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A Singular Trudinger-Moser Inequality in Hyperbolic Space

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Abstract. In this paper, we establish a singular Trudinger-Moser inequality for the whole hyperbolic space $\mathbb{H}^n$:

$$\sup_{u\in W^{1,n}(\mathbb{H}^n),\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^n d\mu\le 1}\int_{\mathbb{H}^n}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}}{\rho^\beta}d\mu<\infty\Longleftrightarrow\frac{\alpha}{\alpha_n}+\frac{\beta}{n}\le 1,$$

where $\alpha>0,\beta\in[0,n)$, ρ and $d\mu$ are the distance function and volume element of $\mathbb{H}^n$ respectively.

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Key Words: Singular Trudinger-Moser inequlity; hyperbolic space.

1 Introduction

In the past forty years, Trudinger-Moser inequality has play an important role in analysis and geometry. People call it Trudinger-Moser inequality because it was first proposed by Trudinger [1] in 1967: $\exists\alpha,C>0$, s.t.

$$\sup_{u\in W_0^{1,n}(\Omega),\int_\Omega|\nabla u|^n dx\le 1}\int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}}dx\le C|\Omega|, \tag{1.1}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω, and then improved by Moser [2] in 1971: the best constant for α is $\alpha_n=n\omega_{n-1}^{\frac{1}{n-1}}$, $\omega_{n-1}=|S^{n-1}|$. Here the best constant means that: if

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$\alpha \le \alpha_n$, then inequality (1.1) holds; if $\alpha > \alpha_n$, then there exists a sequence $\{u_k\} \subset W_0^{1,n}(\Omega)$ with $\int_\Omega |\nabla u_k|^n \mathrm{d}x \le 1$, but $\int_\Omega e^{\alpha u_k^2} \mathrm{d}x \to \infty$ as $k \to \infty$. As limit case of the Sobolev embedding theorem, there is no need to say the importance of (1.1) in analysis. One more word we want to say here it that, using a similar inequality, Moser [3] solved the prescribing Gauss curvature problem on $\mathbb{R}P^2$.

Roughly speaking, the classical Trudinger-Moser inequality ((1.1) with $\alpha = \alpha_n$) has the following four kinds of generalizations:

(1) To high order derivatives, i.e., to $W_0^{m,\frac{n}{m}}(\Omega)$, this work was done by Adams [4] in 1988.

(2) To compact manifolds with or without boundary, this problem was first attempted by Aubin [5] in 1970, then studied by Cherrier [6] in 1979 and solved by Fontana [7] in 1993.

(3) To the whole Euclidean spaces, this problem was first attempted by Cao [8] in 1992, then studied by Panada [9] in 1995, do Ó [10] in 1997, Ruf [11] in 2005, Li-Ruf [12] in 2008, Adimurthi-Yang [13] in 2010, and Yang-Zhu [14] in 2013.

(4) To the whole complete noncompact manifolds, this problem was first attempted by Yang [15] in 2012 for general manifolds. When manifold is $\mathbb{H}^n$, the hyperbolic space with constant sectional curvature -1, this problem was studied by Mancini-Sandeep [16] in 2010, Adimurthi-Tintarev [17] in 2010, Battaglia [18] and Mancini [19] in 2011, Wang-Ye [20] in 2012, Tintarev [21] and Mancini-Sandeep-Tintarev [22] in 2013, and Yang-Zhu [23] in 2014.

In this paper, we will establish a singular Trudinger-Moser inequality on the whole hyperbolic space $\mathbb{H}^n$. Before stating the main result, let us review some relevant results in the past few years. In 2007, Aimurthi-Sandeep [24] first derived a singular Trudinger-Moser inequality on a bounded domain in $\mathbb{R}^n$ containing the origin, they proved

$$\int_\Omega \frac{e^{\alpha |u|^{\frac{n}{n-1}}}}{|x|^\beta} \mathrm{d}x < \infty \tag{1.2}$$

and

$$\sup_{u \in W_0^{1,n}(\Omega), \int_\Omega |\nabla u|^n \mathrm{d}x \le 1} \int_\Omega \frac{e^{\alpha |u|^{\frac{n}{n-1}}}}{|x|^\beta} \mathrm{d}x < \infty \Longleftrightarrow \frac{\alpha}{\alpha_n} + \frac{\beta}{n} \le 1, \tag{1.3}$$

where $\alpha > 0, \beta \in [0,n)$. In 2010, Adimurthi-Yang [13] generalized (1.3) to the whole Euclidean space $\mathbb{R}^n$, they obtained

$$\sup_{||u||_{1,\tau} \le 1} \int_{\mathbb{R}^n} \frac{e^{\alpha |u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{\frac{nk}{n-1}}}{k!}}{|x|^\beta} \mathrm{d}x < \infty \Longleftrightarrow \frac{\alpha}{\alpha_n} + \frac{\beta}{n} \le 1, \tag{1.4}$$

where $||u||_{1,\tau} = \left(\int_{\mathbb{R}^n} (|\nabla u|^n + \tau |u|^n) \mathrm{d}x \right)^{\frac{1}{n}}$, $\alpha > 0$ and $\beta \in [0,n)$. Then in 2012, with the help of (1.4), Yang [25] obtained some existence results of positive solutions to quasi-linear elliptic equations with exponential growth in $\mathbb{R}^n$. Recently, Yang [26] generalized (1.4) to the entire Heisenberg group.

In this paper, we will establish the corresponding inequalities of (1.3) and (1.4) in $\mathbb{H}^n$. In fact, we obtain

Theorem 1.1. *$\forall \alpha > 0, \beta \in [0,n)$, we have*

$$\int_{\mathbb{H}^n} \frac{e^{\alpha|u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{\frac{nk}{n-1}}}{k!}}{\rho^\beta} d\mu < \infty.$$

Moreover,

$$\sup_{u\in W^{1,n}(\mathbb{H}^n), \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^n d\mu \le 1} \int_{\mathbb{H}^n} \frac{e^{\alpha|u|^{\frac{n}{n-1}}} - \sum_{k=0}^{n-2} \frac{\alpha^k |u|^{\frac{nk}{n-1}}}{k!}}{\rho^\beta} d\mu < \infty \Longleftrightarrow \frac{\alpha}{\alpha_n} + \frac{\beta}{n} \le 1,$$

where ρ and $d\mu$ are the distance function and volume element of $\mathbb{H}^n$ respectively.

This paper is organized as follows: In Section 2, we will state some preliminaries, including the Poincaré ball model of the hyperbolic space $\mathbb{H}^n$, the symmetric decreasing rearrangements of functions in $W^{1,n}(\mathbb{H}^n)$, and an important radial lemma; In Section 3, we will complete the proof of Theorem 1.1.

2 Preliminaries

In this section, we will recall the Poincaré ball model of the hyperbolic space $\mathbb{H}^n$ and state an important radial lemma which will be used in the next section.

2.1 Poincaré ball

Denote $B_1 \subset \mathbb{R}^n$ the unit ball centered at the origin, equipped B_1 with

$$ds^2 = \frac{4\sum_{i=1}^n dx_i^2}{(1-r^2)^2},$$

where $r = \sqrt{\sum_{i=1}^n x_i^2}$ is the Euclidean distance from the origin to $x \in B_1$. Using $d\mu$ to denote the volume element with respect to ds^2, i.e.,

$$d\mu = \frac{2^n}{(1-r^2)^n} dx.$$

Direct calculation tells us,

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^n d\mu = \int_{B_1} |\nabla u|^n dx.$$

So

$$W^{1,n}(\mathbb{H}^n)=\{u|\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^n d\mu<\infty\}$$

can be identified with $W_0^{1,n}(B_1)$.

Denote $W_{0,r}^{1,n}(B_1)$ the subspace of radially symmetric functions of $W_0^{1,n}(B_1)$.

We will use symmetric decreasing rearrangements of functions in $W^{1,n}(\mathbb{H}^n)$. Let $u\in W^{1,n}(\mathbb{H}^n)$, denote u^* its symmetric decreasing rearrangement function. It is well known that

$$\int_{\mathbb{H}^n}|u^*|^n d\mu=\int_{\mathbb{H}^n}|u|^n d\mu, \quad \text{and} \quad \int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u^*|^n d\mu\le\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^n d\mu.$$

For more details about symmetric decreasing rearrangements of functions in $W^{1,n}(\mathbb{H}^n)$, we refer the reader to [27,28].

2.2 Radial lemma

Lemma 2.1. ([22, Lemma 2.2]) *Let* $u\in W_{0,r}^{1,n}(B_1)$, *then*

$$|u(r)|\le\frac{\ln^{\frac{n-1}{n}}(\frac{1}{r})}{\omega_{n-1}^{\frac{1}{n}}}||\nabla u||_n, \qquad r\in(0,1).$$

3 Proof of Theorem 1.1

The proof of Theorem 1.1 mainly depends on Lemma 2.1 and the singular Trudinger-Moser inequality (1.3) of Adimurth and Sandeep .

By the standard rearrangement argument [27,28] applied on $\mathbb{H}^n$, it suffices to consider the inequality only for radial functions on the Poincaré ball. Let $u\in W_{0,r}^{1,n}(B_1)$ be an arbitrary function satisfying $\int_{B_1}|\nabla u|^n dx\le 1$. We divide the integral into two parts: (i) $0<r<\frac{1}{2}$ and (ii) $r\ge\frac{1}{2}$.

$$\begin{aligned}
&\int_{\mathbb{H}^n}\frac{1}{\rho^\beta}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)d\mu=\int_{B_1}\frac{1}{(\ln\frac{1+r}{1-r})^\beta}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\frac{2^n}{(1-r^2)^n}dx\\
\le&\int_{B_1}\frac{1}{r^\beta}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\frac{2^n}{(1-r^2)^n}dx=\int_{B_{\frac{1}{2}}}\frac{1}{r^\beta}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\frac{2^n}{(1-r^2)^n}dx\\
&+\int_{B_1\setminus B_{\frac{1}{2}}}\frac{1}{r^\beta}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\frac{2^n}{(1-r^2)^n}dx\\
\le& C\int_{B_{\frac{1}{2}}}\frac{1}{r^\beta}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)dx
\end{aligned}$$

$$+C\int_{B_1\backslash B_{\frac{1}{2}}}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\frac{2^n}{(1-r^2)^n}\mathrm{d}x. \tag{3.1}$$

For the first integral in (3.1), by (1.2) we have

$$\int_{B_{\frac{1}{2}}}\frac{1}{r^\beta}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\mathrm{d}x\le\int_{B_1}\frac{1}{r^\beta}e^{\alpha|u|^{\frac{n}{n-1}}}\mathrm{d}x<\infty,\qquad\forall\alpha>0,\ \beta\in[0,n).$$

When $\alpha/\alpha_n+\beta/n\le 1$, by (1.3) we have

$$\begin{aligned}&\int_{B_{\frac{1}{2}}}\frac{1}{r^\beta}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\mathrm{d}x\le\int_{B_1}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}}{r^\beta}\mathrm{d}x\\&\le\sup_{v\in W_0^{1,n}(B_1),\int_{B_1}|\nabla v|^n dx\le 1}\int_{B_1}\frac{e^{\alpha|v|^{\frac{n}{n-1}}}}{r^\beta}\mathrm{d}x<\infty.\end{aligned} \tag{3.2}$$

For the second integral in (3.1), first noticing

$$e^t-\sum_{k=0}^{n-2}\frac{t^k}{k!}\le Ct^{n-1}e^t,\quad(t\ge 0). \tag{3.3}$$

From Lemma 2.1 we know that

$$e^{\alpha|u|^{\frac{n}{n-1}}}\le C,\quad r\in[\frac{1}{2},1]. \tag{3.4}$$

By (3.3) and (3.4) we have

$$\begin{aligned}&\int_{B_1\backslash B_{\frac{1}{2}}}\left(e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}\right)\frac{2^n}{(1-r^2)^n}\mathrm{d}x\le C\int_{B_1\backslash B_{\frac{1}{2}}}|u|^ne^{\alpha|u|^{\frac{n}{n-1}}}\frac{2^n}{(1-r^2)^n}\mathrm{d}x\\&\le C\int_{\mathbb{H}^n}|u|^n\mathrm{d}\mu\le C\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^n\mathrm{d}\mu\le C,\end{aligned} \tag{3.5}$$

where the third inequality is due to the Poincaré-Sobolev inequality (c.f. in [22, Lemma 2.1]).

By combining (3.2) and (3.5), we have obtained

$$\sup_{u\in W^{1,n}(\mathbb{H}^n),\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^n\mathrm{d}\mu\le 1}\int_{\mathbb{H}^n}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}}{\rho^\beta}\mathrm{d}\mu<\infty,\quad\text{when }\frac{\alpha}{\alpha_n}+\frac{\beta}{n}\le 1. \tag{3.6}$$

Next, we use the Moser function to show that (3.6) does not hold if $\alpha/\alpha_n+\beta/n>1$. For $0<l<1/2$, let u_l be the Moser function

$$u_l(x)=\frac{1}{\omega_{n-1}^{\frac{1}{n}}}\begin{cases}\left(\ln\frac{1}{2l}\right)^{\frac{n-1}{n}}, & 0\leqslant|x|\leqslant l,\\ \dfrac{\ln\frac{1}{2|x|}}{\left(\ln\frac{1}{2l}\right)^{\frac{1}{n}}}, & l\leqslant|x|\leqslant\frac{1}{2},\\ 0, & |x|>\frac{1}{2}.\end{cases}$$

Then,

$$\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u_l|^n\mathrm{d}\mu=\int_{B_1}|\nabla u_l|^n\mathrm{d}x=1,$$

and

$$\begin{aligned}&\int_{\mathbb{H}^n}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}}{\rho^\beta}\mathrm{d}\mu=\int_{B_1}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}}{\left(\ln\frac{1+r}{1-r}\right)^\beta}\frac{2^n}{(1-r^2)^n}\mathrm{d}x\\
\geqslant&\int_{B_l}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}}{\left(\ln\frac{1+r}{1-r}\right)^\beta}\frac{2^n}{(1-r^2)^n}\mathrm{d}x\geqslant\int_{B_l}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}}{\left(\frac{2r}{1-r}\right)^\beta}\frac{2^n}{(1-r^2)^n}\mathrm{d}x\\
\geqslant&2^{-\beta}\int_{B_l}\frac{1}{r^\beta}\mathrm{d}x\left(e^{\alpha\omega_{n-1}^{-\frac{1}{n-1}}\ln\frac{1}{2l}}+O\left(\left(\ln\frac{1}{2l}\right)^{n-2}\right)\right)\\
=&2^{-\beta}\omega_{n-1}\frac{l^{n-\beta}}{n-\beta}\left(e^{\alpha\omega_{n-1}^{-\frac{1}{n-1}}\ln\frac{1}{2l}}+O\left(\left(\ln\frac{1}{2l}\right)^{n-2}\right)\right)\\
=&2^{-n\left(\frac{\alpha}{\alpha_n}+\frac{\beta}{n}\right)}\omega_{n-1}\frac{1}{n-\beta}\left(l^{-n\left(\frac{\alpha}{\alpha_n}+\frac{\beta}{n}-1\right)}+o(1)\right),\qquad\text{as } l\to 0+.\end{aligned}$$

Hence, when $\alpha/\alpha_n+\beta/n>1$, we have

$$\begin{aligned}&\sup_{u\in W^{1,n}(\mathbb{H}^n),\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u|^n\mathrm{d}\mu\leq 1}\int_{\mathbb{H}^n}\frac{e^{\alpha|u|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u|^{\frac{nk}{n-1}}}{k!}}{\rho^\beta}\mathrm{d}\mu\\
\geqslant&\sup_{\int_{\mathbb{H}^n}|\nabla_{\mathbb{H}^n}u_l|^n\mathrm{d}\mu=1}\int_{\mathbb{H}^n}\frac{e^{\alpha|u_l|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha^k|u_l|^{\frac{nk}{n-1}}}{k!}}{\rho^\beta}\mathrm{d}\mu=\infty.\end{aligned}$$

This completes the proof of Theorem 1.1.

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