Analysis in Theory and Applications
Anal. Theory Appl., Vol. **32**, No. 4 (2016), pp. 333-338
DOI: 10.4208/ata.2016.v32.n4.2

A Perturbation of Jensen $*$-Derivations from $K(H)$ into $K(H)$

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Received 6 January 2014; Accepted (in revised version) 28 July 2016

Abstract. Let's take H as an infinite–dimensional Hilbert space and $K(H)$ be the set of all compact operators on H. Using Spectral theorem for compact self–adjoint operators, we prove the Hyers–Ulam stability of Jensen $*$-derivations from $K(H)$ into $K(H)$.

Key Words: Jensen $*$-derivation, C^*-algebra, Hyers–Ulam stability.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

1 Introduction

In a Hilbert space H, an operator T in $B(H)$ is called a compact operator if the image of unit ball of H under T is a compact subset of H. Note that if the operator $T:H\longrightarrow H$ is compact, then the adjoint of T is compact, too. The set of all compact operators on H is shown by $K(H)$. It is easy to see that $K(H)$ is a C^*-algebra [1]. Moreover, every operator on H with finite range is compact. The set of all finite range projections on Hilbert space H is denoted by $P(H)$.

An approximate unit for a C^*-algebra $\mathcal{A}$ is an increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of $\mathcal{A}$ such that $a=\lim_\lambda au_\lambda=\lim_\lambda u_\lambda a$ for all $a\in\mathcal{A}$. Every C^*-algebra admits an approximate unit [2].

Example 1.1. Let H be a Hilbert space with orthonormal basis $(e_n)_{n=1}^\infty$. The C^*-algebra $K(H)$ is non–unital since $dim(H)=\infty$. If P_n is a projection on $\mathbb{C}e_1+\cdots+\mathbb{C}e_n$, then the increasing sequence $(P_n)_{n=1}^\infty$ is an approximate unit for $K(H)$.

Theorem 1.1 (see [2]). *Let $T:H\longrightarrow H$ be a compact self–adjoint operator on Hilbert space H. Then there is an orthonormal basis of H consisting of eigenvectors of T. The nonzero eigenvalues of T are from finite or countably infinite set $\{\lambda_k\}_{k=1}^\infty$ of real numbers and $T=\sum_{k=1}^\infty\lambda_kP_k$, where P_k is the orthogonal projection on the finite–dimensional space of eigenvectors corresponding to eigenvalues. If the number of nonzero eigenvalues is countably infinite, then the series converges to T in the operator norm.*

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The problem of stability of functional equations originated from a question of Ulam [5] concerning the stability of group homomorphisms: let $(G1,*)$ be a group and let $(G2,\star,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta(\varepsilon)>0$ such that if a mapping $h:G_1\longrightarrow G_2$ satisfies the inequality

$$d(h(x*y),h(x)\star h(y))<\delta$$

for all $x,y\in G_1$, then there exists a homomorphism $H:G_1\to G_2$ with

$$d(h(x),H(x))<\varepsilon$$

for all $x\in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x*y)=H(x)\star H(y)$ is stable. Thus, the stability question of functional equations is that how the solutions of the inequality differ from those of the given functional equation.

Hyers [3] gave the first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f:X\longrightarrow Y$ satisfies

$$\|f(x+y)-f(x)-f(y)\|\le\varepsilon$$

for all $x,y\in X$ and some $\varepsilon>0$. Then, there exists a unique additive mapping $T:X\longrightarrow Y$ such that

$$\|f(x)-T(x)\|\le\varepsilon$$

for all $x\in X$. Also, if the function $t\longmapsto f(tx)$ from $\mathbb{R}$ to Y is continuous for each fixed $x\in X$, then T is an $\mathbb{R}$-linear function. This method is called the direct method or Hyers–Ulam stability of functional equations.

Note that if f is continuous, then the function $r\mapsto f(rx)$ from $\mathbb{R}$ into Y is continuous for all $x\in X$. Therefore T is $\mathbb{R}$-linear.

Definition 1.1. Let X and Y be real linear spaces. For $n\in\{2,3,4,\cdots\}$ the mapping $f:X\longrightarrow Y$ is called a Jensen mapping of n-variable, if f for each $x_1,\cdots,x_n\in X$ satisfies the following equation

$$f\left(\frac{x_1+\cdots+x_n}{n}\right)=\frac{1}{n}\big(f(x_1)+\cdots+f(x_n)\big).$$

In 2003, J. M. Rassias and M. J. Rassias [4] investigated the Ulam stability of Jensen and Jensen type mappings by applying the Hyers method. In 2012, M. Eshaghi Gordji and S. Abbaszadeh [6] investigated the Hyers–Ulam stability of Jensen type and generalized n-variable Jensen type functional equations in fuzzy Banach spaces.

Definition 1.2. Let $\mathcal{A}$ be a C^*-algebra. A mapping $d:\mathcal{A}\longrightarrow\mathcal{A}$ with $d(a^*)=d(a)^*$ for all $a\in\mathcal{A}$ ($*$-preserving property) is called a Jensen $*$-derivation if d satisfies

$$d(x_1x_2)=x_1d(x_2)+d(x_1)x_2$$

and

$$d\left(\frac{\lambda x_1+\cdots+\lambda x_n}{n}\right)=\frac{\lambda}{n}\big(d(x_1)+\cdots+d(x_n)\big)$$

for all $\lambda\in\mathbb{C}$, $x_1,\cdots,x_n\in\mathcal{A}$ and $n\in\{2,3,4,\cdots\}$.

Definition 1.3. Let $\mathcal{A}$ be a C^*-algebra. A mapping $d:\mathcal{A}\longrightarrow\mathcal{A}$ with $d(a^*)=d(a)^*$ for all $a\in\mathcal{A}$ ($*$-preserving property) and $d(0)=0$ is called a Jensen Jordan $*$-derivation if d satisfies

$$d(x_1x_2+x_2x_1)=x_1d(x_2)+d(x_1)x_2+x_2d(x_1)+d(x_2)x_1$$

and

$$d\left(\frac{\lambda x_1+\cdots+\lambda x_n}{n}\right)=\frac{\lambda}{n}\big(d(x_1)+\cdots+d(x_n)\big)$$

for all $\lambda\in\mathbb{C}$, $x_1,\cdots,x_n\in\mathcal{A}$ and $n\in\{2,3,4,\cdots\}$.

B. E. Johnson [7] investigated almost algebra $*$-homomorphism between Banach $*$-algebras. Recently, M. Eshaghi Gordji et al. have investigated several stability results on homomorphisms and Jordan homomorphisms on C^*-algebras (see [8]).

In the present paper, using spectral theorem for compact self–adjoint operators, we prove that every almost-Jensen $*$-preserving map $\varphi:K(H)\longrightarrow K(H)$ satisfying $\varphi(T2^nP)=T2^n\varphi(P)+\varphi(T)2^nP$ for all $P\in P(H)$, can be Jensen $*$-derivation. Also, we show that every almost-Jensen $*$-preserving map $\varphi:K(H)\longrightarrow K(H)$ satisfying

$$\varphi(T2^nP+2^nPT)=T2^n\varphi(P)+\varphi(T)2^nP+2^nP\varphi(T)+2^n\varphi(P)T$$

for all $P\in P(H)$, can be Jensen Jordan $*$-derivation.

2 Jensen $*$-derivations and Jensen Jordan $*$-derivations

From now on, we suppose that H is an infinite dimensional Hilbert space, $K(H)$ is the set of all compact operators and $P(H)$ is the set of all finite range projections on H.

It is easy to see that if a Jensen mapping φ satisfies the condition $\varphi(0)=0$, then φ is additive. We use this fact in the main results of this paper.

Lemma 2.1. *Assume that X and Y be linear spaces. If a mapping $f:X\longrightarrow Y$ is additive and for each fixed $x\in X$, $f(\lambda x)=\lambda f(x)$ for all $\lambda\in\mathbb{T}^1_{\theta_0}:=\{e^{i\theta}:0\le\theta<\theta_0\le 2\pi\}$. Then f is $\mathbb{C}$-linear.*

Proof. If λ belongs to $\mathbb{T}^1$, then there exists $\theta\in[0,2\pi]$ such that $\lambda=e^{i\theta}$. It follows from $\frac{\theta}{n}\to 0$ as $n\to\infty$ that there exists $n_0\in\mathbb{N}$ such that $\lambda_1=e^{i\frac{\theta}{n_0}}$ belongs to $\mathbb{T}^1_{\theta_0}$ and $f(\lambda x)=f(\lambda_1^{n_0}x)=\lambda_1^{n_0}f(x)=\lambda f(x)$ for all $x\in X$. Let $t\in(0,1)$. Putting $t_1=t+i(1-t^2)^{\frac{1}{2}}$, $t_2=t-i(1-t^2)^{\frac{1}{2}}$, then we have $t=\frac{t_1+t_2}{2}$ and $t_1,t_2\in\mathbb{T}^1$. It follows that

$$f(tx)=f(\frac{t_1+t_2}{2}x)=\frac{t_1}{2}f(x)+\frac{t_2}{2}f(x)=tf(x).$$

If $\lambda\in B_1:=\{\lambda\in\mathbb{C};|\lambda|\le 1\}$, then there exists $\theta\in[0,2\pi]$ such that $\lambda=|\lambda|e^{i\theta}$. It follows that

$$f(\lambda x)=f(|\lambda|e^{i\theta}x)=|\lambda|f(e^{i\theta}x)=\lambda f(x)$$

for all $x\in X$. If $\lambda\in\mathbb{C}$ then there exist $n_0\in\mathbb{N}$ (from $\frac{\lambda}{n}\to 0$ as $\to\infty$) such that $\lambda_0=\frac{\lambda}{n_0}\in B_1$ and for all $x\in X$

$$f(\lambda x)=f(n_0\lambda_0 x)=n_0\lambda_0 f(x)=\lambda f(x).$$

Thus, we complete the proof. □

Theorem 2.1. *Let H be an infinite dimensional Hilbert space and $\varepsilon>0$ is given; if a continuous mapping $\varphi:K(H)\longrightarrow K(H)$ with $\varphi(0)=0$ satisfies the following conditions:*
(1) $\varphi(TP)=T\varphi(P)+\varphi(T)P$ for all $T\in K(H)$ and $P\in P(H)$;
(2) $\left\|\varphi\left(\frac{\lambda T_1+\cdots+\lambda T_n}{n}\right)-\frac{\lambda}{n}\left(\varphi(T_1)+\cdots+\varphi(T_n)\right)\right\|<\varepsilon$ for all $\lambda\in\mathbb{T}^1_{\theta_0}$ with $0<\theta_0\le 2\pi$;
(3) $\left\|\varphi(T^*)-\varphi(T)^*\right\|<\varepsilon$ for all $T\in K(H)$.
Then there exists a unique Jensen $*$-derivation $D:K(H)\longrightarrow K(H)$ such that

$$\|D(T)-\varphi(T)\|<\varepsilon$$

for all $T\in K(H)$.

Proof. From condition (2), there exists a unique Jensen mapping $D:K(H)\longrightarrow K(H)$ with $D(T)=\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nT)$ such that

$$\|D(T)-\varphi(T)\|<\varepsilon$$

for all $T\in K(H)$ (see [4,6]). Note that $D(0)=0$, thus D is additive. Now, it follows from condition (2) and Lemma 2.1 that D is $\mathbb{C}$-linear.

It follows from condition (1) that

$$D(TP)=\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nTP)=\lim_{n\to\infty}\frac{1}{2^n}\left[T\varphi(2^nP)+\varphi(T)2^nP\right]=TD(P)+\varphi(T)P \tag{2.1}$$

for all $T\in K(H)$ and $P\in P(H)$. So, since D is linear we get

$$D(TP)=\frac{D\big((2^nT)P\big)}{2^n}=TD(P)+\frac{1}{2^n}\varphi(2^nT)P$$

for all $T\in K(H)$ and $P\in P(H)$. By tending n to infinity in the last equality above, we obtain

$$D(TP)=TD(P)+D(T)P \tag{2.2}$$

for all $T\in K(H)$ and $P\in P(H)$. By (2.1) and (2.2), we have $\varphi(T)P=D(T)P$ for all $T\in K(H)$ and $P\in P(H)$.

Now, we show that $D\equiv\varphi$. Let $\{P_m\}\subset P(H)$ be an approximate unit of $K(H)$, then we get

$$D(T)=\lim_m D(T)P_m=\lim_m \varphi(T)P_m=\varphi(T)$$

for all $T\in K(H)$.

Given $S,T\in K(H)$, there are compact self adjoint operators S_1 and S_2 such that $S=S_1+iS_2$. According to Theorem 1.1 we have

$$S=S_1+iS_2=\sum_{i=1}^{\infty}\alpha_iP_i+i\sum_{j=1}^{\infty}\beta_jP_j,$$

where $P_k\in P(H)$ and $\alpha_k,\beta_k\in\mathbb{C}$ for all $k\in\{1,2,3,\cdots\}$. It follows from linearity and continuity of D and T that

$$\begin{aligned}
D(TS)=&D\Big(T\Big\{\sum_{i=1}^{\infty}\alpha_iP_i+i\sum_{j=1}^{\infty}\beta_jP_j\Big\}\Big)\\
=&\sum_{i=1}^{\infty}D\big(T\alpha_iP_i\big)+i\sum_{j=1}^{\infty}D\big(T\beta_jP_j\big)\\
=&\sum_{i=1}^{\infty}\Big[TD\big(\alpha_iP_i\big)+D(T)\alpha_iP_i\Big]+i\sum_{j=1}^{\infty}\Big[TD\big(\beta_jP_j\big)+D(T)\beta_jP_j\Big]\\
=&T\left\{\sum_{i=1}^{\infty}D\big(\alpha_iP_i\big)+i\sum_{j=1}^{\infty}D\big(\beta_jP_j\big)\right\}+D(T)\left\{\sum_{i=1}^{\infty}\alpha_iP_i+i\sum_{j=1}^{\infty}\beta_jP_j\right\}\\
=&TD(S)+D(T)S.
\end{aligned}$$

The last equality obtained by continuity of φ. Indeed, $\sum_{i=1}^{m}P_i\to S_1$ uniformly. Hence

$$\begin{aligned}
\sum_{i=1}^{\infty}D\big(\alpha_iP_i\big)=&\lim_m\sum_{i=1}^{m}D\big(\alpha_iP_i\big)=\lim D\Big(\sum_{i=1}^{m}\alpha_iP_i\Big)\\
=&\lim_m\lim_n\frac{1}{2^n}\varphi\Big(2^n\sum_{i=1}^{m}\alpha_iP_i\Big)=\lim_n\lim_m\frac{1}{2^n}\varphi\Big(2^n\sum_{i=1}^{m}\alpha_iP_i\Big)\\
=&\lim_n\frac{1}{2^n}\varphi\Big(2^n\lim_m\sum_{i=1}^{m}\alpha_iP_i\Big)=\lim_n\frac{1}{2^n}\varphi\Big(2^n\sum_{i=1}^{\infty}\alpha_iP_i\Big)\\
=&D\Big(\sum_{i=1}^{\infty}\alpha_iP_i\Big).
\end{aligned}$$

From the condition (3) we conclude that

$$\|D(T^*)-D(T)^*\|=\lim_{n\to\infty}\frac{1}{2^n}\|\varphi(2^nT^*)-\varphi(2^nT)^*\|\to 0.$$

Hence, D is $*$-preserving. This means that D is a Jensen $*$-derivation. $\square$

Corollary 2.1. Let H be an infinite dimensional Hilbert space and $\varepsilon>0$ is given; if a mapping $\varphi:K(H)\longrightarrow K(H)$ with $\varphi(0)=0$ satisfies the following conditions:
(1) $\varphi(TP+PT)=T\varphi(P)+\varphi(T)P+P\varphi(T)+\varphi(P)T$ for all $T\in K(H)$ and $P\in P(H)$;
(2) $\left\|\varphi\left(\frac{\lambda T_1+\cdots+\lambda T_n}{n}\right)-\frac{\lambda}{n}\big(\varphi(T_1)+\cdots+\varphi(T_n)\big)\right\|<\varepsilon$ for all $\lambda\in\mathbb{T}^1_{\frac{1}{n_0}}$;
(3) $\|\varphi(T^*)-\varphi(T)^*\|<\varepsilon$ for all $T\in K(H)$.
Then φ is Jensen Jordan $*$-derivation.

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