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Existence and Regularity of Solution for Strongly Nonlinear $p(x)$-Elliptic Equation with Measure Data

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Abstract. The first part of this paper is devoted to study the existence of solution for nonlinear $p(x)$ elliptic problem $A(u)=\mu$ in Ω, $u=0$ on $\partial\Omega$, with a right-hand side measure, where Ω is a bounded open set of $\mathbb{R}^N$, $N\geqslant 2$ and $A(u)=-\mathrm{div}(a(x,u,\nabla u))$ is a Leray-Lions operator defined from $W_0^{1,p(x)}(\Omega)$ in to its dual $W^{-1,p'(x)}(\Omega)$. However the second part concerns the existence solution, of the following setting nonlinear elliptic problems $A(u)+g(x,u,\nabla u)=\mu$ in Ω, $u=0$ on $\partial\Omega$. We will give some regularity results for these solutions.

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, ($N\geqslant 2$). In this paper, we deal with the following Direchlet problem:

$$\begin{cases} Au+g(x,u,\nabla u)=\mu, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{P})$$

with non-standard p-structure which involves a variable growth exponent $p(\cdot)$.

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The principal part of the above equation is the operator A defined by

$$Au=-\mathrm{div}(a(x,u,\nabla u)),$$

which satisfies the classical Leray-Lions conditions with some variable exponent $p(\cdot)$. However the non linearity g is supposed to satisfy the sign condition and some natural growth with respect to ∇u, but no growth with u is supposed. The second member μ considered in this paper is not regular that is a Radon measure.

The study of partial differential equation involving $p(x)$ growth conditions has received specific attention in recent decades. This is a consequence of the fact that such equations can be used to model phenomena which arise in mathematical physics. Electro rheological fluids and elastic mechanics are two examples of physical fields which benefit from such kinds of studies.

Problem $(\mathcal{P})$ is studied in [1] where $a=a(x,\nabla u)$ which satisfying the classical Leray-Lions conditions with some constant exponent p such that: $2-\frac{1}{N}<p<N$. The solution obtained in [1] admits the following regularity $u\in W_0^{1,q}(\Omega)$ for all:

$$1\leqslant q<\frac{N(p-1)}{N-1}.$$

The approach used by the authors in [1] is to approximate the measure μ by a sequence (f_n) in $W^{-1,p'(x)}\cap L^1(\Omega)$ which converge to μ. The limiting process hinges of the proof of the almost pointwise convergence of the sequence $(\bigtriangledown u_n)$, where u_n is the weak solution of the problem $(\mathcal{P})$ with $\mu=f_n$.

When trying to relax the coefficients of a, that is $a(\cdot)$ have not a polynomial growths, then the problem $(\mathcal{P})$ is formulated is the general setting of Orlicz spaces generated by an N-function M which appears in the non classical growths of $a(\cdot)$. In this case, we found the work [2] which treat the study of a problem $(\mathcal{P})$ in Orlicz-Sobolev spaces. The solutions obtained in [2] and [3] belongs to $T_0^{1,M}(\Omega)\cap W_0^1L_B(\Omega)$ for any $B\in P_M$, where $B\in P_M$ is a special class of N-functions. For others works, we refer the reader [4–8] and [9].

In the recent years, variable exponent Sobolev spaces have attracted an increasing amount attention, the impulse, for this mainly comes from there physical applications, such in image processing (underline the borders, eliminate the noise) and electro-rheological fluids.

In the framework of variable exponent Sobolev spaces, we list the works [6,7,10,11] and others, where the second member μ of the problem $(\mathcal{P})$ is taking as an element of $L^1(\Omega)$ or is a measure which admits the composition $\mu=f-\mathrm{div}(F)$, with $f\in L^1(\Omega)$ and $F\in\prod L^{p(x)}(\Omega)$.

Our purpose in this paper is to study the existence of solution of the problem $(\mathcal{P})$, in the case where the datum μ is a finite Radon measure and in the case of variable exponent, (that is A is a Leray-Lions operator acting from $W_0^{1,p(x)}(\Omega)$ in to $W^{-1,p'(x)}(\Omega)$), with the regularity $u\in W_0^{1,q}(\Omega)$ for all: $1\leqslant q^-\leqslant q^+<(N/(N-1))(p^- -1)$.

This paper can be seen as a generalization to the variable exponent of the works [1] and as a continuation of the works [6] ($f\in L^1(\Omega)$, $f\in W^{-1,p'(x)}$, $p=p(x)$).

The paper is organized as follows: In the Section 1, we recall some important definitions and some results of variable exponent Lebesgue and Sobolev spaces. In the Section 2 we give and proof ours mains results.

2 Framework of the spaces

In this paragraph, we recall some definition and basic results about the Sobolev spaces with variable exponent.

Let Ω be a bounded open subset of IR^N, $(N\geq 2)$, we denote

$$C_+(\overline{\Omega})=\Big\{\text{continuous}\quad p(\cdot):\overline{\Omega}\longrightarrow IR\quad\text{such that}\quad 1<p_-\leqslant p(x)\leqslant p_+\leqslant N\Big\},$$

where

$$p_-=\min\{p(x)\,/\,x\in\overline{\Omega}\}\qquad\text{and}\qquad p_+=\max\{p(x)\,/\,x\in\overline{\Omega}\}.$$

We define the variable exponent Lebesgue space for $p(\cdot)\in C_+(\overline{\Omega})$ by

$$L^{p(x)}(\Omega)=\Big\{u:\Omega\to IR\quad\text{measurable}\Big/\int_\Omega |u(x)|^{p(x)}\mathrm{d}x<\infty\Big\}.$$

The space $L^{p(x)}(\Omega)$ under the norm

$$\|u\|_{p(x)}=\inf\left\{\lambda>0,\quad \int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}\mathrm{d}x\leq 1\right\},$$

is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ (see [12,13]).

Proposition 2.1. ([12,13]) (Generalized Hölder inequality)

(i) For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$, we have

$$\left|\int_\Omega uv\,\mathrm{d}x\right|\leq\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)\|u\|_{p(x)}\|v\|_{p'(x)}.$$

(ii) For all $p_1(\cdot),p_2(\cdot)\in C_+(\overline{\Omega})$ such that $p_1(x)\leq p_2(x)$, we have $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.2. ([12,13]). *If we denote*

$$\rho(u)=\int_{\Omega}|u|^{p(x)}\mathrm{d}x, \qquad \forall u\in L^{p(x)}(\Omega),$$

then, the following assertions holds

(i) $\|u\|_{p(x)}<1 \quad (resp,=1,>1) \quad \Longleftrightarrow \quad \rho(u)<1 \quad (resp,=1,>1),$

(ii) $\|u\|_{p(x)}>1 \Rightarrow \|u\|_{p(x)}^{p_-}\le\rho(u)\le\|u\|_{p(x)}^{p_+},$
$\|u\|_{p(x)}<1 \Rightarrow \|u\|_{p(x)}^{p_+}\le\rho(u)\le\|u\|_{p(x)}^{p_-},$

(iii) $\|u\|_{p(x)}\to 0 \Leftrightarrow \rho(u)\to 0$ *and* $\|u\|_{p(x)}\to\infty \Leftrightarrow \rho(u)\to\infty.$

Now, we define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega) \quad \text{and} \quad |\nabla u|\in L^{p(x)}(\Omega)\},$$

normed by

$$\|u\|_{1,p(x)}=\|u\|_{p(x)}+\|\nabla u\|_{p(x)}, \qquad \forall u\in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and we define the Sobolev exponent by $p^*(x)=\frac{Np(x)}{N-p(x)}$ for $p(x)<N$.

Proposition 2.3. ([12,14])

(i) Assuming $1<p_-\le p_+<\infty$, *the spaces* $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*

(ii) If $q(\cdot)\in C_+(\bar{\Omega})$ *and* $q(x)<p^*(x)$ *for any* $x\in\Omega$, *then the embedding* $W_0^{1,p(x)}(\Omega)\hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *is continuous and compact.*

(iii) Poincaré inequality: there exists a constant $C>0$, *such that*

$$\|u\|_{p(x)}\le C\|\nabla u\|_{p(x)}, \qquad \forall u\in W_0^{1,p(x)}(\Omega).$$

(iv) Sobolev inequality: there exists an other constant $C>0$, *such that*

$$\|u\|_{p*(x)}\le C\|\nabla u\|_{p(x)}, \qquad \forall u\in W_0^{1,p(x)}(\Omega).$$

Remark 2.1. By (iii) of the Proposition 2.3, we deduce that $\|\nabla u\|_{p(x)}$ and $\|u\|_{1,p(x)}$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$.

Definition 2.1. *For all $k>0$ and $s\in IR$, the truncation function $T_k(\cdot)$ can be defined by*

$$T_k(s)=\begin{cases} s & \text{if} \quad |s|\le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s|>k, \end{cases}$$

and we define

$$T_0^{1,p(x)}(\Omega):=\Big\{u \text{ measurable such that } T_k(u)\in W_0^{1,p(x)}(\Omega), \quad \forall k>0\Big\}.$$

Lemma 2.1. ([15]) *Let $g\in L^{r(x)}(\Omega)$ and $g_n\in L^{r(x)}(\Omega)$ with $\|g_n\|_{r(x)}\le C$ for $1<r(x)<\infty$. If $g_n(x)\to g(x)$ a.e. on Ω, then $g_n\rightharpoonup g$ weakly in $L^{r(x)}(\Omega)$.*

Lemma 2.2. ([15]) *Let $u\in W_0^{1,p(x)}(\Omega)$, then $T_k(u)\in W_0^{1,p(x)}(\Omega)$ for all $k>0$. Moreover, we have*

$$T_k(u)\to u \qquad \text{in } W_0^{1,p(x)}(\Omega) \quad \text{as } k\to\infty.$$

Lemma 2.3. ([10]) *Let $p(\cdot)$ be a continuous function in $C_+(\overline{\Omega})$ and u a function in $W_0^{1,p(x)}(\Omega)$. Suppose $2-\frac{1}{N}<p^-\leqslant p^+\leqslant N$, and that there exists a constant C_1 such that*

$$\int_{\{k\leqslant|u|\leqslant k+1\}}|\nabla u|^{p(x)}\mathrm{d}x\leqslant C_1, \qquad \forall\, k>0.$$

Then, there exists a constant C_2, depending on C_1, such that

$$\|u\|_{W_0^{1,q(x)}(\Omega)}\leqslant C_2,$$

for all continuous functions $q(.)$ on $\overline{\Omega}$ satisfying

$$1\leqslant q(x)<\frac{N}{N-1}(p(x)-1), \qquad \text{for all } x\in\overline{\Omega}.$$

3 Main result

3.1 Preliminary lemma

Lemma 3.1. *Let $(v_n)_n$ be a sequence of functions in $W_0^{1,p(x)}(\Omega)$. Suppose that there exists a constant $C>0$ such that, for all $k>0$:*

$$\int_\Omega|\nabla T_k(v_n)|^{p(x)}\mathrm{d}x\le Ck.$$

Then, there exists a subsequence still denoted by $(v_n)_n$ and a function v, such that

$$\begin{aligned} &v_n(x)\longrightarrow v \text{ a.e. in } \Omega, \qquad T_kv_n\rightharpoonup T_kv \text{ weakly in } W_0^{1,p(\cdot)}(\Omega),\\ &T_kv_n\longrightarrow T_kv \text{ strongly in } L^{p(\cdot)}(\Omega). \end{aligned}$$

Proof. We have

$$||\nabla T_k(v_n)||^{\gamma}_{p(x)} \leqslant Ck,$$

where

$$\gamma = \begin{cases} P^-, & \text{if} \quad ||\nabla T_k(v_n)||_{p(x)} > 1, \\ p^+, & \text{if} \quad ||\nabla T_k(v_n)||_{p(x)} \leqslant 1. \end{cases}$$

Thus

$$||\nabla T_k(v_n)||_{p(x)} \leqslant C_1 k^{\frac{1}{\gamma}}, \tag{3.1}$$

where C_1 is a constant which does not depend on k.

From (3.1), it easily follows that $(v_n)_n$ is Cauchy sequence in measure. Indeed, we have

$$k \text{ meas}\{|v_n|>k\} = \int_{\{|v_n|>k\}} |T_k(v_n)| \mathrm{d}x \leqslant \int_\Omega |T_k(v_n)| \mathrm{d}x.$$

By Hölder inequality, we obtain

$$k \text{ meas}\{|v_n|>k\} \leqslant \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) ||1||_{p'(x)} ||T_k(v_n)||_{p(x)}.$$

Therefore

$$k \text{ meas}\{|v_n|>k\} \leqslant \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) (\text{meas}(\Omega)+1)^{\frac{1}{p'_-}} ||T_k(v_n)||_{p(x)}.$$

By the Poincaré inequality we have

$$\text{meas}\{|v_n|>k\} \leqslant C_2 k^{\frac{1}{\gamma}-1} \to 0, \qquad \text{as } k \to +\infty. \tag{3.2}$$

For all n and m in $\mathbb{N}$, for all $\varepsilon > 0$ and for every $k > 0$, we get

$$\begin{aligned} \{|v_n - v_m| \geqslant \varepsilon\} &= (\{|v_n - v_m| \geqslant \varepsilon, \ |v_n| > k\}) \cup (\{|v_n - v_m| \geqslant \varepsilon, \ |v_n| \leqslant k\}) \\ &= (\{|v_n - v_m| \geqslant \varepsilon, \ |v_n| > k\}) \cup (\{|v_n - v_m| \geqslant \varepsilon, \ |v_n| \leqslant k, \ |v_m| \leqslant k\}) \\ &\quad \cup (\{|v_n - v_m| \geqslant \varepsilon, \ |v_n| \leqslant k, \ |v_m| > k\}). \end{aligned} \tag{3.3}$$

Thus

$$\{|v_n - v_m| \geqslant \varepsilon\} \subset \{|v_n| > k\} \cup \{|T_k(v_n) - T_k(v_m)| \geqslant \varepsilon\} \cup \{|v_m| > k\}.$$

Therefore

$$\begin{aligned} \text{meas}(\{|v_n - v_m| \geqslant \varepsilon\}) \leqslant &\text{meas}(\{|v_n| > k\}) + \text{meas}(\{|T_k(v_n) - T_k(v_m)| \geqslant \varepsilon\}) \\ &+ \text{meas}(\{|v_m| > k\}). \end{aligned} \tag{3.4}$$

By (3.2), we have for all $\delta > 0$ there exists k_0 such that

$$\text{meas}(\{|v_n| > k\}) < \frac{\delta}{3}, \qquad \text{meas}(\{|v_m| > k\}) < \frac{\delta}{3}, \qquad \forall k \geqslant k_0(\delta).$$

By (3.1), we have $(T_k(v_n))_n$ is bounded in $W_0^{1,\,p(\cdot)}(\Omega)$, then there exists a subsequence still denoted $(T_k(v_n))_n$ which is strongly compact in $L^{p(x)}(\Omega)$. This means, in particular, that the sequence $(T_k(v_n))_n$ is Cauchy in measure. We then choose n and m, such that

$$\text{meas}(\{|T_k(v_n)-T_k(v_m)|\geqslant\varepsilon\})<\frac{\delta}{3}.$$

Therefore the sequence $(v_n)_n$ is Cauchy in measure, we thus have (up to subsequence, still denoted by $(v_n)_n$) which converge almost everywhere in Ω, to some function v.

On the other, we have, $T_k(v_n)\to T_k(v)$ a.e. in Ω, by combining (3.1) and (Lemma 2.1), we have

$$T_kv_n \rightharpoonup T_kv \ \text{ weakly in } \ W_0^{1,p(\cdot)}(\Omega);$$

and therefore

$$T_kv_n \to T_kv \ \text{ strongly in } \ L^{p(\cdot)}(\Omega).$$

□

3.2 Nonlinear elliptic problem: $-\mathbf{div}\Big(a(x,u,\nabla u)\Big)=\mu$

Let Ω be a bounded open subset of $I\!R^N\,(N\geq 2)$, and let $p\in C_+(\bar{\Omega})$, such that

$$2-\frac{1}{N}<p^-\leqslant p^+\leqslant N. \tag{3.5}$$

We consider a Leray-Lions operator A from $W_0^{1,p(x)}(\Omega)$ into its dual $W^{-1,p'(x)}(\Omega)$, defined by

$$Au=-\text{div}\Big(a(x,u,\nabla u)\Big); \tag{3.6}$$

where $a:\Omega\times I\!R\times I\!R^N\to I\!R^N$ is a Carathéodory function, satisfying the following conditions

$$|a(x,s,\xi)|\leq\beta(K(x)+|s|^{p(x)-1}+|\xi|^{p(x)-1}), \tag{3.7}$$

$$a(x,s,\xi)\xi\geq\alpha|\xi|^{p(x)}, \tag{3.8}$$

$$[a(x,s,\xi)-a(x,s,\overline{\xi})](\xi-\overline{\xi})>0, \quad \text{for all } \ \xi\neq\overline{\xi} \text{ in } I\!R^N, \tag{3.9}$$

for almost every x in Ω, for every (s,ξ) in $I\!R\times I\!R^N$, where $K(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and $\beta>0,\alpha>0$. Let

$$B_{p(\cdot)}=\left\{q\in\mathcal{C}_+(\bar{\Omega}):\,1\leqslant q(x)<\frac{N}{N-1}(p^- -1)\right\}. \tag{3.10}$$

Let $\mu\in M_b(\Omega)$, $M_b(\Omega)$ denotes the set of bounded measures on Ω (finite Radon measures). Consider the nonlinear elliptic problem:

$$(\mathcal{P})\qquad \begin{cases} -\text{div}\Big(a(x,u,\nabla u)\Big)=\mu & \text{in } \ \Omega,\\ u=0 & \text{on } \ \partial\Omega.\end{cases}$$

Definition 3.1. *A measurable function u will be called a weak solution of the problem $(\mathcal{P})$, if*

$$u\in T_0^{1,p(\cdot)}(\Omega)\cap W_0^{1,\,q(\cdot)},\quad \forall\, q(\cdot)\in B_{p(\cdot)},$$
$$\int_\Omega a(x,u,\nabla u)\nabla\varphi \mathrm{d}x=<\mu,\varphi>,\quad \forall\varphi\in D(\Omega).$$

3.2.1 Approximate problem

Consider the approximate equation

$$(\mathcal{P}_n)\qquad \begin{cases} -\mathrm{div}\Big(a(x,u_n,\nabla u_n)\Big)=f_n & \text{in}\quad \Omega,\\ u_n\in W_0^{1,\,p(\cdot)}(\Omega), \end{cases} \tag{3.11}$$

where (f_n) is a smooth function which converges to μ in the distributional sense such that,

$$||f_n||_{L^1(\Omega)}\leqslant ||\mu||_{M_b(\Omega)}.$$

Lemma 3.2. *The problem $(\mathcal{P}_n)$ has at last one weak solution u_n and there exists a sub-sequence denoted again (u_n) and a function u such that*

$$u_n\to u \;\; a.e \;\; in \;\; \Omega \quad and \quad \nabla u_n\to\nabla u \;\; a.e. \;\; in \;\; \Omega. \tag{3.12}$$

Proof. By [6], the problem $(\mathcal{P}_n)$ has at last one weak solution u_n.

For $k>0$, by taking $T_k(u_n)$ as a test function in (3.11), we deduce that

$$(\exists C>0):\ \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\mathrm{d}x\le Ck.$$

In view of (3.8), we get

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}\mathrm{d}x\le Ck. \tag{3.13}$$

By Lemma 3.1 there exists a subsequence still denoted by u_n and u such that:

$$\begin{aligned} &u_n\to u \;\text{ a.e. in } \Omega; \qquad T_k(u_n)\rightharpoonup T_k(u) \;\text{ weakly in }\; W_0^{1,p(\cdot)}(\Omega);\\ &T_k(u_n)\to T_k(u) \;\text{ strongly in }\; L^{p(\cdot)}(\Omega). \end{aligned} \tag{3.14}$$

We can write

$$\begin{aligned} 0\leqslant &\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\mathrm{d}x\\ =&\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n))\Big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\mathrm{d}x\\ &-\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u))\Big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\mathrm{d}x. \end{aligned} \tag{3.15}$$

We have

$$a(x,T_k(u_n),\nabla T_k(u)) \to a(x,T_k(u),\nabla T_k(u)) \text{ a.e. in } \Omega. \tag{3.16}$$

By combining (3.7), (3.16) and the dominated convergence theorem we get

$$a(x,T_k(u_n),\nabla T_k(u)) \to a(x,T_k(u),\nabla T_k(u)) \quad \text{stongly in } (L^{P'(x)}(\Omega))^N.$$

Therefore

$$\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u))\Big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\mathrm{d}x \to 0, \qquad \text{as } n\to\infty.$$

Let $\eta>0$ and taking $T_\eta(u_n-T_k(u))$ as test function in (3.11), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(u))\mathrm{d}x \le C\eta.$$

For the sake of simplicity, we write only $\varepsilon(n,\eta)$ to mean all quantities (possibly different) such that

$$\lim_{n\to\infty}\lim_{\eta\to 0}\varepsilon(n,\eta)=0.$$

On the other hand

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(u))\mathrm{d}x\\
=&\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|\leqslant k\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))\mathrm{d}x\\
&\quad+\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)(\nabla u_n-\nabla T_k(u))\mathrm{d}x\\
=&\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|\leqslant k\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))\mathrm{d}x\\
&\quad+\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla u_n\mathrm{d}x\\
&\quad-\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)\mathrm{d}x.
\end{aligned}\tag{3.17}$$

By (3.8) the second term of the right-hand side of (3.17) satisfies

$$\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla u_n\mathrm{d}x \geqslant 0,$$

the third term of the right side of (3.17) satisfies

$$\begin{aligned}
&\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)\mathrm{d}x\\
=&\int_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\nabla T_k(u)\mathrm{d}x.
\end{aligned}$$

Since $a(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L^{p'(x)}(\Omega))^N$, there exists $h_{\eta+k}\in(L^{p'(x)}(\Omega))^N$ such that

$$a(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))\rightharpoonup h_{\eta+k},\ \text{ weakly in } (L^{p'(x)}(\Omega))^N.$$

Since,

$$\nabla T_k(u)\chi_{\{|u_n-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}}\to 0 \text{ strongly in } (L^{p(x)}(\Omega))^N,$$

as $n\to\infty$ and $\eta\to 0$. Thus

$$\begin{aligned}&\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|\leqslant k\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))\mathrm{d}x\\ &\leqslant C\eta+\varepsilon(n,\eta).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))\mathrm{d}x\\ &\leqslant\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(u)|\mathrm{d}x.\end{aligned}\tag{3.18}$$

We have

$$\begin{aligned}&|\nabla T_k(u)|\chi_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}}\to 0,\ \text{ a.e. in } \Omega \text{ as } n\to\infty,\\ &|T_k(u)|\chi_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}}\leqslant|T_k(u)|,\end{aligned}$$

thus by Lebesgue dominated convergence theorem, we deduce that

$$|\nabla T_k(u)|\chi_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}}\to 0,\ \text{ in } L^{p(\cdot)}(\Omega),\qquad \text{as } n\to\infty$$

Since the sequence $(|a(x,T_k(u_n),\nabla T_k(u_n))|)_n$ is bonded in $(L^{p'(\cdot)}(\Omega))^N$,

$$\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(u)|\mathrm{d}x\to 0,\qquad \text{as } n\to\infty$$

Thus

$$\int_{\{|T_k(u_n)-T_k(u)|\leqslant\eta\}\cap\{|u_n|>k\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))\mathrm{d}x\leqslant\varepsilon(n).$$

On the other hand,

$$\int_{\{|T_k(u_n)-T_k(u)|>\eta\}\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))\mathrm{d}x\to 0 \text{ as } n\to\infty,$$

since

$$\text{meas}(\{|T_k(u_n)-T_k(u)|>\eta\})\to 0 \text{ as } n\to\infty$$

and by Hölder inequality we have, $a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(u))$ is bounded in $L^1(\Omega)$.

Thus by passing to the limit over n and η, we get

$$\lim_{n\longrightarrow\infty}\int_{\Omega}\Big(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big)\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\mathrm{d}x=0.$$

We conclude by [11, Lemma 4.4] that there exists a sub-sequence denoted again (u_n) such that

$$\nabla u_n\to\nabla u \text{ a.e. in } \Omega. \tag{3.19}$$

This completes the proof of the lemma. □

3.2.2 Result of existence

Theorem 3.1. *Assume that* (3.5) *and* (3.7)-(3.9) *hold and* $\mu\in M_b(\Omega)$, *then there exists at last one weak solution of the problem* $(\mathcal{P})$.

Proof. Let $k>0$, we define the function ψ_k as:

$$\begin{cases}\psi_k(x)=1, & \text{if } x>k+1,\\ \psi_k(x)=x-k, & \text{if } k\leqslant x\leqslant k+1,\\ \psi_k(x)=0, & \text{if } -k\leqslant x\leqslant k,\\ \psi_k(x)=x+k, & \text{if } -k-1\leqslant x\leqslant -k,\\ \psi_k(x)=-1, & \text{if } x<-k-1.\end{cases}$$

By taking $\psi_k(u_n)$ as test function in (3.11) and using (3.8) one has:

$$\int_{D_{k,n}}|\nabla u_n|^{p(x)}\mathrm{d}x\leqslant\frac{1}{\alpha}||\mu||_{M_b(\Omega)}, \tag{3.20}$$

with

$$D_{k,n}=\{x\in\Omega,\ k\leqslant|u_n(x)|\leqslant k+1\}. \tag{3.21}$$

In view of Lemma 2.3, there exists a constant C that does not depend on n such that

$$\|u_n\|_{W_0^{1,q(x)}(\Omega)}\leqslant C, \qquad \forall\, q(\cdot)\in B_{p(\cdot)}. \tag{3.22}$$

Hence (u_n) is relatively compact in $W_0^{1,\,q(x)}$ $\forall\, q(\cdot)\in B_{p(\cdot)}$, thus there exists a sub-sequence denoted again by (u_n) such that

$$u_n\rightharpoonup u \text{ weakly in } W_0^{1,\,q(x)}, \qquad u_n\to u \text{ strongly in } L^{q(x)}, \qquad u_n\to u \text{ a.e. in } \Omega.$$

By Lemma 3.2, we conclude that

$$a(x,u_n,\nabla u_n)\to a(x,u,\nabla u), \qquad \text{a.e in } \Omega.$$

Combining (3.7) and (3.22), yields

$$||a(x,u_n,\nabla u_n)||_{L^{r(x)}} \leqslant C,$$

for every $1<r(x)<N/(N-1)$. Moreover, using Lemma 2.1, we can write

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } L^{r(x)}(\Omega).$$

It is now possible to pass to the limit in (3.11), we conclude that u is the weak solution of the problem $(\mathcal{P})$. □

3.3 Strongly nonlinear elliptic problem: $-\mathrm{div}(a(x,u,\nabla u))+g(x,u,\nabla u)=\mu$

Consider the equation

$$(\mathcal{P}') \qquad \begin{cases} -\mathrm{div}\Big(a(x,u,\nabla u)\Big)+g(x,u,\nabla u)=\mu & \text{in} \quad \Omega, \\ u=0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.23}$$

where a satisfies (3.6)-(3.9), μ lie in $M(\Omega)$, and the non linear term g: $\Omega\times IR\times IR^N \to IR$, is a Carathédory function satisfying for almost every $x\in\Omega$ and for all $s\in IR$, $\zeta\in IR^N$ the following conditions:

$$g(x,s,\xi).s \geqslant 0, \tag{3.24}$$

$$|g(x,s,\xi)| \leq b(|s|)(c(x)+|\xi|^{p(x)}), \tag{3.25}$$

where b: $IR^+ \to IR^+$ is a continuous growth function and $c:\Omega\to IR^+$ with $c\in L^1(\Omega)$.

Definition 3.2. *We say that u is a weak solution of the problem* (3.23) *if*

$$u\in W_0^{1,\,q(\cdot)},\ \ \forall\, q(\cdot)\in B_{p(\cdot)},\ \ a(x,u,\nabla u)\in L^1(\Omega),\ \ g(x,u,\nabla u)\in L^1(\Omega),$$

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi \mathrm{d}x+\int_\Omega g(x,u,\nabla u)\varphi \mathrm{d}x=<\mu,\varphi>, \qquad \forall \varphi\in D(\Omega). \tag{3.26}$$

Theorem 3.2. *Let a satisfy* (3.6)-(3.9)*, g satisfy* (3.24)-(3.25) *and μ lies in $M(\Omega)$. Then there exists a weak solution of* (3.23).

Proof. Let (f_n) be a sequence of $L^1(\Omega)\cap W^{-1,p'(\cdot)}(\Omega)$ which converges to μ in the distributional sense and such that

$$||f_n||_{L^1(\Omega)} \leqslant ||\mu||_{M_b(\Omega)}, \qquad \forall n\in\mathbb{N}. \tag{3.27}$$

By [6] there exists a weak solution (u_n) of the problem (3.23) with $f_n=\mu$ which satisfies:

$$u_n\in W_0^{1,p(x)}(\Omega), \quad g(x,u_n,\nabla u_n)\in L^1(\Omega),$$

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v\mathrm{d}x+\int_\Omega g(x,u_n,\nabla u_n)v\mathrm{d}x=<f_n,v>,$$
$$\forall v\in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega). \tag{3.28}$$

Following the lines of [1], it is easy to deduce that

$$||g(x,u_n,\nabla u_n)||_{L^1(\Omega)}\leqslant||f_n||_{L^1(\Omega)}. \tag{3.29}$$

Setting $h_n=f_n-g(x,u_n,\nabla u_n)$, by (3.27) and (3.29) we have

$$||h_n||_{L^1(\Omega)}\leqslant 2||\mu||_{M_b(\Omega)}.$$

Note that u_n is the solution of the problem (3.11) with a right-hand side is h_n. By the Section 2.2 the sequence (u_n) is relatively compact in $W_0^{1,q(x)}(\Omega)$, $\forall\, q(\cdot)\in B_{p(\cdot)}$. Then we can assume (after extraction of a sub-sequence, still denoted by (u_n))

$$\begin{aligned}&u_n\rightharpoonup u \text{ weakly in } W_0^{1,\,q(x)},\ \text{ for all } q(\cdot)\in B_{p(\cdot)},\\ &u_n\to u \text{ in } L^{q(x)},\quad u_n\to u \text{ a.e. in } \Omega,\\ &a(x,u_n,\nabla u_n)\rightharpoonup a(x,u,\nabla u) \text{ weakly in } L^{r(x)}(\Omega),\quad \text{for } 1<r(x)<N/(N-1).\end{aligned} \tag{3.30}$$

Now, we prove that

$$g(x,u_n,\nabla u_n)\to g(x,u,\nabla u)\quad\text{strongly in } L^1(\Omega),$$

we have $g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)$ a.e. in Ω, using the Vitali convergence theorem, it sufficient to show that $g_n(x,u_n,\nabla u_n)$ is uniformly equi-integrable. Indeed, let $h>0$, taking $T_1(u_n-T_h(u_n))$ as a test function in (3.28), we obtain

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla T_1(u_n-T_h(u_n))\mathrm{d}x+\int_\Omega g(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))\mathrm{d}x\\ =&\int_\Omega f_nT_1(u_n-T_h(u_n))\mathrm{d}x,\end{aligned}$$

it follows that

$$\int_{\{|u_n|\geqslant h\}}g(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))\mathrm{d}x\leqslant\int_{\{|u_n|\geqslant h\}}f_nT_1(u_n-T_h(u_n))\mathrm{d}x.$$

Then

$$\begin{aligned}&\int_{\{|u_n|\geqslant h+1\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x\\ \leqslant&\int_{\{|u_n|\geqslant h\}}g(x,u_n,\nabla u_n)T_1(u_n-T_h(u_n))\mathrm{d}x\\ \leqslant&\int_{\{|u_n|\geqslant h\}}f_nT_1(u_n-T_h(u_n))\mathrm{d}x\end{aligned}$$

$$\leqslant\Big(\frac{1}{p_-}+\frac{1}{p'_-}\Big)||f_n||_{-1,p'(\cdot)}||T_1(u_n-T_h(u_n))||_{1,p(\cdot)}\to 0,\qquad \text{as } h\to\infty.$$

Thus, for all $\varepsilon>0$, there exists $h(\varepsilon)>0$ such that

$$\int_{\{|u_n|\geqslant h(\varepsilon)\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x\leqslant\frac{\varepsilon}{2}. \tag{3.31}$$

On the other hand, for any measurable subset $D\subset\Omega$, we have

$$\begin{aligned}&\int_D|g(x,u_n,\nabla u_n)|\mathrm{d}x\\ \leqslant&\int_D b(h)(c(x)+|\nabla T_h(u_n)|^{p(x)})\mathrm{d}x+\int_{\{|u_n|\geqslant h(\varepsilon)\}}|g(x,u_n,\nabla u_n)|\mathrm{d}x.\end{aligned}$$

There exists $\beta(\varepsilon)>0$ such that

$$b(h)\int_D(c(x)+|\nabla T_h(u_n)|^{p(x)})\mathrm{d}x\leqslant\frac{\varepsilon}{2},\qquad \text{for meas}(D)\leqslant\beta(\varepsilon).$$

Thus

$$\int_D|g(x,u_n,\nabla u_n)|\mathrm{d}x\leqslant\varepsilon,\qquad \text{with meas}(D)\leqslant\beta(\varepsilon).$$

By Vitali convergence theorem we deduce that $g(x,u_n,\nabla u_n)\to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$. Now, taking $\varphi\in D(\Omega)$ as test function in (3.28), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla\varphi\mathrm{d}x+\int_\Omega g(x,u_n,\nabla u_n)\varphi\mathrm{d}x=<f_n,\varphi>.$$

By letting n tends to ∞, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi\mathrm{d}x+\int_\Omega g(x,u,\nabla u)\varphi\mathrm{d}x=<\mu,\varphi>,\qquad\forall\varphi\in D(\Omega).$$

We have show that

$$\begin{aligned}&u\in W_0^{1,\,q(\cdot)},\ \forall\, q(\cdot)\in B_{p(\cdot)};\quad g(x,u,\nabla u)\in L^1(\Omega),\\ &a(x,u,\nabla u)\in L^{r(x)}(\Omega),\quad\text{for } 1<r(x)<\frac{N}{N-1},\\ &\int_\Omega a(x,u,\nabla u)\nabla\varphi\mathrm{d}x+\int_\Omega g(x,u,\nabla u)\varphi\mathrm{d}x=<\mu,\varphi>,\ \forall\varphi\in D(\Omega),\end{aligned}$$

thus u is the weak solution of problem (3.23). □

Exemple 3.1. We consider the following functions

$$a(x,u,\nabla u)=|\nabla u|^{p(x)-2}\nabla u,\quad\text{and}\quad g(x,u,\nabla u)=(1+|\nabla u|^{p(x)})|u|^{p(x)-2}u.$$

It is clear that $a(x,u,\nabla u)$ and $g(x,u,\nabla u)$ verifies (3.6)-(3.9) and (3.24)-(3.25) respectively, then by Theorem 3.2 for all $\mu\in M(\Omega)$ there exists a weak solution of the problem

$$\begin{cases} -\mathrm{div}\Big(|\nabla u|^{p(x)-2}\nabla u\Big)+\Big(1+|\nabla u|^{p(x)}\Big)|u|^{p(x)-2}u=\mu, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{3.32}$$

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