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Existence and Properties of Radial Solutions of a Sub-linear Elliptic Equation

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Abstract. A non lipschitzian nonlinear elliptic equation is reviewed and results of existence, uniqueness, positivity and classification are proved using direct methods derived from the equation.

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1 Main results

In this paper we focus on the study of the equation

$$\Delta u+u-|u|^{-2\theta}u=0, \quad \text{on } \mathbb{R}^d, \tag{1.1}$$

with $d>1$, and $0<\theta<1/2$.

Such a problem and more generally $\Delta u+f(x,u)=0$ has been the object of numerous studies because of its interests. Indeed, it can be understood as a time-dependent problem such as

$$\begin{cases} \mathcal{L}(t,u)+f(t,u)=0, & \text{on } \Omega\times[0,T], \\ u(x,t)=0, & \text{on } \partial\Omega\times[0,T], \\ u(x,0)=u_0(x), \end{cases} \tag{1.2}$$

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where $\mathcal{L}(t,u) = i\partial u/\partial t - \Delta u$ for the famous Schrödinger operator for example, $\mathcal{L}(t,u) = \partial u/\partial t - \Delta u$ for Heat equation, $\mathcal{L}(t,u) = \partial^2 u/\partial t^2 - \Delta u$ for Wave equation, ..., and with a suitable domain $\Omega \subset \mathbb{R}^d$. The easy case is when non linear term $f(u)$ is assumed to be locally Lipschitz continuous which is not the case in many interesting cases in physics such as nonlinear waves, Shrödinger equation when dealing with complex case, chemical reaction models, population genetics problems, reactor dynamics and heat conduction. For backgrounds on (1.2), we refer to [1] and the references therein.

Problem (1.1) has been firstly studied in [2] using a shooting method and a phase plane analysis to prove the existence of nodal radial compactly supported solutions. In [3], a classification of the solutions of the problem $\Delta u + |u|^{p-1}u + \lambda u = 0$, in the unit ball B_1 in $\mathbb{R}^d$, with $d \geq 2$ and $p > 1$ have been developed. The authors analyzed in subcritical, critical and supercritical cases the possible singularities of the solution at the origin and obtained thus three different classes. The critical behavior is related to the exponent p whom critical value is $p_c = (d+2)/(d-2)$ for $d \geq 3$. In [4] and [5], the authors have focused on the mixed case $f(u) = |u|^{p-1}u + \lambda |u|^{q-1}u$ with $0 < q < 1 < p$ depending as usual on p_c. A classification of solutions has been established and nodal solutions has been proved to exist by using variational methods as well as shooting ones already emerged with ODEs.

In [6], the original famous known as Brezis-Nirenberg problem has been considered. The authors established the existence of positive solutions of $\Delta u + u^{p_c} + f(x,u) = 0$ on Ω, and $u = 0$ on $\partial\Omega$ where Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 3$ and $f(x,u)$ is a lower-order perturbation of u^{p_c} in the sense that $\lim_{u\to\infty} f(u)/u^{p_c} = 0$. Next, an interesting study was developed in [7] by considering the problem $\alpha(\Delta u + u) - u^{1-\alpha} = 0$ on a ball B_R in $\mathbb{R}^d$ and $u = 0$ on ∂B_R and with a parameter $\alpha \in]1,2[$. In [8], R. Kajikiya studied (1.1) in the sub-linear case $f(s) = |s|^{p-1}$ with $0 < p < 1$. Under suitable assumptions extracted from f, the author established a sufficient and necessary condition for the existence and uniqueness of radially symmetric nodal solution. Next, a famous study of problem (1.1) has been developed in [9] where the author originally considered the problem and proved the existence of a ground state and infinitely many radial solutions which are precisely compactly supported. The author proved also that other solutions exist and these are oscillating on ± 1 with a finite number of zeros. Positive solutions which are tending to zero at infinity are necessarily compactly supported. Finally, a more complicated situation has been considered recently in [10] where the author have considered the problem

$$\Delta u + \lambda u^p - \chi_{[u>0]} u^{-\beta} = 0, \quad u \geq 0 \quad \text{on } \Omega,$$

with $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with smooth boundary, $\lambda > 0$, $0 < \beta < 1$, $1 \leq p < p_c$. The authors proved the existence of nontrivial solutions without restrictions on λ and studied the behavior of solutions according to it. For example, as $\lambda \to \infty$, they proved that the least energy solutions concentrate around a point that maximizes the distance to the boundary.

In the present work we reconsider the nonlinear problem (1.1) and we focus essentially on radial solutions. The radial version of problem (1.1) provided with the value of the solution u at the origin is

$$\begin{cases} u''+\dfrac{d-1}{r}u'+u-|u|^{-2\theta}u=0, & r\in(0,+\infty),\\ u(0)=a, \qquad u'(0)=0, \end{cases} \tag{1.3}$$

where $a\in\mathbb{R}$. In the rest of the whole paper we denote

$$g(s)=1-|s|^{-2\theta},\quad f(s)=sg(s)=s-s|s|^{-2\theta}\quad\text{and}\quad F(s)=\frac{s^2}{2}\left(1-\frac{|s|^{-2\theta}}{1-\theta}\right).$$

We now recall some results already known or easy to handle on the study of problem (1.1) or one of the problems related. To do this we recall the properties of g, f and F. Because of the parity properties of these functions, we only provide their behaviors on $(0,+\infty)$.

- **H1**. The function g is non-decreasing on $(0,+\infty)$ with $g(1)=0$, $\lim\limits_{u\to 0}g(u)=-\infty$ and $\lim\limits_{u\to+\infty}g(u)=1$.
- **H2**. The function f is non-increasing on $(0,u_\theta)$ and non-decreasing on $(u_\theta,+\infty)$ with $u_\theta=(1-2\theta)^{1/2\theta}$, $f(0)=f(1)=0$, $f(u_\theta)=-2\theta u_\theta^{1-2\theta}<0$, and $\lim\limits_{u\to+\infty}f(u)=+\infty$.
- **H3**. The function F is non-increasing on $(0,1)$ and non-decreasing on $(1,+\infty)$ with $f(0)=f(p)=0$, $p=(1-\theta)^{-1/2\theta}$, and $\lim\limits_{u\to+\infty}F(u)=+\infty$.

The parameter $p=1/(1-\theta)^{\frac{1}{2\theta}}$ is the unique real number in $(1,+\infty)$ such that $F(p)=0$. We recall finally that we shall use many times the energy of the solution u defined for $r\geq 0$ by

$$E(r)=\frac{1}{2}u'^2(r)+\int_0^{u(r)}sg(s)\mathrm{d}s=\frac{1}{2}u'^2(r)+\int_0^{u(r)}f(s)\mathrm{d}s=\frac{1}{2}u'^2(r)+F(u(r)).$$

The first result is stated as follows.

Theorem 1.1. *The solution u of problem* (1.3) *is oscillating around* 1 *or* -1 *for any* $a\in]-1,1[\setminus\{0\}$ *with no zeros in* $(0,\infty)$.

Next we study the case where the origin value $u(0)=a$ is not in the ±1-attractive zone. We prove that there are also different zones to be distinguished. We obtained the following result.

Theorem 1.2. i). *For $1<a<p$, the solution u of problem* (1.3) *is oscillating around 1 with no zeros.*

ii). *For $a>p$, the solution u of problem* (1.3) *is oscillating around 0 with finite number of zeros on its support being compact.*

2 On the existence and uniqueness of solutions

Lemma 2.1. *For all $a \in (0,p)$, the solution u of* (1.3) *satisfies the assertion*

$$u(\zeta)=0, \text{ for some } \zeta \Longrightarrow u'(\zeta) \neq 0,$$

except if $u \equiv 0$.

Lemma 2.2. *For all $a \in]0,p[$, with $p = 1/(1-\theta)^{\frac{1}{2\theta}}$, problem* (1.3) *has a unique positive solution u.*

Indeed, denote for $r \in (0,+\infty)$ and consider the system

$$\begin{cases} u(r) = a + \displaystyle\int_0^r v(s)\mathrm{d}s, \\ v(r) = -\dfrac{1}{r^{d-1}} \displaystyle\int_0^r s^{d-1} u(s) g(u(s))\mathrm{d}s. \end{cases} \tag{2.1}$$

Using standard arguments from iterative methods in functional analysis, we observe that such a system has a unique local solution (u,v) on $r \in (0,\delta)$ for $\delta > 0$ small enough. The solution satisfies $u(0) = a$, $v(0) = 0$. Furthermore, on $(0,\delta)$, $u > 0$, $v < 0$, and u and v are $\mathcal{C}^2$ and

$$u'(r) = v(r) \quad \text{and} \quad v'(r) = -\frac{d-1}{r} v(r) - u(r) g(u(r)).$$

We now study the differentiability at 0. Using L'Hospital rule, we obtain

$$u''(0) = v'(0) = \lim_{r \to 0} \frac{v(r)}{r} = -\frac{a g(a)}{d}.$$

On the other hand,

$$\begin{aligned} \lim_{r\to 0} v'(r) = \lim_{r \to 0} u''(r) = -\lim_{r \to 0} \left[(d-1) \frac{v(r)}{r} + u(r) g(u(r)) \right] \\ = -\left[(d-1) \frac{-a g(a)}{d} + a g(a) \right] = -\frac{a g(a)}{d}. \end{aligned}$$

Hence, u is $\mathcal{C}^2$ at 0. It suffices then to prove that $u > 0$ on $(0,+\infty)$ to guarantee the existence and uniqueness on $(0,+\infty)$. We suppose by contrast that $u(\zeta) = 0$ for some $\zeta > 0$. The evaluation of the energy E gives

$$E(\zeta) = \frac{1}{2} u'^2(\zeta) < E(0) = F(a) < 0,$$

because of the fact $0 < a < p$. Which leads to a contradiction.

Let u be a compactly supported solution of problem (1.3) already with $a>1$ and let $R=\inf\{r\in(0,\infty), u(s)=0, \forall s\geq r\}$. Henceforth, u is a solution of the problem

$$\begin{cases} \Delta u+u-|u|^{-2\theta}u=0, & \text{in } B(0,R),\\ u=0, & \text{on } \partial B(0,R). \end{cases} \tag{2.2}$$

Recall that for $R<\sqrt{\lambda_1(B(0,1))}$ the first eigenvalue of $-\Delta$ on the unit ball, problem (2.2) has no positive solution. See [7, 10] and the references therein. Consequently, we will assume for the rest of this part that $R\geq\sqrt{\lambda_1(B(0,1))}$ and consider a modified version of the radial expression of (2.2),

$$\begin{cases} u''+\dfrac{d-1}{r}u'+u-|u|^{-2\theta}u=0, & r\in(0,+\infty),\\ u(R)=0. \end{cases} \tag{2.3}$$

We will discuss the behavior of the solution u relatively to the values $u'(R)$. Two situations can occur. First, $u'(R)<0$. It results that $u'(r)<0$ on a small interval $(R-\varepsilon,R+\varepsilon)$. Therefore, $u(r)<0$ on $(R,R+\varepsilon)$ which contradicts the definition of R. Next, for $u'(R)=0$, we get $E(R)=0$. Therefore, proceeding as in [11] and [9] we obtain $u\equiv 0$ for $r\geq R$.

3 Proof of Theorem 1.1

We recall firstly that some situations which are somehow more general are developed in [4]. The proof developed here is inspired from there. Let $a\in(0,1)$ and u be the solution of problem (1.3). It holds that $u''(r)>0$ on a small interval $(0,\varepsilon)$ for ε small enough positive. Consequently, u' is strictly increasing on $(0,\varepsilon)$. Which yields that $u'(r)>0$ on $(0,\varepsilon)$. Thus u is strictly increasing on $(0,\varepsilon)$ for ε small enough positive. So that, $u(r)>a$ on $(0,\varepsilon)$. We will prove that the value a is taken only for $r=0$. Indeed, suppose not, and let $\zeta>0$ be the first point satisfying $u(\zeta)=a$. The evaluation of the energy $E(r)$ at 0 and ζ yields that

$$E(0)=F(a)>E(\zeta)=\frac{1}{2}u'^2(\zeta)+F(a),$$

which is contradictory. So, the solution u starts increasing with origin point $u(0)=a$ and did not reach it otherwise. We next prove that it can not continue to increase on its whole domain $(0,+\infty)$. Suppose contrarily that it is increasing on $(0,+\infty)$ and denote L its limit as $r\to+\infty$. Of course, such a limit can not be infinite because of the energy of the solution. Next, the finite limit is a zero of the function $f(s)$. Therefore, $L=1$. But, this yields $u''(r)>0$ as $r\to+\infty$ (Recall that $f(s)<0$ on $(0,1)$). In the other hand, Eq. (1.3) guaranties that $u''(r)/u'(r)\sim-(d-1)/r<0$ as $r\to+\infty$ which means that $u''(r)<0$ as $r\to+\infty$ leading to a contradiction. We therefore conclude that u is oscillatory. Let t_1 be the first point in $(0,+\infty)$ such that $u'(t_1)=0$. It holds that $u(t_1)>1$. If not, by multiplying Eq. (1.3) by r^{d-1}

and integrating from 0 to t_1 we obtain $0=-\int_0^{t_1} r^{d-1} f(u(r))>0$ which is contradictory. Thus, u crosses the line $y=1$ once in $(0,t_1)$ leading to a unique point $r_1\in(0,t_1)$ such that $u(r_1)=1$. Next, using similar techniques, we prove that u can not remain greater than 1 in the rest of its domain. (Consider the same equation on $(t_1,+\infty)$ with initial data $u(t_1)$ and $u'(t_1)$). Consequently we prove that there exists unique sequences $(t_k)_k$ and $(r_k)_k$ such that

$$r_k<t_k<r_{k+1},\quad u(r_k)=1,\quad u'(\zeta_k)=0,\quad k\geq 1. \tag{3.1}$$

Next, observing that E is decreasing as a function of r, we deduce that the sequence of maxima $(u(t_k))_k$ goes to 1 and therefore u.

4 Proof of Theorem 1.2

The proof is based on a series of preliminary results. We recall first that it suffices to study the case $a>0$ due to the parity properties of the function g and/or f.

Lemma 4.1. *For $a>1$, the solution u satisfies $\big(u(r)<a,\forall r>0\big)$.*

Proof. From Eq. (1.3), we obtain $du''(0)=-ag(a)<0$. Consequently, $u''(r)<0$ for $r\in(0,\varepsilon)$ for some $\varepsilon>0$ small enough. Thus, u' is decreasing strictly on $(0,\varepsilon)$ and then, $u'(r)<0$ for $r\in(0,\varepsilon)$. Therefore, u is decreasing strictly on $(0,\varepsilon)$ and then, $u(r)<a$ for $r\in(0,\varepsilon)$. Let next $\zeta>0$ be the first point such that $u(\zeta)=a$, if possible. Using the energy E we obtain $E(\zeta)<E(r)<E(0)$, for all $r\in(0,\zeta)$. Whenever $u'(\zeta)=0$, we obtain $E(0)<E(0)$ which is impossible. So $u'(\zeta)\neq 0$, which implies that $u'^2(\zeta)/2+E(0)<E(0)$ which is also impossible. As a conclusion, there is no positive points for which the solution u reaches a again. □

Lemma 4.2. *For $a>1$, the solution u is not strictly decreasing on $(0,+\infty)$.*

Proof. Assume contrarily that u is strictly decreasing on $(0,+\infty)$. Thus, it has a limit L as $r\to+\infty$. A first case is where u is unbounded with limit $-\infty$. This yields that $f(u)$ has the same limit $-\infty$ as $r\to+\infty$. A careful computation yields that $u'(r)\to+\infty$ as $r\to+\infty$. Which is contradictory. Thus, two cases remain to be possible, $L=0$ or $L=1$.

Case 1. $L=0$. Consider the dynamical system in the phase plane defined for $r\in(0,+\infty)$ by

$$\begin{cases} v=u', \\ v'=-\dfrac{d-1}{r}v+u-|u|^{-2\theta}u, \\ u(0)=a, \qquad v(0)=0. \end{cases} \tag{4.1}$$

A careful study for $r\to+\infty$, yields the estimation $u\sim A\cos(r)+B\sin(r)$ for r large enough, which is contradictory.

Case 2. $L=1$. Using Eq. (1.3) or (4.1), we obtain for r large enough, $2^{d-1}v(2r)-v(r)\sim\dfrac{2^d-1}{d}r$ which leads to a contradiction. $\square$

Lemma 4.3. *Let for $a>1$, $r_1(a)$ be the first critical point of the solution u of* (1.3) *in* $(0,+\infty)$. *Then $u(r_1)<1$.*

Proof. Suppose not, i.e, $u(r_1)=1$ or $u(r_1)>1$. When $u(r_1)>1$, we obtain

$$u''(r_1)=-u(r_1)g(u(r_1))<0.$$

Hence, u' is decreasing on $(r_1-\varepsilon,r_1+\varepsilon)$ for some ε small enough. Thus, u is increasing near r_1 at the left and decreasing near r_1 at the right, which is contradictory. When $u(r_1)=1$, then u is a solution of the problem $u''(r)+\frac{d-1}{r}u'(r)+ug(u)=0$ on $(r_1,+\infty)$ with the initial condition $u(r_1)=1$ and $u'(r_1)=0$. Consequently, $u\equiv 1$ which is contradictory. $\square$

Lemma 4.4. *Let $a>1$ and $r_1(a)$ be the first critical point of the solution u in $(0,+\infty)$. Then*

a. *for $r_1(a)\in]0,1[$, the solution u of* (1.3) *oscillates around 1, with limit 1, and thus has a finite number of zeros.*

b. *for $r_1(a)\in]-1,0[$, the solution u of* (1.3) *oscillates around -1, with limit -1, and thus has a finite number of zeros.*

Proof. In the situation **a.** u is a solution of the problem

$$\begin{cases} u''+\dfrac{d-1}{r}u'+u-|u|^{-2\theta}u=0, & r\in(r_1(a),+\infty),\\ u(r_1(a))\in]0,1[, \quad u'(r_1(a))=0. \end{cases} \tag{4.2}$$

Hence, by applying Theorem 1.1, the solution oscillates around 1, with limit 1 and thus, it has a finite number of zeros. In the situation **b.** u is a solution of the problem

$$\begin{cases} u''+\dfrac{d-1}{r}u'+u-|u|^{-2\theta}u=0, & r\in(r_1(a),+\infty),\\ u(r_1(a))\in]-1,0[, \quad u'(r_1(a))=0. \end{cases} \tag{4.3}$$

Hence, for the same reasons, it oscillates around -1, with limit -1 and thus with a finite number of zeros. $\square$

Lemma 4.5. *Let $a>1$ and u the solution of* (1.3) *in* $(0,+\infty)$. *The following situation can not occur. There exists sequences (r_k), (t_k), (z_k) and (ζ_k) satisfying*

i. $t_{2k-1}<z_{2k-1}<\zeta_{2k-1}<z_{2k}<t_{2k}<r_{2k}<\zeta_{2k}<r_{2k+1}, \qquad \forall k.$

ii. $u(r_k)=-u(z_k)=1, \quad u(t_k)=u'(\zeta_k)=0, \quad \forall k.$

iii. *u is increasing strictly on $(\zeta_{2k-1},\zeta_{2k})$ and decreasing strictly on $(\zeta_{2k},\zeta_{2k+1})$, $\forall k$.*

Proof. Suppose by contrast that the situation occurs. Using the functional energy $E(r)$, it is straightforward that $|u(\zeta_k)|\downarrow 1$. Observe next that for r large enough and $k\in\mathbb{N}$ unique such that $\zeta_{2k}\le r<\zeta_{2k+1}$ or $\zeta_{2k+1}\le r<\zeta_{2k+2}$, we have $E(\zeta_{2k})\le E(r)<E(\zeta_{2k+1})$ or $E(\zeta_{2k+1})\le E(r)<E(\zeta_{2k+2})$ which means that $\lim_{r\to+\infty}E(r)=-\theta/2(1-\theta)$. In particular we get $\lim_{k\to+\infty}E(t_k)=-\theta/2(1-\theta)$, which means that $\lim_{k\to+\infty}u'^2(t_k)=-\theta/(1-\theta)<0$, which is a contradiction. □

Denote for the rest of the paper ρ_a the first zero of the solution u of problem (1.3) for $a>p$. We have

Lemma 4.6. *For all $a>p$, $\rho_a<\infty$ except if u is trivial.*

Proof. Suppose u a solution of problem (1.3) with $a>p$ and $\rho_a=\infty$. The solution u starts as decreasing from $a=u(0)$. Suppose that it remains decreasing on its whole domain $(0,+\infty)$. Thus it has a limit L as $r\to+\infty$. Thus $L=0$ or $L=1$. For $L=0$ and r large enough, we obtain $u(r)=A\cos(r)+B\sin(r)$ which is contradictory. The case where $L=1$ is analogous. Consequently $\rho_a<+\infty$.

We now study the behavior of the solution u on the whole domain $(0,+\infty)$. Denote r_0 the first critical point of the solution u of problem (1.3) with $a>p$. There are four possible situations. The case $u(r_0)>1$ with Eq. (1.3) implies that

$$0=\int_0^{r_0}\big(s^{d-1}u'(s)\big)'\mathrm{d}s=-\int_0^{r_0}s^{d-1}u(s)g(u(s))\mathrm{d}s<0, \tag{4.4}$$

which is impossible. Next, for $u(r_0)=1$, the solution u will be a solution of the problem

$$\begin{cases} u''+\dfrac{d-1}{r}u'+u-|u|^{-2\theta}u=0, \qquad r\in(r_0,+\infty),\\ u(r_0)=1, \qquad u'(r_0)=0. \end{cases} \tag{4.5}$$

Therefore, $u\equiv 1$, for any $r\ge r_0$, which is contradictory by the same argument as above. We now assume that $0<u(r_0)<1$. Using again Eq. (4.4) we obtain a contradiction. It remains to examine the last case where $u(r_0)=0$. In this case, we obtain $u(r_0)=u'(r_0)=0$ and $\rho_a>\sqrt{\lambda_1(B(0,1))}$. So as the proof of the lemma. □

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