Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **6**, No. 6, pp. 783-796

DOI: 10.4208/aamm.2013.m206
December 2014

On the Generalized Thermoelasticity Problem for an Infinite Fibre-Reinforced Thick Plate under Initial Stress

Ahmed E. Abouelregal[1,3] and Ashraf M. Zenkour[2,4,*]

[1] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
[2] *Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[3] *Department of Mathematics, College of Science and Arts, University of Aljouf, El-Qurayat, Saudi Arabia*
[4] *Department of Mathematics, Faculty of Science, Kafrelsheikh University, Kafr El-Sheikh 33516, Egypt*

Received 29 April 2013; Accepted (in revised version) 3 July 2014

Available online 7 August 2014

Abstract. In this paper, the generalized thermoelasticity problem for an infinite fiber-reinforced transversely-isotropic thick plate subjected to initial stress is solved. The lower surface of the plate rests on a rigid foundation and temperature while the upper surface is thermally insulated with prescribed surface loading. The normal mode analysis is used to obtain the analytical expressions for the displacements, stresses and temperature distributions. The problem has been solved analytically using the generalized thermoelasticity theory of dual-phase-lags. Effect of phase-lags, reinforcement and initial stress on the field quantities is shown graphically. The results due to the coupled thermoelasticity theory, Lord and Shulman's theory, and Green and Naghdi's theory have been derived as limiting cases. The graphs illustrated that the initial stress, the reinforcement and phase-lags have great effects on the distributions of the field quantities.

AMS subject classifications: 73B, 73C, 73K

Key words: Dual-phase-lag theory, fiber-reinforced, initial stress, normal mode analysis, thick plate.

1 Introduction

In modern times, attention has been given to the problems of generation and propagation of elastic waves in an anisotropic elastic solids or layers of different configurations.

*Corresponding author.
Email: zenkour@kaf-sci.edu.eg (A. M. Zenkour)

There is a different between the isotropic and anisotropic media for the propagation of elastic waves. The information obtained from such studies is important to seismologists and geophysicists to find the location of the earthquakes as well as their energy, mechanism etc. and thereby gives valuable insight into the global tectonics. Available information suggests that the layered media, crystals and some other materials such as fiber-reinforced materials, fluid saturated porous materials etc. exhibits anisotropy. Belfied et al. [1] gave the idea of introducing a continuous self-reinforcement at every point of an elastic solid. Different problems concerning the surface waves in a fibre-reinforced anisotropic elastic media have been discussed in the literature [2–7].

The inclusion of the temperature change yields what so called the classical theory of thermoelasticity (Nowacki [8, 9]). The next step is to present the theory of coupled thermoelasticity. This was done by Biot [10] to overcome the first shortcoming of the classical theory. The third step is to modify the coupled thermoelasticity theory and to introduce a generalized thermoelasticity theory with one thermal relaxation (Lord and Shulman [11]). An extension is made by Green and Lindsay [12] to introduce two thermal relaxations for the generalized thermoelasticity theory. The fourth step is made by Green and Naghdi [13] to formulate the generalized theory of thermoelasticity without energy dissipation. The important step is made by Tzou [14–16] when he proposed the dual-phase-lag (DPL) model. This model includes two phase-lags, one of them is the heat flux τ_q and the other is the temperature gradient τ_θ. Many investigators have applied the DPL heat transfer model for different structures [17–20].

The wave propagation in solids subjected to initial stresses has been investigated by many authors for various models [4, 21]. In this article, the dual-phase-lag (DPL) generalized thermoelasticity theory is applied to study the 2-D problem of a fiber-reinforced thick plate subjected to initial stress. The problem is solved numerically using a normal mode analysis method. Numerical results for the temperature, displacements, and stresses distributions are illustrated graphically. The results obtained for field quantities may be used as benchmarks for future comparisons. They offer a significant theoretical basis and suggestions for the design of various fiber-reinforced thermoelastic elements under load to meet special engineering needs.

2 Basic equations

The linear governing equations of homogeneous, transversely isotropic, fiber-reinforced solid are presented here. The solid subjected to hydrostatic initial stress and treated without the inclusion of incremental body forces and heat sources. The basic equations in the context of generalized thermoelasticity with dual-phase-lags take the following form.

The equations of motion are given by

$$\sigma_{ij,j}+\left(u_{i,k}\sigma^0_{kj}\right)_{,j}=\rho\frac{\partial^2 u_i}{\partial t^2}, \tag{2.1}$$

where σ_{ij} are the stresses, σ^0_{kj} denotes the initial stress tensor, ρ is the density, u_i are the displacements and $i,j,k=1,2,3$. The comma followed by an index denotes space-coordinate differentiation with respect to this index and the repeated indices in the subscript implies summation.

The heat conduction equation corresponding to DPL model proposed by Tzou takes the form [14–16]

$$\Big(1+\tau_\theta\frac{\partial}{\partial t}\Big)(K_{ij}T_{,i})_{,i}=\Big(\delta+\tau_q\frac{\partial}{\partial t}\Big)\Big(\rho C_E\frac{\partial T}{\partial t}+T_0\frac{\partial}{\partial t}(\beta_{ij}u_{i,j})\Big), \tag{2.2}$$

where $K_{ij}=K_i\delta_{ij}$ is the thermal conductivity, C_E denotes the specific heat at constant strain, T_0 is the reference temperature, assumed to be such as $|(T-T_0)/T|=1$, $\beta_{ij}=\beta_i\delta_{ij}$ denotes the thermoelastic coupling tensor, τ_q is the PL of the heat flux, τ_θ is the PL of gradient of temperature where $0\le\tau_\theta<\tau_q$ and δ_{ij} is Kronecker's delta.

The constitutive equations are given by

$$\begin{aligned}\sigma_{ij}=&\lambda e_{kk}\delta_{ij}+2\mu_T e_{ij}+\alpha(b_kb_me_{km}\delta_{ij}+b_ib_je_{kk})+2(\mu_L-\mu_T)(b_kb_ie_{kj}+b_kb_je_{ki})\\&+\beta b_kb_me_{km}b_ib_j-\beta_{ij}(T-T_0),\end{aligned} \tag{2.3}$$

where e_{ij} are the strains, λ and μ_T are elastic constants, and α, β, $\mu_L-\mu_T$ are reinforcement parameters. The above relations are presented for a fibre-reinforced linearly elastic anisotropic medium with respect to the reinforcement direction $\mathbf{b}\equiv(b_1,b_2,b_3)$, with $b_1^2+b_2^2+b_3^2=1$.

Strain-displacement relations are given by

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}). \tag{2.4}$$

3 Formulation of the problem

Let us consider an infinite thick plate with traction free surfaces at $x=\pm L$ (layer of thickness $2L$), which consists of homogeneous, transversely isotropic thermoelastic material. The origin of the coordinate system (x,y,z) is located on the middle surface of the layer. The $y-z$ plane is considered as the middle surface and x axis is normal to it along the thickness. Then, the components of the displacement vector and temperature are independent of z and can be given by

$$u=u(x,y,t),\quad v=v(x,y,t),\quad w=0,\quad T=T(x,y,t). \tag{3.1}$$

The constitutive relations in the present case are reduced to

$$\sigma_{xx}=\lambda_1\frac{\partial u}{\partial x}+(\lambda+\alpha)\frac{\partial v}{\partial y}-\beta_1(T-T_0),\quad \sigma_{xy}=\mu_L\Big(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\Big), \tag{3.2a}$$

$$\sigma_{yy}=(\lambda+2\mu_T)\frac{\partial v}{\partial y}+(\lambda+\alpha)\frac{\partial u}{\partial x}-\beta_2(T-T_0), \tag{3.2b}$$

where **b** is chosen so that its components are $(1,0,0)$ and

$$\lambda_1=\lambda+2\alpha+4\mu_L-2\mu_T+\beta, \tag{3.3a}$$
$$\beta_1=(\lambda+\alpha+\lambda_1)\alpha_1+(\lambda+\alpha)\alpha_2, \tag{3.3b}$$
$$\beta_2=(2\lambda+\alpha)\alpha_1+(\lambda+2\mu_T)\alpha_2, \tag{3.3c}$$

in which α_1 and α_2 are the coefficients of linear thermal expansion. The equations of motion can be summarized as follows:

$$(\lambda_1+\sigma_0)\frac{\partial^2 u}{\partial x^2}+(\sigma_0+\mu_L)\frac{\partial^2 u}{\partial y^2}+(\alpha+\lambda+\mu_L)\frac{\partial^2 v}{\partial x\partial y}-\beta_1\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{3.4a}$$
$$(\lambda+2\mu_T+\sigma_0)\frac{\partial^2 v}{\partial y^2}+(\sigma_0+\mu_L)\frac{\partial^2 v}{\partial x^2}+(\alpha+\lambda+\mu_L)\frac{\partial^2 u}{\partial x\partial y}-\beta_2\frac{\partial T}{\partial y}=\rho\frac{\partial^2 v}{\partial t^2}, \tag{3.4b}$$

where σ_0 is the initial pressure. Also, the heat equation can be written as

$$\Big(1+\tau_\theta\frac{\partial}{\partial t}\Big)\Big(K_1\frac{\partial^2 T}{\partial x^2}+K_2\frac{\partial^2 T}{\partial y^2}\Big)=\Big(\delta+\tau_q\frac{\partial}{\partial t}\Big)\Big[\rho C_E\frac{\partial T}{\partial t}+T_0\frac{\partial}{\partial t}\Big(\beta_1\frac{\partial u}{\partial x}+\beta_2\frac{\partial v}{\partial y}\Big)\Big]. \tag{3.5}$$

For simplification, the following dimensionless variables are used

$$\{x',y'\}=c_0\eta\{x,y\},\qquad \{u',v'\}=c_0\eta\{u,v\},\qquad c_0^2=\frac{\lambda_1}{\rho},\qquad \eta=\frac{\rho C_E}{K_1}, \tag{3.6a}$$
$$\theta=\frac{\beta_1(T-T_0)}{\lambda_1},\qquad \{t',\tau'_\theta,\tau'_q\}=c_0^2\eta\{t,\tau_\theta,\tau_q\},\qquad \sigma'_0=\frac{\sigma_0}{\lambda_1},\qquad \sigma'_{ij}=\frac{\sigma_{ij}}{\rho c_0^2}. \tag{3.6b}$$

The governing equations, with the help of the above equations after suppressing the primes, are given by

$$(1+\sigma_0)\frac{\partial^2 u}{\partial x^2}+(\sigma_0+B_4)\frac{\partial^2 u}{\partial y^2}+(B_1+B_4)\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial\theta}{\partial x}=\frac{\partial^2 u}{\partial t^2}, \tag{3.7a}$$
$$(B_2+\sigma_0)\frac{\partial^2 v}{\partial y^2}+(\sigma_0+B_4)\frac{\partial^2 v}{\partial x^2}+(B_1+B_4)\frac{\partial^2 u}{\partial x\partial y}-B_3\frac{\partial\theta}{\partial y}=\frac{\partial^2 v}{\partial t^2}, \tag{3.7b}$$
$$\Big(1+\tau_\theta\frac{\partial}{\partial t}\Big)\Big(\frac{\partial^2\theta}{\partial x^2}+\varepsilon_1\frac{\partial^2\theta}{\partial y^2}\Big)=\Big(\delta+\tau_q\frac{\partial}{\partial t}\Big)\Big[\frac{\partial\theta}{\partial t}+\frac{\partial}{\partial t}\Big(\varepsilon_2\frac{\partial u}{\partial x}+\varepsilon_3\frac{\partial v}{\partial y}\Big)\Big], \tag{3.7c}$$
$$\sigma_{xx}=\frac{\partial u}{\partial x}+B_1\frac{\partial v}{\partial y}-\theta,\quad \sigma_{yy}=B_1\frac{\partial u}{\partial x}+B_2\frac{\partial v}{\partial y}-B_3\theta,\quad \sigma_{xy}=B_4\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big), \tag{3.7d}$$

where

$$B_1=\frac{\alpha+\mu}{\lambda_1},\qquad B_2=\frac{\lambda+2\mu_T}{\lambda_1},\qquad B_3=\frac{\beta_2}{\beta_1},\qquad B_4=\frac{\mu_L}{\lambda_1}, \tag{3.8a}$$
$$\varepsilon_1=\frac{K_2}{K_1},\qquad \varepsilon_2=\frac{\beta_1^2 T_0}{\rho C_E\lambda_1},\qquad \varepsilon_3=\frac{\beta_1\beta_2 T_0}{\rho C_E\lambda_1}. \tag{3.8b}$$

4 Initial and boundary conditions

The initial and boundary conditions for the present problem are given, respectively, by

$$\begin{cases} u(x,y,0)=v(x,y,0)=\theta(x,y,0)=0, \\ \dfrac{\partial u}{\partial t}\Big|_{t=0}=\dfrac{\partial v}{\partial t}\Big|_{t=0}=\dfrac{\partial \theta}{\partial t}\Big|_{t=0}=0, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \sigma_{xx}(L,y,t)=-P, \quad \sigma_{xy}(L,y,t)=0, \quad \dfrac{\partial \theta}{\partial x}\Big|_{x=L}=0, \\ u(-L,y,t)=0, \qquad v(-L,y,t)=0, \quad \theta(-L,y,t)=0. \end{cases} \tag{4.2}$$

5 Normal mode analysis

The solution of the present problem for the field quantities is decomposed in terms of normal modes as (see Cheng and Zhang [22])

$$[u,v,\theta,\sigma_{ij}](x,y,t)=[u^*,v^*,\theta^*,\sigma_{ij}^*](x)e^{\omega t+iay}, \tag{5.1}$$

where ω denotes the frequency parameter, $i=\sqrt{-1}$, a denotes the wave number in the y direction, and $u^*(x)$, $v^*(x)$, $\theta^*(x)$ and $\sigma_{ij}^*(x)$ are the amplitudes of the field quantities.

Using Eq. (5.1), Eqs. (3.7) and (3.8) take the forms

$$\Big(\frac{d^2}{dx^2}-g_1\Big)u^*+g_2\frac{dv^*}{dx}=g_3\frac{d\theta^*}{dx}, \tag{5.2a}$$

$$\Big(\frac{d^2}{dx^2}-g_4\Big)v^*+g_5\frac{du^*}{dx}=g_6\theta^*, \tag{5.2b}$$

$$\Big(\frac{d^2}{dx^2}-g_7\Big)\theta^*=g_8\frac{du^*}{dx}+g_9 v^*, \tag{5.2c}$$

and

$$\sigma_{xx}^*=\frac{du^*}{dx}+iaB_1v^*-\theta^*, \quad \sigma_{xy}^*=B_4\Big(iau^*+\frac{dv^*}{dx}\Big), \quad \sigma_{yy}^*=B_1\frac{du^*}{dx}+iaB_2v^*-B_3\theta^*, \tag{5.3}$$

where

$$g_1=\frac{a^2(\sigma_0+B_4)+\omega^2}{1+\sigma_0}, \qquad g_2=\frac{ia(B_1+B_4)}{1+\sigma_0}, \qquad g_3=\frac{1}{1+\sigma_0}, \tag{5.4a}$$

$$g_4=\frac{a^2(\sigma_0+B_2)+\omega^2}{\sigma_0+B_4}, \qquad g_5=\frac{ia(B_1+B_4)}{\sigma_0+B_4}, \qquad g_6=\frac{iaB_3}{\sigma_0+B_4}, \tag{5.4b}$$

$$g_7=\omega^2\varepsilon_1+\frac{\omega(\delta+\tau_q\omega)}{1+\tau_\theta\omega}, \qquad g_8=\frac{\varepsilon_2\omega(\delta+\tau_q\omega)}{1+\tau_\theta\omega}, \qquad g_9=\frac{ia\varepsilon_3\omega(\delta+\tau_q\omega)}{1+\tau_\theta\omega}. \tag{5.4c}$$

Eliminating $\theta^*(x)$ and $v^*(x)$ in Eqs. (5.2), one obtains

$$(D^6-AD^4+BD^2-C)u^*(x)=0, \tag{5.5}$$

where

$$A=\frac{h_1h_5-g_3h_4h_6-g_2h_3}{h_6-g_2g_8}, \quad B=\frac{h_2h_5-h_1h_6-h_3h_4}{h_5-g_2g_8}, \quad C=\frac{-h_2h_6}{h_5-g_2g_8}, \tag{5.6}$$

in which

$$h_1=g_4+g_7, \qquad h_2=g_4g_7-g_9g_6, \qquad h_3=g_8g_4+g_5g_9, \tag{5.7a}$$

$$h_4=g_9g_3+g_2g_7, \qquad h_5=g_2g_8-g_9, \qquad h_6=g_1g_9. \tag{5.7b}$$

So, one can factorize Eq. (5.5) as

$$(D^2-k_1^2)(D^2-k_2^2)(D^2-k_3^2)u^*(x)=0, \tag{5.8}$$

where k_n^2, $(n=1,2,3)$, denote the roots of the characteristic equation

$$k^6-Ak^4+Bk^2-C=0. \tag{5.9}$$

The solution of Eq. (5.8) is given by

$$u^*(x)=\sum_{n=1}^{3}\Big(M_{1n}(a,\omega)e^{-k_nx}+M_{2n}(a,\omega)e^{k_nx}\Big). \tag{5.10}$$

In a similar manner, one gets

$$\theta^*(x)=\sum_{n=1}^{3}\Big(M'_{1n}(a,\omega)e^{-k_nx}+M'_{2n}(a,\omega)e^{k_nx}\Big), \tag{5.11a}$$

$$v^*(x)=\sum_{n=1}^{3}\Big(M''_{1n}(a,\omega)e^{-k_nx}+M''_{2n}(a,\omega)e^{k_nx}\Big), \tag{5.11b}$$

where M_{in}, M'_{in} and M''_{in} are different parameters. Substituting Eqs. (5.10) and (5.11) into Eqs. (5.2) and (5.3), one gets the following relations:

$$M'_{1n}(a,\omega)=H_{1n}M_{1n}(a,\omega), \qquad M'_{2n}(a,\omega)=H_{2n}M_{2n}(a,\omega), \tag{5.12a}$$

$$M''_{1n}(a,\omega)=H_{3n}M_{1n}(a,\omega), \qquad M''_{2n}(a,\omega)=H_{4n}M_{2n}(a,\omega), \quad n=1,2,3, \tag{5.12b}$$

where

$$H_{1n}=\frac{-g_8k_n^3+h_3k_n}{k_n^4-h_1k_n^2+h_2}, \qquad H_{3n}=\frac{1}{g_9}[(k_n^2-g_7)H_{1n}+g_8k_n], \tag{5.13a}$$

$$H_{2n}=-H_{1n}, \qquad H_{4n}=\frac{1}{g_9}[(k_n^2-g_7)H_{1n}-g_8k_n]. \tag{5.13b}$$

Thus, one obtains

$$\theta^*(x)=\sum_{n=1}^{3}\Big(H_{1n}M_{1n}(a,\omega)e^{-k_nx}+H_{2n}M_{2n}(a,\omega)e^{k_nx}\Big), \tag{5.14a}$$

$$v^*(x)=\sum_{n=1}^{3}\Big(H_{3n}M_{1n}(a,\omega)e^{-k_nx}+H_{4n}M_{2n}(a,\omega)e^{k_nx}\Big). \tag{5.14b}$$

Substituting Eqs. (5.10) and (5.11) into Eqs. (5.3), one obtains

$$\sigma^*_{xx}(x)=\sum_{n=1}^{3}\Big(H_{5n}M_{1n}(a,\omega)e^{-k_nx}+H_{6n}M_{2n}(a,\omega)e^{k_nx}\Big), \tag{5.15a}$$

$$\sigma^*_{yy}(x)=\sum_{n=1}^{3}\Big(H_{7n}M_{1n}(a,\omega)e^{-k_nx}+H_{8n}M_{2n}(a,\omega)e^{k_nx}\Big), \tag{5.15b}$$

$$\sigma^*_{xy}(x)=\sum_{n=1}^{3}\Big(H_{9n}M_{1n}(a,\omega)e^{-k_nx}+H_{10n}M_{2n}(a,\omega)e^{k_nx}\Big), \tag{5.15c}$$

where

$$H_{5n}=-k_n+iaB_1H_{3n}-H_{1n}, \qquad H_{6n}=k_n+iaB_1H_{4n}-H_{2n}, \tag{5.16a}$$

$$H_{7n}=-B_1k_n+iaB_2H_{3n}-B_3H_{1n}, \qquad H_{10n}=iaB_4+B_4k_nH_{4n}, \tag{5.16b}$$

$$H_{9n}=iaB_4-B_4k_nH_{3n}, \qquad H_{8n}=B_1k_n+iaB_2H_{4n}-B_3H_{2n}. \tag{5.16c}$$

The boundary conditions given in Eq. (4.2), with the aid of the field quantities, are summarized by

$$\sigma^*_{xx}\big|_{x=L}=\sum_{n=1}^{3}\Big(H_{5n}M_{1n}(a,\omega)e^{-k_nL}+H_{6n}M_{2n}(a,\omega)e^{k_nL}\Big)=-P^*, \tag{5.17a}$$

$$\sigma^*_{xy}\big|_{x=L}=\sum_{n=1}^{3}\Big(H_{9n}M_{1n}(a,\omega)e^{-k_nL}+H_{10n}M_{2n}(a,\omega)e^{k_nL}\Big)=0, \tag{5.17b}$$

$$\frac{\partial\theta^*}{\partial x}\Big|_{x=L}=\sum_{n=1}^{3}\Big(-k_nH_{1n}M_{1n}(a,\omega)e^{-k_nL}+k_nH_{2n}M_{2n}(a,\omega)e^{k_nL}\Big)=0, \tag{5.17c}$$

$$u^*|_{x=-L}=\sum_{n=1}^{3}\Big(M_{1n}(a,\omega)e^{k_nL}+M_{2n}(a,\omega)e^{-k_nL}\Big)=0, \tag{5.17d}$$

$$\theta^*|_{x=-L}=\sum_{n=1}^{3}\Big(H_{1n}M_{1n}(a,\omega)e^{k_nL}+H_{2n}M_{2n}(a,\omega)e^{-k_nL}\Big)=0, \tag{5.17e}$$

$$v^*|_{x=-L}=\sum_{n=1}^{3}\Big(H_{3n}M_{1n}(a,\omega)e^{k_nL}+H_{3n}M_{2n}(a,\omega)e^{-k_nL}\Big)=0. \tag{5.17f}$$

The above system of equations can be written in the matrix form. That is

$$[E]\{M\}=\{F\}, \tag{5.18}$$

where $\{M\}=\{M_{11},M_{12},M_{13},M_{21},M_{22},M_{23}\}^p$ and $\{F\}=\{-P^*,0,0,0,0,0\}^p$ in which the supper index $''p''$ denotes the transpose of its vector. The elements E_{ij} of the coefficient matrix $[E]$ are given in Appendix. After applying the inverse of matrix method, one can get the values of the six constants M_{ij}, $(i=1,2,\ j=1,2,3)$. Hence, on can obtained the expressions for the temperature, the displacements, and stresses in the plate muscles.

6 Particular cases

6.1 Generalized thermoelastic isotropic medium with hydrostatic initial stress

Substituting $\mu_L=\mu_T=\mu$, $K_1=K_2=K$, $\beta_1=\beta_2$ and $\alpha=\beta=0$, one obtains the corresponding expressions of displacements, stresses, and temperature in this case.

6.2 Generalized thermoelastic fiber-reinforced medium

Letting $\sigma_0\to 0$, the present medium reduces to the case of a fiber-reinforced generalized thermoelastic medium.

6.3 Generalized thermoelastic isotropic medium

Substituting $\mu_L=\mu_T=\mu$, $K_1=K_2=K$, $\beta_1=\beta_2$ and $\alpha=\beta=0$, and letting $\sigma_0\to 0$, the present medium reduces to an isotropic generalized thermoelastic medium.

7 Special cases of thermoelasticity theory

7.1 Coupled thermoelasticity (CTE) theory

The equations of the coupled thermoelasticity theory are obtained when $\tau_\theta=\tau_q=0$ and $\delta=1$.

7.2 Lord-Shulman (LS) thermoelasticity theory

The Lord-Shulman theory is given by setting $\tau_\theta=0$, $\delta=1$ and $\tau_q=\tau_0>0$, where τ_0 is the first relaxation time.

7.3 Green-Naghdi (GN) theory

The equations of the Green-Naghdi generalized thermoelasticity theory without energy dissipation are obtained when $\tau_\theta=0$, $\delta=0$ and $\tau_q=1$.

7.4 Equations of the dual-phase-lag (DPL) model

For dual-phase-lag generalized thermoelasticity theory, one putts $\delta=1$ and $\tau_q \geq \tau_\theta > 0$.

8 Numerical results

The numerical results depict the variations of normal displacement, normal force stress and temperature distributions in the context of thermoelasticity theory with phase-lags. To study the effect of reinforcement on wave propagation, we use the following physical constants for generalized fibre-reinforced thermoelastic materials:

$$
\begin{aligned}
&\lambda = 5.65\times 10^{10}\mathrm{N/m^2}, && \mu_T = 2.46\times 10^{10}\mathrm{N/m^2}, && \mu_L = 5.66\times 10^{10}\mathrm{N/m^2},\\
&\alpha = -1.28\times 10^{10}\mathrm{N/m^2}, && \beta = 220.9\times 10^{10}\mathrm{N/m^2}, && C_E = 0.787\times 10^{3}\mathrm{J/(kgK)},\\
&\rho = 2660\mathrm{kg/m^3}, && \alpha_1 = 0.017\times 10^{-4}(1/\mathrm{K}), && \alpha_2 = 0.015\times 10^{-4}(1/\mathrm{K}),\\
&K_1 = 0.0921\times 10^{3}\mathrm{J/(msK)}, && K_2 = 0.0963\times 10^{3}\mathrm{J/(msK)}, && T_0 = 293\mathrm{K}.
\end{aligned}
$$

The results depict the variations of the real part of the thermal temperature θ, the displacements u and v, the stresses σ_{xx}, σ_{yy} and σ_{xy}. These quantities depend not only on space x and time t, but also on phase-lags τ_θ and τ_q. It is assumed that $\tau_0=0.02$, $a=1$, $L=1$, $P=0.5$, and $\omega=\omega_0+i\xi$ in which $\omega_0=2$ and $\xi=1$.

Figs. 1-6 compared the results obtained for temperature, displacements and stresses against the x direction for different values of τ_q and τ_θ at $y=1$. The computations are performed for one value of time, namely $t=0.3$ and various values of the parameters τ_q and τ_θ. The graphs represent six curves predicted by LS and GN theories of thermoelasticity obtained as special cases of the general DPL model. The results of LS theory ($\tau_\theta=0$, $\tau_q=0.05$, $\delta=1$), the GN theory ($\tau_\theta=0$, $\tau_q=0.05$, $\delta=0$) and the generalized theory of thermoelasticity proposed by Tzou ($\tau_q=0.05 \geq \tau_\theta=0.02>0$) are all presented in

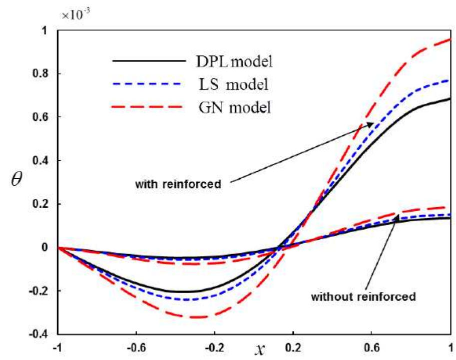

Figure 1: The temperature θ distribution for various models with and without fiber-reinforcement.

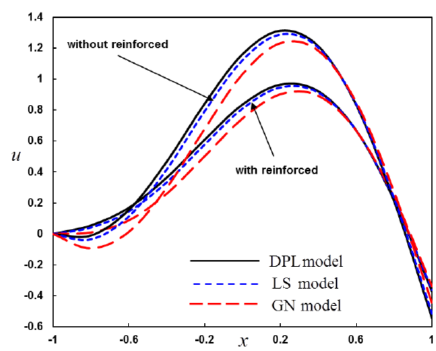

Figure 2: The displacement u distribution for various models with and without fiber-reinforcement.

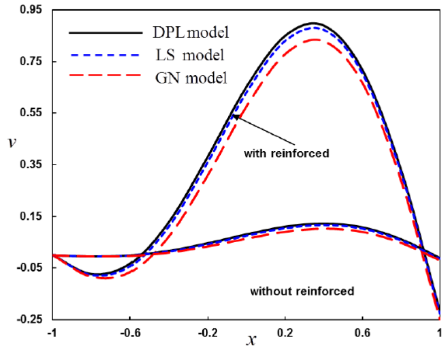

Figure 3: The displacement v distribution for various models with and without fiber-reinforcement.

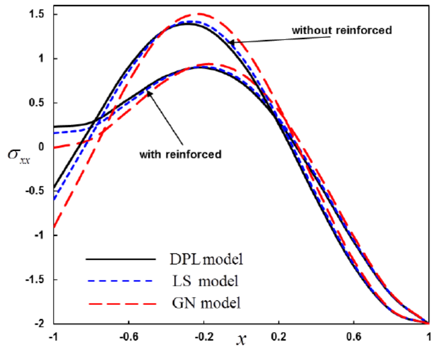

Figure 4: The stress σ_{xx} distribution for various models with and without fiber-reinforcement.

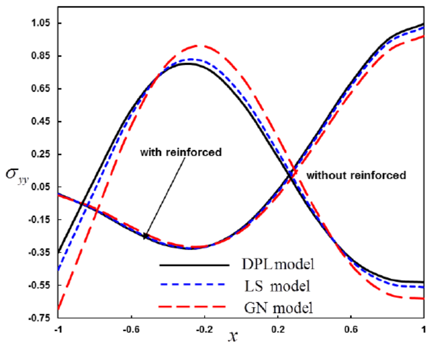

Figure 5: The stress σ_{yy} distribution for various models with and without fiber-reinforcement.

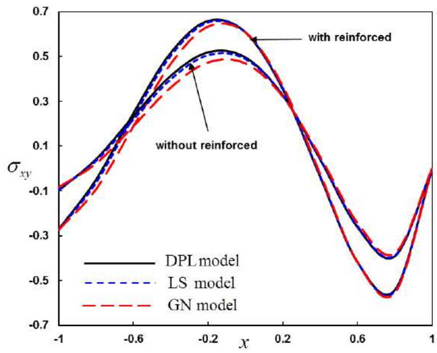

Figure 6: The stress σ_{xy} distribution for various models with and without fiber-reinforcement.

Figs. 1-6. In addition, the variation of the physical quantities under two types of the inclusion of reinforcement and without ($\alpha=0$, $\beta=0$ and $\mu_L-\mu_T=0$) the reinforcement is also illustrated.

Figs. 1-3 show that the temperature and displacements are zero at the rigid base $x=-1$, which confirms the assumed boundary conditions. The upper surface of the plate, $x=1$, is assumed to be thermally insulated and the displacements are minimum which supports the physical fact. The values of u and v due to GN model are smaller than those for other theories in most positions through-the-thickness of the plate. Fig. 2 shows the variations of u initially start with sharp increase in the range $-1\leq x\leq 0.2$ and then follows pattern with reference to x. Also the value of v increase slowly in the range $-1\leq x\leq 0.4$, decrease sharply in the range $0.4\leq x\leq 1$and as x.

Figs. 4 and 6 show that the stress components σ_{xx} and σ_{xy} satisfy the boundary condition at $x=1$. These trends obey elastic and thermoelastic properties of the solid under investigation. Fig. 5 shows that at $x=1$, the stress σ_{yy} is maximum with the reinforced thermoelastic material and minimum without it. The distribution in LS theory is closed to that in DPL theory, whereas the distribution in GN theory is a little different. The

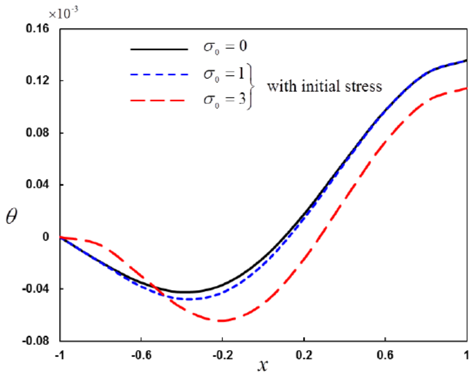

Figure 7: The temperature θ distribution with and without the initial stress.

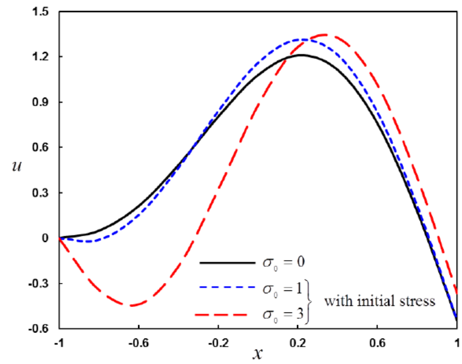

Figure 8: The displacement u distribution with and without the initial stress.

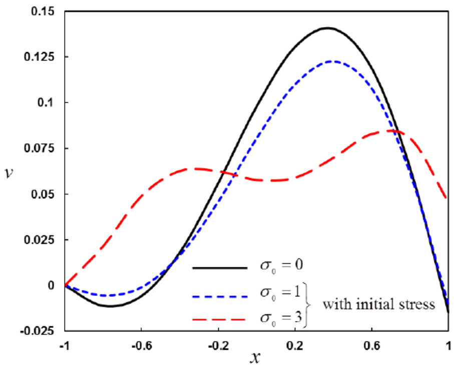

Figure 9: The displacement v distribution with and without the initial stress.

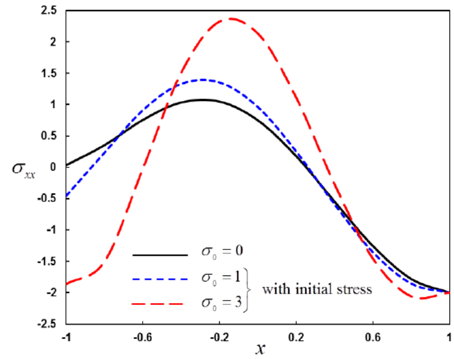

Figure 10: The stress σ_{xx} distribution with and without the initial stress.

temperature distribution decays along the direction of the transmitted wave propagation for the effects of diffusion. The values of τ_q and τ_θ can judge whether the wavelike behavior in the DPL heat conduction is dominant or not. However, it can found from the numerical results that the shift times τ_q and τ_θ may play a more important role in this task. Also, it is clear that the surface waves in the fibre-reinforced medium are affected by the reinforced parameters.

Figs. 7-12 show the sensitivity of the inclusion of the initial stress σ_0. The figures exhibit the variation of the temperature θ, displacement components u, v, and the stress components σ_{xx}, σ_{yy} and σ_{xy} with distance x under the DPL theory at $y=1$ for three different values of initial stress ($\sigma_0 = 0,1,3$). The field quantities depend not only on the state and space variables t, x, and y, but also on the variation of initial stress σ_0. It has been observed that the initial stress σ_0 plays a vital role on the development of temperature, displacement, and stress fields. It is clear from these figures that the surface waves in the fibre-reinforced medium are very sensitive to the inclusion of the initial stress σ_0.

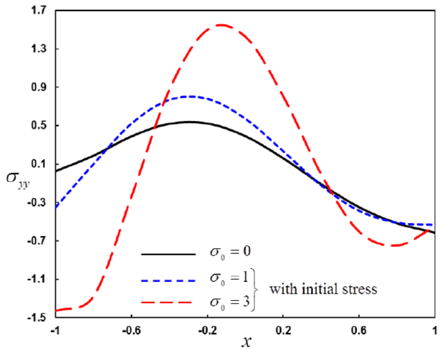

Figure 11: The stress σ_{yy} distribution with and without the initial stress.

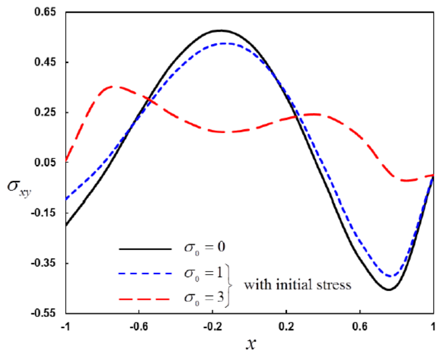

Figure 12: The stress σ_{xy} distribution with and without the initial stress.

9 Conclusions

Analytical solutions have been developed and utilized. The stress distributions and the temperature are evaluated as functions of the distance based on the normal mode analysis for the generalized thermoelastic problem in solids. The effects of reinforcement, hydrostatic initial stress and phase-lags are studied on all quantities. The computations have revealed that:

- The presence of phase-lags parameters plays a significant role in all the physical quantities. In addition, the presence of initial stress in the current model is highly significant.
- The fibre-reinforcement is significant to the behavior of the distributions of the field quantities.
- The method used in the present article is applicable to a wide range of problems in thermodynamics and thermoelasticity.
- It is observed that all theories of coupled and generalized thermoelasticity can be obtained as limited cases from the present one.
- The variations of all quantities show appreciable effect with and without dependence of initial stress.
- According to the numerical results and graphs, a conclusion about the new theory of thermoelasticity has been constructed. The result provides a motivation to investigate conducting materials as a new class of applicable materials.
- The DPL model is near to LS model. This gives good agreement with the conclusion of Hetnarski and Ignaczak [23] that the DPL model is an extension to the LS one.
- Results obtained in this paper may be considered as more general in the sense that they include the combined effect of fibre-reinforcing and initial stress field.

Appendix

The elements E_{ij} of the coefficient matrix $[E]$ given in Eq. (5.18) are

$$
\begin{aligned}
&E_{11}=H_{51}e^{-k_1L}, && E_{12}=H_{52}e^{-k_2L}, && E_{13}=H_{53}e^{-k_3L}, && E_{14}=H_{61}e^{k_1L},\\
&E_{15}=H_{62}e^{k_2L}, && E_{16}=H_{63}e^{k_3L}, && E_{21}=H_{91}e^{-k_1L}, && E_{22}=H_{92}e^{-k_2L},\\
&E_{23}=H_{93}e^{-k_3L}, && E_{24}=H_{101}e^{k_1L}, && E_{25}=H_{102}e^{k_2L}, && E_{26}=H_{103}e^{k_3L},\\
&E_{31}=-k_1H_{11}e^{-k_1L}, && E_{32}=-k_2H_{12}e^{-k_2L}, && E_{33}=-k_3H_{13}e^{-k_3L}, && E_{34}=k_1H_{21}e^{k_1L},\\
&E_{35}=k_2H_{22}e^{k_2L}, && E_{36}=k_3H_{23}e^{k_3L}, && E_{41}=e^{k_1L}, && E_{42}=e^{k_2L},\\
&E_{43}=e^{k_3L}, && E_{44}=e^{-k_1L}, && E_{45}=e^{-k_2L}, && E_{46}=e^{-k_3L},\\
&E_{51}=H_{11}e^{k_1L}, && E_{52}=H_{12}e^{k_2L}, && E_{53}=H_{13}e^{k_3L}, && E_{54}=H_{21}e^{-k_1L},\\
&E_{55}=H_{22}e^{-k_2L}, && E_{56}=H_{23}e^{-k_3L}, && E_{61}=H_{31}e^{k_1L}, && E_{62}=H_{32}e^{k_2L},\\
&E_{63}=H_{33}e^{k_3L}, && E_{64}=H_{41}e^{-k_1L}, && E_{65}=H_{42}e^{-k_2L}, && E_{66}=H_{43}e^{-k_3L}.
\end{aligned}
$$

References

[1] A. J. BELFIELD, T. G. ROGERS AND A. J. M. SPENCER, *Stress in elastic plates reinforced by fibres lying in concentric circles*, J. Mech. Phys. Solids, 31 (1983), pp. 25–54.

[2] P. D. S. VERMA AND O. H. RANA, *Rotation of a circular cylindrical tube reinforced by fibres lying along helices*, Mech. Mater., 2 (1983), pp. 353–359.

[3] P. R. SENGUPTA AND S. NATH, *Surface waves in fibre-reinforced anisotropic elastic media*, Sadhana, 26 (2001), pp. 363–370.

[4] A. M. ZENKOUR, *Thermal effects on the bending response of fiber-reinforced viscoelastic composite plates using a sinusoidal shear deformation theory*, Acta. Mech., 171 (2004), pp. 171–187.

[5] I. A. ABBAS AND A. N. ABD-ALLA, *Effect of initial stress on a fiber-reinforced anisotropic thermoelastic thick plate*, Int. J. Thermophys., 32 (2011), pp. 1098–1110.

[6] A. E. ABOUELREGAL AND A. M. ZENKOUR, *Effect of fractional thermoelasticity on a two-dimensional problem of a mode I crack in a rotating fibre-reinforced thermoelastic medium*, Chinese Phys. B, 22 (2013), 108102.

[7] I. A. ABBAS AND A. M. ZENKOUR, *The effect of rotation and initial stress on thermal shock problem for a fiber-reinforced anisotropic half-space using Green-Naghdi theory*, J. Comput. Theor. Nanosci., 11 (2014), pp. 331–338.

[8] W. NOWACKI, Dynamic Problems of Thermoelasticity, Noordhoff, Leyden, The Netherlands, 1975.

[9] W. NOWACKI, Thermoelasticity, 2nd edition, Pergamon Press, Oxford, 1986.

[10] M. A. BIOT, *Thermoelasticity and irreversible thermodynamics*, J. Appl. Phys., 27 (1956), pp. 240–253.

[11] H. W. LORD AND Y. SHULMAN, *A generalized dynamical theory of thermoelasticity*, J. Mech. Phys. Solids, 15 (1967), pp. 299–309.

[12] A. E. GREEN AND K. A. LINDSAY, *Thermoelasticity*, J. Elast., 2 (1972), pp. 1–7.

[13] A. E. GREEN AND P. M. NAGHDI, *Thermoelasticity without energy dissipation*, J. Elast., 31 (1993), pp. 189–209.

[14] D. Y. TZOU, Macro-to-Microscale Heat Transfer: the Lagging Behavior, Washington, DC, Taylor & Francis, 1996.

[15] D. Y. TZOU, *A unified approach for heat conduction from macro-to-micro scales*, J. Heat Trans., 117 (1995), pp. 8–16.

[16] D. Y. TZOU, *Experimental support for the lagging behavior in heat propagation*, J. Thermophys. Heat Trans., 9 (1995), pp. 686–693.

[17] R. QUINTANILLA AND P. M. JORDAN, *A note on the two temperature theory with dual-phase-lag delay: some exact solutions*, Mech. Res. Commun., 36 (2009), pp. 796–803.

[18] A. E. ABOUELREGAL, *Generalized thermoelasticity for an isotropic solid sphere in dual-phase-lag of heat transfer with surface heat flux*, Int. J. Comput. Meth. Eng. Sci. Mech., 12 (2011), pp. 96–105.

[19] A. M. ZENKOUR, D. S. MASHAT AND A. E. ABOUELREGAL, *The effect of dual-phase-lag model on reflection of thermoelastic waves in a solid half space with variable material properties,* Acta. Mech. Solida Sinica, 26 (2013), pp. 659–670.

[20] I. A. ABBAS AND A. M. ZENKOUR, *Dual-phase-lag model on thermoelastic interactions in a semi-infinite medium subjected to a ramp-type heating*, J. Comput. Theoretical Nanosci., 11 (2014), pp. 642–645.

[21] B. SINGH, *Effect of hydrostatic initial stresses on waves in a thermoelastic solid half-space*, Appl. Math. Comput., 198 (2008), pp. 494–505.

[22] J. C. CHENG AND S. Y. ZHANG, *Normal mode expansion method for laser generated ultrasonic Lamb waves in orthotropic thin plates*, Appl. Phys. B, 70 (2000), pp. 57–63.

[23] R. B. HETNARSKI AND J. IGNACZAK, *Generalized thermoelasticity*, J. Thermal Stresses, 22 (1999), pp. 451–476.