfrac{\partial(v-\phi)}{\partial y_N}<0, & \text{on } \partial(B_R^+\setminus B_\epsilon)\cap\partial\mathbb{R}^N_+,\\ v-\phi\ge 0, & \text{on } (\partial B_R\cup\partial B_\epsilon)\cap\overline{\mathbb{R}^N_+}.\end{cases}\tag{5.1}$$

According to the maximum principle and the third item of (5.1), if $v-\phi$ is negative at any point of $\overline{B_R^+}\setminus B_\epsilon$, then there is $x_0\in\partial\mathbb{R}^N_+\cap\{\epsilon<|y|<R\}$ such that

$$\min_{\overline{B_R^+}\setminus B_\epsilon}(v-\phi)=(v-\phi)(x_0)<0.$$

Moreover, since $x_0\in\partial\mathbb{R}^N_+$ is a minimizer of $v-\phi$, we have $\frac{\partial}{\partial y_N}(v-\phi)(x_0)\ge 0$. This violates the second item of (5.1). We conclude that $v\ge\phi$ in $\overline{B_R^+}\setminus B_\epsilon$. Finally, if $y\in\overline{B_R^+}\setminus\{0\}$, and if $0<\epsilon<|y|/2$ we have

$$v(y)\ge m_R\frac{\epsilon^{2-N}-|y|^{2-N}}{\epsilon^{2-N}-R^{2-N}}.$$

Letting $\epsilon\to 0$ in this inequality gives the desired result. □

The proofs of the following two lemmas can be found in [20,21] or [22].

Lemma 5.2. *Let $f\in C^1(\mathbb{R}^N_+)$, $N\ge 2$ and $b>0$. If f satisfies*

$$f(y)\ge\left(\frac{\lambda}{|y-x|}\right)^b f\left(x+\frac{\lambda^2(y-x)}{|y-x|^2}\right),\qquad \text{for all } y\in\mathbb{R}^N_+,\ x\in\partial\mathbb{R}^N_+ \text{ and } \lambda>0,$$

then $f(y)=f(y_Ne_N)$ for all $y\in\mathbb{R}^N_+$, where $e_N=(0,\cdots,0,1)$.

Lemma 5.3. *Let $f\in C^1(\mathbb{R}^N)$, $N\ge 1$ and $b>0$. Suppose that for every $x\in\mathbb{R}^N$, there exists $\lambda(x)>0$ such that*

$$\left(\frac{\lambda(x)}{|y-x|}\right)^b f\left(x+\frac{\lambda(x)^2(y-x)}{|y-x|^2}\right)=f(y),\qquad \text{for all } y\in\mathbb{R}^N\setminus\{x\}.$$

Then there exists $a\ge 0$, $d>0$ and $\bar{x}\in\mathbb{R}^N$ such that

$$f(x)=\pm\left(\frac{a}{d+|\bar{x}-x|^2}\right)^{b/2}.$$

Lemma 5.4. *Let μ, $\alpha_1,\cdots,\alpha_m$ be as in* (4.13)*, let $P=-de_N$ and $Q=de_N$ and let T be as in* (4.7)*. If σ^2, $\beta_1,\cdots,\beta_m$ and y^0 are given by*

$$\beta_i=\left(\frac{4d^2}{\mu^2+4d^2}\right)^{(N-2)/2}\alpha_i,\qquad \sigma^2=\mu^2\left(\frac{4d^2}{\mu^2+4d^2}\right)^2\qquad\text{and}\qquad y^0=\overline{x}-d\frac{\mu^2-4d^2}{\mu^2+4d^2}e_N,$$

then

$$\left(\frac{|Ty-P|}{2d}\right)^{N-2}\frac{\alpha_i}{\left(\mu^2+|Ty-Q|^2\right)^{(N-2)/2}}=\frac{\beta_i}{\left(\sigma^2+|y-y^0|^2\right)^{(N-2)/2}}.$$

Proof. The computation is elementary. Some details are provided for the convenience of the reader. First, since $|Ty-P|/(2d)=2d/|y-P|$, we consider the denominator on the right-hand side of the equation

$$\left(\frac{|Ty-P|}{2d}\right)^2\frac{\alpha_i^{2/(N-2)}}{\mu^2+|Ty-Q|^2}=\frac{(2d)^2\alpha_i^{2/(N-2)}}{|y-P|^2\left(\mu^2+|Ty-Q|^2\right)}. \tag{5.2}$$

Using the definition of T in Eq. (4.7), the equality $P-Q=-2de_N$ and performing elementary computations yields

$$|y-P|^2|Ty-Q|^2=4d^2\left(|y-P|^2-4d\langle y-P,e_N\rangle+4d^2\right)=4d^2\left(|y-P|^2-4dy_N\right),$$

where $\langle\cdot,\cdot\rangle$ is the usual Euclidean inner product and y_N is the N^{th} component of y. Next, we use $|y-P|^2=|y'-\bar{x}|^2+(y_N+d)^2$ and the above equality to see that the denominator of the right-hand side of (5.2) is

$$\begin{aligned}|y-P|^2\left(\mu^2+|Ty-Q|^2\right)&=(\mu^2+4d^2)|y-P|^2-16d^3\\&=(\mu^2+4d^2)\left[|y'-\bar{x}|^2+\left(y_N+d\frac{\mu^2-4d^2}{\mu^2+4d^2}\right)^2+\mu^2\left(\frac{4d^2}{\mu^2+4d^2}\right)^2\right]\\&=(\mu^2+4d^2)\left[\left|y-\left(\bar{x}-d\frac{\mu^2-4d^2}{\mu^2+4d^2}e_N\right)\right|^2+\mu^2\left(\frac{4d^2}{\mu^2+4d^2}\right)^2\right].\end{aligned}$$

Using this in (5.2) completes the proof of the lemma. □

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