

Inequalities and Separation for a Biharmonic Laplace-Beltrami Differential Operator in a Hilbert Space Associated with the Existence and Uniqueness Theorem

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Abstract. In this paper, we have studied the separation for the biharmonic Laplace-Beltrami differential operator

$$Au(x) = -\Delta\Delta u(x) + V(x)u(x),$$

for all $x \in R^n$, in the Hilbert space $H = L_2(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$, where $L(H_1)$ is the space of all bounded linear operators on the Hilbert space H_1 , while $\Delta\Delta u$ is the biharmonic differential operator and

$$\Delta u = - \sum_{i,j=1}^n \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1}(x) \frac{\partial u}{\partial x_j} \right]$$

is the Laplace-Beltrami differential operator in R^n . Here $g(x) = (g_{ij}(x))$ is the Riemannian matrix, while $g^{-1}(x)$ is the inverse of the matrix $g(x)$. Moreover, we have studied the existence and uniqueness Theorem for the solution of the non-homogeneous biharmonic Laplace-Beltrami differential equation $Au = -\Delta\Delta u + V(x)u(x) = f(x)$ in the Hilbert space H where $f(x) \in H$ as an application of the separation approach.

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1 Introduction

The concept of separation for differential operators was first introduced by Everitt and Giertz [1,2]. They have obtained the separation results for the Sturm Liouville differential

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operator

$$Au(x) = -u''(x) + V(x)u(x), \quad x \in R, \quad (1.1)$$

in the space $L_2(R)$. They have studied the following question: What are the conditions on $V(x)$ such that if $u(x) \in L_2(R)$ and $Au(x) \in L_2(R)$ imply that both of $u''(x)$ and $V(x)u(x) \in L_2(R)$? More fundamental results of separation for differential operators were obtained by Everitt and Giertz [3,4]. A number of results concerning the property referred to the separation of differential operators was discussed by Biomatov [5], Otelbaev [6], Zettle [7] and Mohamed et al. [8–13]. The separation for the differential operators with the matrix potential was first studied by Bergbaev [14]. Brown [15] has shown that certain properties of positive solutions of disconjugate second order differential expressions imply the separation. Some separation criteria and inequalities associated with linear second order differential operators have been studied by Brown et al. [16,17]. Mohamed et al. [11] have studied the separation property of the Sturm Liouville differential operator

$$Au(x) = -(\mu(x)u')' + V(x)u(x), \quad x \in R, \quad (1.2)$$

in the Hilbert space $H_p(R)$, ($p=1,2$), where $V(x) \in L(l_p)$ is an operator potential which is a bounded linear operator on l_p and $\mu(x) \in C^1(R)$ is a positive continuous function on R .

Mohamed et al. [9] have studied the separation property for the linear differential operator

$$Au(x) = (-1)^m D^{2m}u(x) + V(x)u(x), \quad x \in R, \quad (1.3)$$

in the Banach space $L_p(R)^l$ where $V(x)$ is an $l \times l$ positive hermitian matrix and $D^{2m} = d^{2m}/dx^{2m}$ is the classical differentiation of order $2m$.

Mohamed et al. [12] have studied the separation of the Schrodinger operator

$$Au(x) = -\Delta u(x) + V(x)u(x), \quad x \in R^n, \quad (1.4)$$

with the operator potential $V(x) \in C^1(R^n, L(H_1))$ in the Hilbert space $L_2(R^n, H_1)$ and $\Delta = \sum_{i=1}^n (\partial^2/\partial x_i^2)$ is the Laplace operator in R^n .

Mohamed et al. [13] have studied the separation for the general second order differential operator

$$Au(x) = - \sum_{i,j=1}^n \alpha_{ij}(x) D_i^j u(x) + V(x)u(x), \quad x \in R^n, \quad (1.5)$$

in the weighted Hilbert space $L_{2,k}(R^n, H_1)$ with a positive weight function $k(x)$ and the operator potential $V(x) \in C^1(R^n, L(H_1))$ where $\alpha_{ij} \in C^2(R^n)$ and $D_i^j = \partial^2/\partial x_i \partial x_j$.

Zayed et al. [18] have obtained recent results on the separation of linear and nonlinear Schrodinger-type operators with the operator potentials in Banach spaces. Furthermore,

Zayed et al. [19] have studied the separation of the elliptic differential operator

$$Au(x) = - \sum_{i,j=1}^n [D_i(P_{ij}(x) D_j u(x)) - P_{ij}(x) b_i(x) b_j(x) u(x)] + V(x)u(x), \quad (1.6)$$

for all $x \in R^n$, in the weighted Hilbert space $L_{2,k}(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$, where $P_{ij}(x)$ and $b_i(x)$ are real-valued continuous function, while $D_i = \partial/\partial x_i$.

Zayed et al. [20] recently have studied the separation for the Laplace Beltrami differential operator

$$Au(x) = - \frac{1}{\sqrt{\det g(x)}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g(x)} g^{-1}(x) \frac{\partial u}{\partial x_j} \right] + V(x)u(x) \quad (1.7)$$

for all $x \in R^n$, in the Hilbert space $L_2(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$ and $g(x) = (g_{ij}(x))$ is the Riemannian matrix, while $g^{-1}(x)$ is the inverse of the matrix $g(x)$.

Recently, Zayed [21] has studied the separation for the biharmonic differential operator

$$Au(x) = \Delta \Delta u + V(x)u(x), \quad x \in R^n, \quad (1.8)$$

in the Hilbert space $H = L_2(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$ and $\Delta \Delta u$ is the biharmonic differential operator, while $\Delta u = \sum_{i=1}^n (\partial^2 u(x) / \partial x_i^2)$ is the Laplace operator in R^n .

Further results for separation of differential operators can be found in [22–30].

The main objective of the present paper is to study the separation for the biharmonic Laplace-Beltrami differential operator

$$Au(x) = -\Delta \Delta u + V(x)u(x), \quad x \in R^n, \quad (1.9)$$

in the Hilbert space $L_2(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$ where $\Delta \Delta u$ is the biharmonic differential operator and

$$\Delta u = - \sum_{i,j=1}^n \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1}(x) \frac{\partial u}{\partial x_j} \right], \quad (1.10)$$

is the Laplace-Beltrami differential operator in R^n . We derive also the coercive estimate for the operator (1.9). The existence and uniqueness Theorem for the solution of the non-homogeneous biharmonic Laplace-Beltrami differential equation

$$Au(x) = -\Delta \Delta u(x) + V(x)u(x) = f(x), \quad (1.11)$$

in the Hilbert space $H = L_2(R^n, H_1)$ is also given, where $f(x) \in H$.

2 Some notations

In this section, we introduce the definitions that will be used in the subsequent section. Let H_1 be a separable Hilbert space with the norm $\|\cdot\|_1$ and the scalar product $\langle \cdot, \cdot \rangle_1$. We introduce the Hilbert space $H = L_2(R^n, H_1)$ of all vector functions $u(x), x \in R^n$ equipped with the norm

$$\|u\|^2 = \int_{R^n} \|u(x)\|_1^2 dx. \quad (2.1)$$

The symbol $\langle u, v \rangle$ where $u, v \in H$ denotes the scalar product in H which is defined by

$$\langle u, v \rangle = \int_{R^n} \langle u, v \rangle_1 dx. \quad (2.2)$$

The space of all vector functions $u(x), x \in R^n$, that have generalized derivatives $D^\alpha u(x)$, $\alpha \leq 2$ such that $u(x)$ and $D^\alpha u(x)$ belong to H is denoted by $W_2^2(R^n, H_1)$. We say that the function $u(x) \in W_{2,loc}^2(R^n, H_1)$ if for all functions $Q(x) \in C_0^\infty(R)$ the vector function $Q(x)u(x) \in W_2^2(R^n, H_1)$.

3 Main results

Definition 3.1. The biharmonic Laplace-Beltrami differential operator

$$Au(x) = -\Delta\Delta u + V(x)u(x),$$

for all $x \in R^n$ where Δu is given by (10) is said to be separated in the Hilbert space $H = L_2(R^n, H_1)$ if the following statement holds: If $u(x) \in H \cap W_{2,loc}^2(R^n, H_1)$ and $Au(x) \in H$ imply both of $\Delta\Delta u$ and $Vu \in H$.

The main results in this paper have been formulated as follows:

Theorem 3.1. If the following conditions are satisfied for all $x \in R^n$:

$$\left\| \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right] - \sum_{i,j=1}^n V_0^{-1/2} \frac{\partial}{\partial x_j} \left[\frac{1}{2} g^{-1} V u \frac{\partial}{\partial x_i} \ln \det g \right] \right\| \leq 2\sigma_1 n^2 \|Vu\|, \quad (3.1)$$

$$\left\| V_0^{-1/2} \frac{\partial}{\partial x_j} \left[g^{-1} u \frac{\partial V}{\partial x_i} \right] \right\| \leq 2\sigma_2 \|Vu\|, \quad (3.2)$$

and

$$\left\| V_0^{-1/2} \frac{\partial}{\partial x_j} \left[g^{-1} V \frac{\partial u}{\partial x_i} \right] \right\| \leq 2\sigma_3 \|Vu\|, \quad (3.3)$$

where $V_0 = \text{Re}V$ while σ_i ($i = 1, 2, 3$) are positive constants satisfying the inequality $0 < \sigma < \frac{1}{n^2}$, and $\sigma = \sum_{i=1}^3 \sigma_i$. Then, the following coercive estimate is true:

$$\|Vu\| + \|\Delta\Delta u\| + \left\| \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right] \right\| \leq N \|Au\|, \quad (3.4)$$

where

$$N = 1 + 2 \left(1 - \frac{n^2 \sigma}{\beta} \right)^{-1} + (1 - n^2 \sigma \beta)^{-1/2} \left(1 - \frac{n^2 \sigma}{\beta} \right)^{-1/2},$$

is a positive constant independent on $u(x)$, while β is a positive constant satisfying the inequality $n^2 \sigma < \beta < \frac{1}{n^2 \sigma}$. That is, the biharmonic Laplace-Beltrami differential operator $Au(x)$ given by (1.9) is separated in the Hilbert space $H = L_2(\mathbb{R}^n, H_1)$.

Proof. From the definition of the scalar product in H and by integrating by parts, we obtain

$$\left\langle \frac{\partial u}{\partial x_i}, v \right\rangle = - \left\langle u, \frac{\partial v}{\partial x_i} \right\rangle \quad \text{for all } u, v \in C_0^\infty(\mathbb{R}^n).$$

From (1.9), we get

$$\langle Au, Vu \rangle = \langle -\Delta\Delta u + Vu, Vu \rangle = \langle -\Delta\Delta u, Vu \rangle + \langle Vu, Vu \rangle.$$

On setting $\Delta u = W(x)$, we have

$$\begin{aligned} \langle Au, Vu \rangle &= \langle -\Delta W, Vu \rangle + \langle Vu, Vu \rangle \\ &= \left\langle \sum_{i,j=1}^n \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial W}{\partial x_j} \right], Vu \right\rangle + \langle Vu, Vu \rangle \\ &= - \sum_{i,j=1}^n \left\langle \sqrt{\det g} g^{-1} \frac{\partial W}{\partial x_j}, \frac{\partial}{\partial x_i} \left[\frac{1}{\sqrt{\det g}} Vu \right] \right\rangle + \langle Vu, Vu \rangle \\ &= \frac{1}{2} \sum_{i,j=1}^n \left\langle g^{-1} \frac{\partial W}{\partial x_j}, Vu \frac{\partial}{\partial x_i} (\ln \det g) \right\rangle - \sum_{i,j=1}^n \left\langle g^{-1} \frac{\partial W}{\partial x_j}, u \frac{\partial V}{\partial x_i} \right\rangle \\ &\quad - \sum_{i,j=1}^n \left\langle g^{-1} \frac{\partial W}{\partial x_j}, V \frac{\partial u}{\partial x_i} \right\rangle + \langle Vu, Vu \rangle \\ &= - \sum_{i,j=1}^n \left\langle W, \frac{\partial}{\partial x_j} \left[\frac{1}{2} g^{-1} Vu \frac{\partial}{\partial x_i} (\ln \det g) \right] \right\rangle + \sum_{i,j=1}^n \left\langle W, \frac{\partial}{\partial x_j} \left[g^{-1} u \frac{\partial V}{\partial x_i} \right] \right\rangle \\ &\quad + \sum_{i,j=1}^n \left\langle W, \frac{\partial}{\partial x_j} \left[g^{-1} V \frac{\partial u}{\partial x_i} \right] \right\rangle + \langle Vu, Vu \rangle. \end{aligned} \quad (3.5)$$

On substituting (1.10) into (3.5) and equating the real parts of both sides of the resultant form, we obtain

$$\begin{aligned}
& \operatorname{Re}\langle Au, Vu \rangle \\
&= \left\langle \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right], \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right] \right\rangle \\
&\quad - \operatorname{Re} \left\langle \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right], \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right] \right\rangle \\
&\quad + \operatorname{Re} \left\langle \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right], \sum_{i,j=1}^n V_0^{-1/2} \frac{\partial}{\partial x_j} \left[\frac{1}{2} g^{-1} V u \frac{\partial}{\partial x_i} (\ln \det g) \right] \right\rangle \\
&\quad - \operatorname{Re} \left\langle \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right], \sum_{i,j=1}^n V_0^{-1/2} \frac{\partial}{\partial x_j} \left[g^{-1} u \frac{\partial V}{\partial x_i} \right] \right\rangle \\
&\quad - \operatorname{Re} \left\langle \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right], \sum_{i,j=1}^n V_0^{-1/2} \frac{\partial}{\partial x_j} \left[g^{-1} V \frac{\partial u}{\partial x_i} \right] \right\rangle \\
&\quad + \langle Vu, Vu \rangle. \tag{3.6}
\end{aligned}$$

Since for any complex number Z , we have

$$-|Z| \leq \operatorname{Re} Z \leq |Z|, \tag{3.7}$$

then by using the Cauchy-Schwarz inequality, we get

$$\operatorname{Re}\langle Au, Vu \rangle \leq |\langle Au, Vu \rangle| \leq \|Au\| \|Vu\|. \tag{3.8}$$

Further, since for any $\beta > 0$ and $y_1, y_2 \in \mathbb{R}^n$, then with reference to [18] we have

$$|y_1| |y_2| \leq \frac{\beta}{2} |y_1|^2 + \frac{1}{2\beta} |y_2|^2. \tag{3.9}$$

Consequently, we deduce from (3.1)-(3.3) and (3.7)-(3.9) that

$$\begin{aligned}
& - \operatorname{Re} \left\langle K_1, K_1 - \sum_{i,j=1}^n V_0^{-1/2} \frac{\partial}{\partial x_j} \left[\frac{1}{2} g^{-1} V u \frac{\partial}{\partial x_i} (\ln \det g) \right] \right\rangle \\
& \geq -n^2 \sigma_1 \beta \left\| \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right] \right\|^2 - \frac{n^2 \sigma_1}{\beta} \|Vu\|^2, \tag{3.10} \\
& - \operatorname{Re} \left\langle \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right], \sum_{i,j=1}^n V_0^{-1/2} \frac{\partial}{\partial x_j} \left[g^{-1} u \frac{\partial V}{\partial x_i} \right] \right\rangle
\end{aligned}$$

$$\geq -n^2\sigma_2\beta \left\| \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right] \right\|^2 - \frac{n^2\sigma_2}{\beta} \|Vu\|^2, \quad (3.11)$$

and

$$\begin{aligned} & -\operatorname{Re} \left\langle \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right], \sum_{i,j=1}^n V_0^{-1/2} \frac{\partial}{\partial x_j} \left[g^{-1} V \frac{\partial u}{\partial x_i} \right] \right\rangle \\ & \geq -n^2\sigma_3\beta \left\| \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right] \right\|^2 - \frac{n^2\sigma_3}{\beta} \|Vu\|^2, \end{aligned} \quad (3.12)$$

On substituting (3.10)-(3.12) into (3.6) we get the inequality

$$\left(1 - \frac{n^2\sigma}{\beta}\right) \|Vu\|^2 + (1 - n^2\sigma\beta) \left\| \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right] \right\|^2 \leq \|Au\| \|Vu\|, \quad (3.13)$$

where $\sigma = \sum_{i=1}^3 \sigma_i$. Choosing $n^2\sigma < \beta < \frac{1}{n^2\sigma}$, we deduce from (3.13) that

$$\|Vu\| \leq \left(1 - \frac{n^2\sigma}{\beta}\right)^{-1} \|Au\|. \quad (3.14)$$

$$\left\| \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial u}{\partial x_j} \right] \right\| \leq (1 - n^2\beta\sigma)^{-1/2} \left(1 - \frac{n^2\sigma}{\beta}\right)^{-1/2} \|Au\|. \quad (3.15)$$

Since $Au = -\Delta\Delta u + Vu$, then we get

$$\|\Delta\Delta u\| \leq \|Au\| + \|Vu\| \leq \left\{1 + \left(1 - \frac{n^2\sigma}{\beta}\right)^{-1}\right\} \|Au\|. \quad (3.16)$$

From the inequalities (3.14)-(3.16) we arrive at the coercive estimate (3.4). Hence, the proof of Theorem 3.1 is completed. \square

Theorem 3.2. *If the biharmonic Laplace-Beltrami differential operator $Au(x)$ given by (1.9) is separated in the Hilbert space $H = L_2(R^n, H_1)$ and if there are positive functions $t(x), \psi(x) \in C^1(R^n)$ such that the following conditions are true:*

$$\left\| t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial u}{\partial x_j} \right\| \leq 2\sqrt{\rho_1} \left\| t^{1/2} \psi^{1/2} V_0^{1/2} u \right\|, \quad (3.17)$$

$$\left\| g^{-1/2} \psi^{-1} \frac{\partial \psi}{\partial x_i} V_0^{-1/2} \right\| \leq 2\sqrt{\rho_2}, \quad (3.18)$$

$$\left\| g^{-1/2} t^{-1} \frac{\partial t}{\partial x_j} V_0^{-1/2} \right\| \leq 2\sqrt{\rho_3}, \quad (3.19)$$

$$\left\| \frac{1}{2} g^{-1/2} V_0^{-1/2} \frac{\partial}{\partial x_j} (\ln \det g) \right\| \leq 2\sqrt{\rho_4}, \quad (3.20)$$

and

$$\left\| t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial W}{\partial x_i} \right\| \leq \left\| t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial \psi}{\partial x_i} \right\|, \quad (3.21)$$

where ρ_i ($i=1-4$) are positive constants satisfying $0 < \rho < \frac{\beta}{2n^2}$ and $\rho = \sum_{i=1}^4 \rho_i$ while $V_0 = \operatorname{Re} V$. Then, the non-homogeneous biharmonic Laplace-Beltrami differential equation

$$Au = -\Delta \Delta u + V(x)u(x) = f(x), \quad (3.22)$$

has a unique solution in the Hilbert space H , where $f(x) \in H$.

Proof. First, we prove that the homogeneous biharmonic Laplace-Beltrami differential equation

$$Au(x) = -\Delta \Delta u(x) + V(x)u(x) = 0, \quad (3.23)$$

has only the zero solution $u(x) = 0$ for all $x \in R^n$. To this end, we assume that $t(x)$ and $\psi(x) \in C^1(R^n)$ are positive functions. Thus, on setting $\Delta u(x) = W(x)$, we have

$$\begin{aligned} \langle Vu, t\psi u \rangle &= \langle \Delta \Delta u, t\psi u \rangle = \langle \Delta W, t\psi u \rangle \\ &= - \left\langle \sum_{i,j=1}^n \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_j} \left[\sqrt{\det g} g^{-1} \frac{\partial W}{\partial x_i} \right], t\psi u \right\rangle \\ &= \sum_{i,j=1}^n \left\langle \sqrt{\det g} g^{-1} \frac{\partial W}{\partial x_i}, \frac{\partial}{\partial x_j} \left(\frac{1}{\sqrt{\det g}} t\psi u \right) \right\rangle \\ &= \sum_{i,j=1}^n \left\langle \sqrt{\det g} g^{-1} \frac{\partial W}{\partial x_i}, \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_j} (t\psi u) \right\rangle \\ &\quad + \sum_{i,j=1}^n \left\langle \sqrt{\det g} g^{-1} \frac{\partial W}{\partial x_i}, t\psi u \frac{\partial}{\partial x_j} \left(\frac{1}{\sqrt{\det g}} \right) \right\rangle \\ &= \sum_{i,j=1}^n \left\langle \frac{\partial W}{\partial x_i}, g^{-1} \frac{\partial}{\partial x_j} (t\psi u) \right\rangle - \frac{1}{2} \sum_{i,j=1}^n \left\langle \frac{\partial W}{\partial x_i}, g^{-1} t\psi u \frac{\partial}{\partial x_j} (\ln \det g) \right\rangle \\ &= \sum_{i,j=1}^n \left\langle \frac{\partial W}{\partial x_i}, g^{-1} t\psi \frac{\partial u}{\partial x_j} \right\rangle + \sum_{i,j=1}^n \left\langle \frac{\partial W}{\partial x_i}, g^{-1} t u \frac{\partial \psi}{\partial x_j} \right\rangle \\ &\quad + \sum_{i,j=1}^n \left\langle \frac{\partial W}{\partial x_i}, g^{-1} \psi u \frac{\partial t}{\partial x_j} \right\rangle - \frac{1}{2} \sum_{i,j=1}^n \left\langle \frac{\partial W}{\partial x_i}, g^{-1} t\psi u \frac{\partial}{\partial x_j} (\ln \det g) \right\rangle. \quad (3.24) \end{aligned}$$

Equating the real parts of both sides of (3.24), we get

$$\langle V_0 u, t\psi u \rangle = \left\langle t^{1/2} \psi^{1/2} V_0^{1/2} u, t^{1/2} \psi^{1/2} V_0^{1/2} u \right\rangle$$

$$\begin{aligned}
 &= \sum_{i,j=1}^n \operatorname{Re} \left\langle t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial W}{\partial x_i}, t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial u}{\partial x_j} \right\rangle \\
 &\quad + \sum_{i,j=1}^n \operatorname{Re} \left\langle t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial W}{\partial x_i}, t^{1/2} \psi^{1/2} \left[g^{-1/2} \psi^{-1} \frac{\partial \psi}{\partial x_j} V_0^{-1/2} \right] V_0^{1/2} u \right\rangle \\
 &\quad + \sum_{i,j=1}^n \operatorname{Re} \left\langle t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial W}{\partial x_i}, t^{1/2} \psi^{1/2} \left[g^{-1/2} t^{-1} \frac{\partial t}{\partial x_j} V_0^{-1/2} \right] V_0^{1/2} u \right\rangle \\
 &\quad - \sum_{i,j=1}^n \operatorname{Re} \left\langle t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial W}{\partial x_i}, t^{1/2} \psi^{1/2} \left[\frac{1}{2} g^{-1/2} V_0^{-1/2} \frac{\partial}{\partial x_j} (\ln \det g) \right] V_0^{1/2} u \right\rangle. \quad (3.25)
 \end{aligned}$$

With the aid of (3.7)-(3.9) together with the inequalities (3.17)-(3.21), we deduce that (3.25) takes the form

$$\begin{aligned}
 \left\| t^{1/2} \psi^{1/2} V_0^{1/2} u \right\|^2 &\leq \frac{n\beta}{2} \sum_{i=1}^n \left\| t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial W}{\partial x_i} \right\|^2 + \frac{2n^2 \rho_1}{\beta} \left\| t^{1/2} \psi^{1/2} V_0^{1/2} u \right\|^2 \\
 &\quad + \frac{n\beta}{2} \sum_{i=1}^n \left\| t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial W}{\partial x_i} \right\|^2 + \frac{2n^2 \rho_2}{\beta} \left\| t^{1/2} \psi^{1/2} V_0^{1/2} u \right\|^2 \\
 &\quad + \frac{n\beta}{2} \sum_{i=1}^n \left\| t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial W}{\partial x_i} \right\|^2 + \frac{2n^2 \rho_3}{\beta} \left\| t^{1/2} \psi^{1/2} V_0^{1/2} u \right\|^2 \\
 &\quad + \frac{n\beta}{2} \sum_{i=1}^n \left\| t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial W}{\partial x_i} \right\|^2 + \frac{2n^2 \rho_4}{\beta} \left\| t^{1/2} \psi^{1/2} V_0^{1/2} u \right\|^2. \quad (3.26)
 \end{aligned}$$

Consequently, if we put $\rho = \sum_{i=1}^4 \rho_i$, then (3.26) becomes in the form

$$\left(1 - \frac{2n^2 \rho}{\beta} \right) \left\| t^{1/2} \psi^{1/2} V_0^{1/2} u \right\|^2 \leq 2n\beta \sum_{i=1}^n \left\| t^{1/2} \psi^{1/2} g^{-1/2} \frac{\partial \psi}{\partial x_i} \right\|^2, \quad (3.27)$$

By choosing $\psi(x) = 1$ for all $x \in R^n$, then if $0 < \rho < \frac{\beta}{2n^2}$ we see that (3.27) becomes in the form

$$0 < \left(1 - \frac{2n^2 \rho}{\beta} \right) \left\| t^{1/2} V_0^{1/2} u \right\|^2 \leq 0. \quad (3.28)$$

From (2.1) and (3.28) we obtain

$$0 < \left(1 - \frac{2n^2 \rho}{\beta} \right) \int_{R^n} \left\| t^{1/2} V_0^{1/2} u \right\|_1^2 dx \leq 0. \quad (3.29)$$

Now, the inequality (3.29) holds only for $u(x) = 0$. This prove that $u(x) = 0$ is the only solution of the homogeneous biharmonic Laplace-Beltrami differential equation (3.23). Furthermore, it is easy to check that the linear manifold

$$L = \{f : Au(x) = f(x), \quad \text{for all } f \in C_0^\infty(R^n)\}$$

is dense in H . So, we can construct a sequence of vector functions $\{y_r\} \in C_0^\infty(R^n)$ where $\|y_r\| \neq 0$ for all r such that $\|Ay_r - f\| \rightarrow 0$ as $r \rightarrow \infty$ for all $f \in H$. On using the coercive estimate (3.4), we have

$$\begin{aligned} & \|V(y_p - y_r)\| + \|\Delta\Delta(y_p - y_r)\| + \left\| \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial(y_p - y_r)}{\partial x_j} \right] \right\| \\ & \leq N \|A(y_p - y_r)\|, \end{aligned} \quad (3.30)$$

where $y_p - y_r = u$ and $p, r = 1, 2, \dots$. As $p, r \rightarrow \infty$ we see from (3.30) that the sequences $\{Vy_r\}$, $\{\Delta\Delta y_r\}$ and $\left\{ \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial y_r}{\partial x_j} \right] \right\}$ are Cauchy sequences in the Hilbert space H and then they are convergent. Therefore, there exists real functions $\mu_0(x)$, $\mu_1(x)$ and $\mu_2(x)$ in H such that

$$\|Vy_r - \mu_0\| \rightarrow 0, \quad \|\Delta\Delta y_r - \mu_1\| \rightarrow 0$$

and

$$\left\| \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial y_r}{\partial x_j} \right] - \mu_2 \right\| \rightarrow 0.$$

Hence these sequences are bounded in H . This implies that as $r \rightarrow \infty$, we have

$$y_r \rightarrow V^{-1}\mu_0 = y, \quad \Delta\Delta y_r \rightarrow \Delta\Delta y,$$

and

$$\sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial y_r}{\partial x_j} \right] \rightarrow \sum_{i,j=1}^n \frac{V_0^{1/2}}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g} g^{-1} \frac{\partial y}{\partial x_j} \right].$$

Hence for a given function $f \in H$ there exists $y \in H \cap W_{2,loc}^2(R^n, H_1)$ such that $Ay = f$. Suppose that \tilde{y} is another solution of the non-homogeneous biharmonic Laplace-Beltrami differential equation $Ay = f$, then we get $A(y - \tilde{y}) = 0$. But $Au = 0$ has only the zero solution $u = 0$. Then $y = \tilde{y}$ and the uniqueness is proved. Hence, the proof of Theorem 3.2 is completed. \square

4 Conclusions

The biharmonic Laplace-Beltrami differential operator (1.9) has been investigated using the separation method in the Hilbert space H with the norm (2.1) and the scalar product (2.2). In this paper we have two interesting results. The first result is Theorem 3.1, which proves that the operator (1.9) is separated in the Hilbert space H . The second result is Theorem 3.2, which shows that the non-homogeneous biharmonic Laplace-Beltrami differential equation (3.22) has a unique solution in the Hilbert space H .

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