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Infinite Sequence Solutions for Space-Time Fractional Symmetric Regularized Long Wave Equation

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Abstract. In this paper, we investigate the space-time fractional symmetric regularized long wave equation. By using the Bäcklund transformations and nonlinear superposition formulas of solutions to Riccati equation, we present infinite sequence solutions for space-time fractional symmetric regularized long wave equation. This method can be extended to solve other nonlinear fractional partial differential equations.

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1 Introduction

In the past decades, much effort has been devoted to study nonlinear partial differential equations [1-25]. Nonlinear fractional partial differential equations (NFPDEs) regarded as the generalization form of nonlinear partial differential equations of integer order have attracted considerable attention in recent years. Moreover, the investigation of exact and approximate solutions for NFPDEs arising in mathematical physics, chemistry, biology, engineering, control theory, signal processing and so forth has become one of the most active and important research areas. A variety of analytical and numerical techniques have been well established and applied to solve NFPDEs, including the homogeneous balance method [6], the fractional sub-equation method [7-11], the exp-function method [12], the (G'/G)-expansion method [13, 14], the first integral method [15], the modified trial equation method [16], the Jacobi elliptic equation method [17], the modified

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Kudryashov method [18], the homotopy analysis transform method [19], the fractional variational iteration method [20], the Adomian decomposition method [21], and so on. In many analytical methods, the fractional complex transformation proposed by Li and He [22] plays a key role in converting NFPDEs into NODEs. The purpose of present article is to examine the space-time fractional symmetric regularized long wave (FSRLW) equation by means of Riccati equation method and symbolic computation. As a result, based on the Bäcklund transformations and nonlinear superposition formulas of solutions to Riccati equation, infinite sequence solutions in terms of trigonometric and hyperbolic functions are established.

For the convenience of a reader, we recall the Jumarie's modified Riemann-Liouville derivative [23] of order α, that is

$$D_t^\alpha f(t)=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^t (t-\xi)^{-\alpha-1}(f(\xi)-f(0))\mathrm{d}\xi, & \alpha<0,\\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))\mathrm{d}\xi, & 0<\alpha<1,\\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, \qquad n\le\alpha<n+1,\ n\ge 1. \end{cases}$$

Some significant properties of fractional modified Riemann-Liouville derivative are

$$\begin{aligned}&D_t^\alpha t^\delta=\frac{\Gamma(1+\delta)}{\Gamma(1+\delta-\alpha)}t^{\delta-\alpha}, \qquad \delta>0,\\ &D_t^\alpha(f(t)g(t))=g(t)D_t^\alpha f(t)+f(t)D_t^\alpha g(t),\\ &D_t^\alpha f[g(t)]=f_g'[g(t)]D_t^\alpha g(t)=D_g^\alpha[g(t)](g'(t))^\alpha.\end{aligned}$$

The layout of this paper is as follows. In Section 2 and Section 3, we present the main steps of Riccati equation method, and list the Bäcklund transformations and nonlinear superposition formulas [24, 25] of solutions to Riccati equation. In Section 4, we apply this method to establish infinite sequence solutions for space-time FSRLW equation. The last section is the conclusion.

2 Method

Consider a NFPDE in three independent variables as

$$P\big(u,\ D_t^\alpha u,\ D_x^\beta u,\ D_y^\gamma u,\ D_t^\alpha D_t^\alpha u,\ D_t^\alpha D_x^\beta u,\ \cdots\big)=0, \qquad 0<\alpha,\ \beta,\ \gamma\le 1, \tag{2.1}$$

where $D_t^\alpha u$, $D_x^\beta u$, $D_y^\gamma u$, $\cdots$ are the modified Riemann-Liouville derivatives, and P is a polynomial in u and its fractional derivatives. We find solutions to Eq. (2.1) in the form

$$u(t,\ x,\ y)=U(\xi),\ \xi=\frac{ct^\alpha}{\Gamma(1+\alpha)}+\frac{kx^\beta}{\Gamma(1+\beta)}+\frac{ly^\gamma}{\Gamma(1+\gamma)}.$$

Then, Eq. (2.1) is reduced to a nonlinear ordinary differential equation

$$O(U, U', U'', U''', \cdots) = 0, \tag{2.2}$$

where the prime represents the derivative with respect to ξ. Next, we aim to find exact solutions for Eq. (2.2). We suppose that formal solution of (2.2) can be expressed by

$$U(\xi) = A_0 + \sum_{i=1}^{m} \left[w^{i-1} \left(A_i w + C_i \sqrt{R + w^2} \right) + B_i w^{-i} \right], \tag{2.3}$$

in which A_0, A_i, B_i, C_i $(i = 1, 2, \cdots, m)$, and R are constants to be determined. And $w = w(\xi)$ satisfies Riccati equation

$$\frac{\mathrm{d}w}{\mathrm{d}\xi} = R + w^2, \tag{2.4}$$

which possesses the following solutions.

For $R < 0$, the expressions of hyperbolic function solutions read

$$w_0(\xi) = -\sqrt{-R} \tanh\left(\sqrt{-R}\xi\right), \tag{2.5}$$

$$w_0(\xi) = -\sqrt{-R} \coth\left(\sqrt{-R}\xi\right), \tag{2.6}$$

$$w_1(\xi) = \frac{b_3 R + a_3 \sqrt{-R} \tanh\left(\sqrt{-R}\xi\right)}{-a_3 + b_3 \sqrt{-R} \tanh\left(\sqrt{-R}\xi\right)}. \tag{2.7}$$

For $R > 0$, the expressions of trigonometric function solutions read

$$w_0(\xi) = \sqrt{R} \tan\left(\sqrt{R}\xi\right), \tag{2.8}$$

$$w_0(\xi) = -\sqrt{R} \cot\left(\sqrt{R}\xi\right), \tag{2.9}$$

$$w_1(\xi) = \frac{\sqrt{R}\left[\cos\left(\sqrt{R}\xi\right) + \sin\left(\sqrt{R}\xi\right)\right]}{\cos\left(\sqrt{R}\xi\right) - \sin\left(\sqrt{R}\xi\right)}, \tag{2.10}$$

$$w_1(\xi) = \frac{-\left(r\sqrt{R} + CR\right)\cos\left(\sqrt{R}\xi\right) + \sqrt{R}\left(r - C\sqrt{R}\right)\sin\left(\sqrt{R}\xi\right)}{\left(r - C\sqrt{R}\right)\cos\left(\sqrt{R}\xi\right) + \left(r + C\sqrt{R}\right)\sin\left(\sqrt{R}\xi\right)}, \tag{2.11}$$

$$w_1(\xi) = \frac{-3b_4 R + 4a_4\sqrt{R} - 5b_4 R \sin\left(2\sqrt{R}\xi\right) - 5a_4\sqrt{R}\cos\left(2\sqrt{R}\xi\right)}{3a_4 + 4b_4\sqrt{R} + 5a_4 \sin\left(2\sqrt{R}\xi\right) - 5b_4\sqrt{R}\cos\left(2\sqrt{R}\xi\right)}, \tag{2.12}$$

$$w_1(\xi) = \frac{-b_5 R + a_5\sqrt{R}\left[\sec\left(2\sqrt{R}\xi\right) + \tan\left(2\sqrt{R}\xi\right)\right]}{a_5 + b_5\sqrt{R}\left[\sec\left(2\sqrt{R}\xi\right) + \tan\left(2\sqrt{R}\xi\right)\right]}, \tag{2.13}$$

$$w_1(\xi) = \frac{\sqrt{R}\left[-2a_6 b_6\sqrt{R} + \left(a_6^2 - b_6^2 R\right)\left(\sec\left(2\sqrt{R}\xi\right) + \tan\left(2\sqrt{R}\xi\right)\right)\right]}{a_6^2 - b_6^2 R + 2a_6 b_6\sqrt{R}\left[\sec\left(2\sqrt{R}\xi\right) + \tan\left(2\sqrt{R}\xi\right)\right]}, \tag{2.14}$$

where r, a_i, b_i $(i = 3, 4, 5, 6)$ and C are arbitrary nonzero constants.

For $R=0$, the rational solution is

$$w_0(\xi)=\frac{1}{-\xi+d_0}, \tag{2.15}$$

where d_0 is free constant.

Step 1. By homogeneous balance between the highest order derivative and nonlinear terms appearing in Eq. (2.2), one can get the value of m easily.

Step 2. Substituting (2.3) together with (2.4) into Eq. (2.2), collecting all terms with the same powers of $w^{j_1}(\xi)$ and $w^{j_2}(\xi)\left(\sqrt{R+w^2(\xi)}\right)^{j_3}$, and equating zero of all the coefficients yield a system of algebraic equations about unknowns $\{A_0, A_i, B_i, C_i, R\}$ $(i=1, 2, \cdots, m)$.

Step 3. Solving the system obtained in Step 2 by symbolic computation system, one can get all the values of unknowns $\{A_0, A_i, B_i, C_i, R\}$ $(i=1, 2, \cdots, m)$. Inserting all the values of unknowns and solutions (2.5)-(2.15) into (2.3), many families of exact solutions to (2.1) can be got.

3 Bäcklund transformations and nonlinear superposition formulas of solutions to Eq. (2.4)

3.1 Bäcklund transformations

Eq. (2.4) admits Bäcklund transformations

$$\tilde{w}(\xi)=\frac{p_2+q_2w(\xi)+m_2w^2(\xi)+r_2w'(\xi)+n_2w^3(\xi)+l_2(w'(\xi))^2}{a_2+b_2w(\xi)+d_2w^2(\xi)+c_2w'(\xi)+f_2w^3(\xi)+k_2(w'(\xi))^2} \tag{3.1}$$

and

$$\tilde{w}(\xi)=\frac{-BR+Aw(\xi)}{A+Bw(\xi)}, \tag{3.2}$$

where A, B, a_2, b_2, c_2, d_2, f_2, k_2, m_2, l_2, r_2, a_6, b_6 are arbitrary nonzero constants, and $w(\xi)$ is known solution of (2.4). The relationships among parameters are given by

$$\begin{aligned}
&p_2=R(-b_2+m_2+f_2R),\\
&q_2=\frac{1}{k_2l_2}\left[b_2l_2^2-(l_2^2+k_2^2R)(m_2+r_2+(f_2+l_2)R)\right],\\
&n_2=\frac{1}{k_2}(f_2l_2-l_2^2-k_2^2R),\\
&d_2=-c_2+\frac{1}{k_2}(f_2l_2-l_2^2)+\frac{1}{l_2}(m_2+r_2+f_2R)k_2-k_2R,\\
&a_2=\frac{1}{k_2}\left[b_2l_2-l_2^2R-l_2(m_2+r_2+f_2R)-k_2R(c_2+k_2R)\right].
\end{aligned}$$

The combinations of an arbitrary solution to (2.4) with transformation (3.1) or (3.2) and iterations can result in infinite sequence solutions for (2.4). Here we only list three groups, and omit others.

$$\begin{cases} w_n(\xi)=\dfrac{p_2+q_2w_{n-1}(\xi)+m_2w_{n-1}^2(\xi)+r_2w_{n-1}'(\xi)+n_2w_{n-1}^3(\xi)+l_2(w_{n-1}'(\xi))^2}{a_2+b_2w_{n-1}(\xi)+d_2w_{n-1}^2(\xi)+c_2w_{n-1}'(\xi)+f_2w_{n-1}^3(\xi)+k_2(w_{n-1}'(\xi))^2}, \\ w_0(\xi)=-\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right),\ R<0,\ n=1,\ 2,\ \cdots \end{cases} \tag{3.3}$$

$$\begin{cases} w_n(\xi)=\dfrac{p_2+q_2w_{n-1}(\xi)+m_2w_{n-1}^2(\xi)+r_2w_{n-1}'(\xi)+n_2w_{n-1}^3(\xi)+l_2(w_{n-1}'(\xi))^2}{a_2+b_2w_{n-1}(\xi)+d_2w_{n-1}^2(\xi)+c_2w_{n-1}'(\xi)+f_2w_{n-1}^3(\xi)+k_2(w_{n-1}'(\xi))^2}, \\ w_0(\xi)=\sqrt{R}\tan\left(\sqrt{R}\xi\right),\ R>0,\ n=1,\ 2,\ \cdots \end{cases} \tag{3.4}$$

and

$$\begin{cases} w_n(\xi)=\dfrac{-BR+Aw_{n-1}(\xi)}{A+Bw_{n-1}(\xi)}, \\ w_1(\xi)=\dfrac{\sqrt{R}\left[-2a_6b_6\sqrt{R}+\left(a_6^2-b_6^2R\right)\left(\sec\left(2\sqrt{R}\xi\right)+\tan\left(2\sqrt{R}\xi\right)\right)\right]}{a_6^2-b_6^2R+2a_6b_6\sqrt{R}\left[\sec\left(2\sqrt{R}\xi\right)+\tan\left(2\sqrt{R}\xi\right)\right]}, \\ R>0,\ n=2,\ 3,\ \cdots \end{cases} \tag{3.5}$$

3.2 Nonlinear superposition formulas

Under the condition of $m_3d_3<0$, Eq. (2.4) possesses the following solutions $\tilde{w}(\xi)$,

$$\tilde{w}(\xi)=\frac{\mathrm{i}R\left[\mathrm{i}m_3\sqrt{R}+(m_3+\mathrm{i}d_3\sqrt{R}+c_3R)w_2(\xi)+(-c_3R+d_3w_2(\xi))w_1(\xi)\right]}{-\sqrt{R^3}(d_3+c_3w_2(\xi))+(m_3\sqrt{R}+\mathrm{i}Rd_3+c_3\sqrt{R^3}-\mathrm{i}m_3w_2(\xi))w_1(\xi)}, \tag{3.6}$$

$$\tilde{w}(\xi)=\frac{m_3+d_3w_2(\xi)+\frac{1}{\sqrt{R}}\left[-\mathrm{i}c_3Rw_1(\xi)+\mathrm{i}(m_3+c_3R+d_3w_1(\xi))w_2(\xi)\right]}{d_3+c_3w_2(\xi)-\frac{1}{\sqrt{R^3}}(m_3\sqrt{R}-\mathrm{i}Rd_3+c_3\sqrt{R^3}+\mathrm{i}m_3w_2(\xi))w_1(\xi)}, \tag{3.7}$$

and

$$\tilde{w}(\xi)=\frac{R\left[-r_3w_1(\xi)+(p_3+r_3)w_2(\xi)-p_3w_3(\xi)\right]}{-r_3w_2(\xi)w_3(\xi)+w_1(\xi)(-p_3w_2(\xi)+(p_3+r_3)w_3(\xi))}, \tag{3.8}$$

where c_3, p_3, r_3 are arbitrary nonzero constants, and $w_1(\xi)$, $w_2(\xi)$, $w_3(\xi)$ are three known solutions of (2.4). Thus, combining nonlinear superposition formulas (3.6) and (3.8) with known solutions, one gets

$$\begin{cases} w_n(\xi)=\frac{\mathrm{i}R\left[\mathrm{i}m_3\sqrt{R}+\left(m_3+\mathrm{i}d_3\sqrt{R}+c_3R\right)w_{n-1}(\xi)+(-c_3R+d_3w_{n-1}(\xi))w_{n-2}(\xi)\right]}{-\sqrt{R^3}(d_3+c_3w_{n-1}(\xi))+(m_3\sqrt{R}+\mathrm{i}Rd_3+c_3\sqrt{R^3}-\mathrm{i}m_3w_{n-1}(\xi))w_{n-2}(\xi)}, \\ w_1(\xi)=-\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right), \\ w_2(\xi)=\frac{b_3R+a_3\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right)}{-a_3+b_3\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right)}, \qquad n=3,\ 4,\ \cdots \end{cases} \tag{3.9}$$

and

$$
\begin{cases}
w_n(\xi)=\dfrac{R\left[-r_3w_{n-3}(\xi)+(p_3+r_3)w_{n-2}(\xi)-p_3w_{n-1}(\xi)\right]}{-r_3w_{n-2}(\xi)w_{n-1}(\xi)+w_{n-3}(\xi)\left(-p_3w_{n-2}(\xi)+(p_3+r_3)w_{n-1}(\xi)\right)},\\
w_1(\xi)=\dfrac{-b_5R+a_5\sqrt{R}\left[\sec\left(2\sqrt{R}\xi\right)+\tan\left(2\sqrt{R}\xi\right)\right]}{a_5+b_5\sqrt{R}\left[\sec\left(2\sqrt{R}\xi\right)+\tan\left(2\sqrt{R}\xi\right)\right]},\\
w_2(\xi)=\dfrac{\sqrt{R}\left[\cos\left(\sqrt{R}\xi\right)+\sin\left(\sqrt{R}\xi\right)\right]}{\cos\left(\sqrt{R}\xi\right)-\sin\left(\sqrt{R}\xi\right)},\\
w_3(\xi)=\sqrt{R}\tan\left(\sqrt{R}\xi\right),\qquad n=4,\,5,\,\cdots
\end{cases}
\tag{3.10}
$$

4 Application to space-time FSRLW equation

Now, we focus on the space-time FSRLW equation [8]

$$D_t^{2\alpha}u+D_x^{2\alpha}u+uD_t^{\alpha}\left(D_x^{\alpha}u\right)+D_x^{\alpha}uD_t^{\alpha}u+D_t^{2\alpha}\left(D_x^{2\alpha}u\right)=0,\qquad 0<\alpha\le 1,\tag{4.1}$$

which arises in several physical applications including ion sound waves in plasma. When $\alpha=1$, it is shown that this equation describes weakly nonlinear ion acoustic and space-charge waves, and the real-valued $u(x,\,t)$ corresponds to the dimensionless fluid velocity with a decay condition.

Applying the transformation

$$u(x,\,t)=U(\xi)=U\left(\frac{kx^{\alpha}}{\Gamma(1+\alpha)}+\frac{ct^{\alpha}}{\Gamma(1+\alpha)}\right),$$

to Eq. (4.1), integrating twice with respect to ξ, and taking the integration constants as zero, we arrive at

$$(c^2+k^2)U+\frac{ck}{2}U^2+c^2k^2U''=0.\tag{4.2}$$

Analyzing U'' and U^2 in (4.2) reveals $m=2$. Therefore, we assume that

$$
\begin{aligned}
U(\xi)=&A_0+A_1w(\xi)+C_1\sqrt{R+w^2(\xi)}+\frac{B_1}{w(\xi)}+A_2w^2(\xi)\\
&+C_2w(\xi)\sqrt{R+w^2(\xi)}+\frac{B_2}{w^2(\xi)},
\end{aligned}
\tag{4.3}
$$

where A_0, A_1, A_2, B_1, B_2, C_1, C_2, and R are undetermined constants.

The substitution of expression (4.3) along with (2.4) into (4.2), then multiplication by $w^4(\xi)\sqrt{R+w^2(\xi)}$ and collection of the same powers of $w^{j_1}(\xi)$ and $w^{j_2}(\xi)\left(\sqrt{R+w^2(\xi)}\right)^{j_3}$ lead to a sequence of nonlinear algebraic equations whose solutions can be computed with the aid of Maple as

Case 1:

$$\begin{cases} R=\dfrac{k^2+c^2}{c^2k^2},\ A_0=-\dfrac{6(k^2+c^2)}{ck},\ A_1=0,\ A_2=-6ck,\\ B_1=0,\ B_2=0,\ C_1=0,\ C_2=\pm 6ck. \end{cases}$$

Case 2:

$$\begin{cases} R=-\dfrac{k^2+c^2}{c^2k^2},\ A_0=\dfrac{4(k^2+c^2)}{ck},\ A_1=0,\ A_2=-6ck,\\ B_1=0,\ B_2=0,\ C_1=0,\ C_2=\pm 6ck. \end{cases}$$

Case 3:

$$\begin{cases} R=\dfrac{k^2+c^2}{4c^2k^2},\ A_0=-\dfrac{3(k^2+c^2)}{ck},\ A_1=0,\ A_2=-12ck,\\ B_1=0,\ B_2=0,\ C_1=0,\ C_2=0. \end{cases}$$

Case 4:

$$\begin{cases} R=-\dfrac{k^2+c^2}{4c^2k^2},\ A_0=\dfrac{k^2+c^2}{ck},\ A_1=0,\ A_2=-12ck,\\ B_1=0,\ B_2=0,\ C_1=0,\ C_2=0. \end{cases}$$

Case 5:

$$\begin{cases} R=-\dfrac{k^2+c^2}{16c^2k^2},\ A_0=-\dfrac{k^2+c^2}{2ck},\ A_1=0,\ A_2=-12ck,\\ B_1=0,\ B_2=-\dfrac{3(c^4+2c^2k^2+k^4)}{64c^3k^3},\ C_1=0,\ C_2=0. \end{cases}$$

Case 6:

$$\begin{cases} R=\dfrac{k^2+c^2}{16c^2k^2},\ A_0=-\dfrac{3(k^2+c^2)}{2ck},\ A_1=0,\ A_2=-12ck,\\ B_1=0,\ B_2=-\dfrac{3(c^4+2c^2k^2+k^4)}{64c^3k^3},\ C_1=0,\ C_2=0. \end{cases}$$

Case 7:

$$\begin{cases} R=\dfrac{k^2+c^2}{4c^2k^2},\ A_0=-\dfrac{3(k^2+c^2)}{ck},\ A_1=0,\ A_2=0,\\ B_1=0,\ B_2=-\dfrac{3(k^2+c^2)^2}{4c^3k^3},\ C_1=0,\ C_2=0. \end{cases}$$

Case 8:

$$\begin{cases} R=-\dfrac{k^2+c^2}{4c^2k^2},\ A_0=\dfrac{k^2+c^2}{ck},\ A_1=0,\ A_2=0,\\ B_1=0,\ B_2=-\dfrac{3(k^2+c^2)^2}{4c^3k^3},\ C_1=0,\ C_2=0. \end{cases}$$

In fact, we note $R=\frac{k^2+c^2}{c^2k^2}>0$ from Case 1, which means that we can get trigonometric function solutions to Eq. (4.1). The expression (4.3) can be rewritten as

$$U_1(\xi)=-\frac{6(k^2+c^2)}{ck}-6ckw^2(\xi)\pm 6ckw(\xi)\sqrt{R+w^2(\xi)}. \tag{4.4}$$

Thus, by substituting (2.8)-(2.14) into (4.4) respectively, a series of general exact solutions can be obtained. Here we only list one of them via inserting (2.8) into (4.4), namely,

$$\begin{aligned} U_{11}(\xi)=&-\frac{6(k^2+c^2)}{ck}-\frac{6(k^2+c^2)}{ck}\tan^2\left(\sqrt{\frac{k^2+c^2}{c^2k^2}}\xi\right)\\ &\pm 6ck\sqrt{\frac{k^2+c^2}{c^2k^2}}\tan\left(\sqrt{\frac{k^2+c^2}{c^2k^2}}\xi\right)\sqrt{\frac{k^2+c^2}{c^2k^2}\sec^2\left(\sqrt{\frac{k^2+c^2}{c^2k^2}}\xi\right)}, \end{aligned}$$

where $\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{ct^\alpha}{\Gamma(1+\alpha)}$.

Moreover, it is clear that many trigonometric function solutions corresponding to Cases 3, 6, and 7 can be given in a similar way. But for brevity we do not list all of them.

According to Case 2, we can get the following expression

$$U_2(\xi)=\frac{4(k^2+c^2)}{ck}-6ckw^2(\xi)\pm 6ckw(\xi)\sqrt{R+w^2(\xi)}. \tag{4.5}$$

In consideration of $R=-\frac{k^2+c^2}{c^2k^2}<0$, some hyperbolic function solutions can be derived by carrying (2.5)-(2.7) into (4.5). The substitution of (2.5) into (4.5) yields

$$\begin{aligned} U_{21}(\xi)=&\frac{4(k^2+c^2)}{ck}-\frac{6(k^2+c^2)}{ck}\tanh^2\left(\sqrt{\frac{k^2+c^2}{c^2k^2}}\xi\right)\mp 6ck\sqrt{\frac{k^2+c^2}{c^2k^2}}\\ &\times\tanh\left(\sqrt{\frac{k^2+c^2}{c^2k^2}}\xi\right)\sqrt{-\frac{k^2+c^2}{c^2k^2}+\frac{k^2+c^2}{c^2k^2}\tanh^2\left(\sqrt{\frac{k^2+c^2}{c^2k^2}}\xi\right)}, \end{aligned}$$

where $\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{ct^\alpha}{\Gamma(1+\alpha)}$.

Similarly, we can express many hyperbolic function solutions from Cases 4, 5, and 8, whose details we omit here.

Next, we would like to present infinite sequence solutions to (4.1) based on Bäcklund transformations (3.1)-(3.2) and nonlinear superposition formulas (3.6) and (3.8). Combining (3.4) with (4.4), we have

$$\begin{cases} u_n(x,\,t)=U_n(\xi)=-\dfrac{6(k^2+c^2)}{ck}-6ckw_n^2(\xi)\pm 6ckw_n(\xi)\sqrt{R+w_n^2(\xi)},\\ w_n(\xi)=\dfrac{p_2+q_2w_{n-1}(\xi)+m_2w_{n-1}^2(\xi)+r_2w'_{n-1}(\xi)+n_2w_{n-1}^3(\xi)+l_2\left(w'_{n-1}(\xi)\right)^2}{a_2+b_2w_{n-1}(\xi)+d_2w_{n-1}^2(\xi)+c_2w'_{n-1}(\xi)+f_2w_{n-1}^3(\xi)+k_2\left(w'_{n-1}(\xi)\right)^2},\\ w_0(\xi)=\sqrt{R}\tan\left(\sqrt{R}\xi\right), \end{cases}$$

where $\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{ct^\alpha}{\Gamma(1+\alpha)}$, $R=\frac{k^2+c^2}{c^2k^2}>0$, $n=1,\,2,\,\cdots$

Combining (3.5) with (4.4), we have

$$\begin{cases} u_n(x,\,t)=U_n(\xi)=-\dfrac{6(k^2+c^2)}{ck}-6ckw_n^2(\xi)\pm 6ckw_n(\xi)\sqrt{R+w_n^2(\xi)}, \\ w_n(\xi)=\dfrac{-BR+Aw_{n-1}(\xi)}{A+Bw_{n-1}(\xi)}, \\ w_1(\xi)=\dfrac{\sqrt{R}\left[-2a_6b_6\sqrt{R}+\left(a_6^2-b_6^2R\right)\left(\sec\left(2\sqrt{R}\xi\right)+\tan\left(2\sqrt{R}\xi\right)\right)\right]}{a_6^2-b_6^2R+2a_6b_6\sqrt{R}\left[\sec\left(2\sqrt{R}\xi\right)+\tan\left(2\sqrt{R}\xi\right)\right]}, \end{cases}$$

where $\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{ct^\alpha}{\Gamma(1+\alpha)}$, $R=\frac{k^2+c^2}{c^2k^2}>0$, $n=2,\,3,\,\cdots$

Combining (3.3) with (4.5), we have

$$\begin{cases} u_n(x,\,t)=U_n(\xi)=\dfrac{4(k^2+c^2)}{ck}-6ckw_n^2(\xi)\pm 6ckw_n(\xi)\sqrt{R+w_n^2(\xi)}, \\ w_n(\xi)=\dfrac{p_2+q_2w_{n-1}(\xi)+m_2w_{n-1}^2(\xi)+r_2w'_{n-1}(\xi)+n_2w_{n-1}^3(\xi)+l_2\left(w'_{n-1}(\xi)\right)^2}{a_2+b_2w_{n-1}(\xi)+d_2w_{n-1}^2(\xi)+c_2w'_{n-1}(\xi)+f_2w_{n-1}^3(\xi)+k_2\left(w'_{n-1}(\xi)\right)^2}, \\ w_0(\xi)=-\sqrt{-R}\tanh\left(\sqrt{-R}\xi\right), \end{cases}$$

where $\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{ct^\alpha}{\Gamma(1+\alpha)}$, $R=-\frac{k^2+c^2}{c^2k^2}<0$, $n=1,\,2,\,\cdots$

Combining (3.10) with (4.4), we have

$$\begin{cases} u_n(x,\,t)=U_n(\xi)=-\dfrac{6(k^2+c^2)}{ck}-6ckw_n^2(\xi)\pm 6ckw_n(\xi)\sqrt{R+w_n^2(\xi)}, \\ w_n(\xi)=\dfrac{R\left[-r_3w_{n-3}(\xi)+(p_3+r_3)w_{n-2}(\xi)-p_3w_{n-1}(\xi)\right]}{-r_3w_{n-2}(\xi)w_{n-1}(\xi)+w_{n-3}(\xi)(-p_3w_{n-2}(\xi)+(p_3+r_3)w_{n-1}(\xi))}, \\ w_1(\xi)=\dfrac{-b_5R+a_5\sqrt{R}\left[\sec\left(2\sqrt{R}\xi\right)+\tan\left(2\sqrt{R}\xi\right)\right]}{a_5+b_5\sqrt{R}\left[\sec\left(2\sqrt{R}\xi\right)+\tan\left(2\sqrt{R}\xi\right)\right]}, \\ w_2(\xi)=\dfrac{\sqrt{R}\left[\cos\left(\sqrt{R}\xi\right)+\sin\left(\sqrt{R}\xi\right)\right]}{\cos\left(\sqrt{R}\xi\right)-\sin\left(\sqrt{R}\xi\right)}, \\ w_3(\xi)=\sqrt{R}\tan\left(\sqrt{R}\xi\right), \end{cases}$$

where $\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{ct^\alpha}{\Gamma(1+\alpha)}$, $R=\frac{k^2+c^2}{c^2k^2}>0$, $n=4,\,5,\,\cdots$

5 Conclusion

To sum up, taking advantage of Bäcklund transformations and nonlinear superposition formulas of solutions to Riccati equation, we have successfully established infinite sequence solutions for space-time fractional symmetric regularized long wave equation through symbolic computation. This method can be extended to deal with other nonlinear fractional partial differential equations.

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