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On Approximation Properties of Modified Sázas-Mirakyan Operators via Jain Operators

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Abstract. In the present manuscript, we propose the modification of Jain operators which the generalization of Szász-Mirakyan operators. These new class operators are linear positive operators of discrete type depending on a real parameters. We give theorem of degree of approximation and the Voronovskaya asymptotic formula.

Key Words: Positive linear operators, Jain operators, Szász-Mirakyan operator.

AMS Subject Classifications: 41A25, 41A30, 41A36

1 Introduction

In [1], Patel and Mishra introduced following sequence of positive linear operators, for $f\in C([0,\infty))$; $0\le\mu<1$; $1<\gamma\le e$

$$P_n^{[\mu,\,\gamma]}(f,x)=\sum_{k=0}^{\infty}\omega_{\mu,\gamma}(k,nx)f\Big(\frac{k}{n}\Big), \tag{1.1}$$

where

$$\omega_{(\mu,\gamma)}(k,nx)=nx(\log\gamma)^k(nx+k\mu)^{k-1}\frac{\gamma^{-(nx+k\mu)}}{k!}.$$

In the particular case, $\gamma=e$ the operators (1.1) equal to Jain operators [2]. Also, for $\gamma=e$ and $\mu=0$, the operators (1.1) turns to classical the Szász-Mirakyan operators. Therefore,

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the above operators is the generalization of Szaász-Mirakyan operators via Jain operators. The relation between the local smoothness of function and local approximation, the degree of approximation and the statistical convergence of the Jain operators was studied by Agratini [3]. Umar and Razi [4] studied Kantorovich-type extension of Jain operators. Durrmeyer type generalization of Jain operators and its approximation properties was elaborated by Tarabie [5], Mishra and Patel [6], Patel and Mishra [7] and Agratini [8]. Some related work in this area can be found in [9–16]. Motivated by such operators, we further generalized following modification of the operators (1.1) as: For $f\in C([0,\infty))$; $n\in\mathbb{N}$; $1<\gamma\leq e$; $0\leq\mu<1$;

$$P_n(f,x):=P_n^{[\mu,a_n,b_n,\gamma]}(f,x)=\sum_{k=0}^{\infty}\omega_{(\mu,\gamma)}(k,a_nx)f\left(\frac{k}{b_n}\right), \tag{1.2}$$

where $\omega_{(\mu,\gamma)}(k,a_nx)$ as defined in (1.1) and $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are given increasing and unbounded numerical sequence such that $a_n\geq 1$, $b_n\geq 1$ and $\left(\frac{a_n}{b_n}\right)$ is nondecreasing and

$$\frac{a_n}{b_n}=1+o\left(\frac{a_n}{b_n}\right). \tag{1.3}$$

Along the paper, when we will deal with approximation results, the parameters $\mu\in[0,1)$ and $\gamma\in(1,e]$ will be assumed to be a sequence μ_n and γ_n which tends to zero and Euler's number e as $n\to\infty$, respectively.

2 Moments of P_n

To discuss moments of the operators (1.2), we need following lemmas:

Lemma 2.1 (see [1]). *For $0<\alpha<\infty$, $0\leq\mu<1$ and $1<\gamma\leq e$. Let*

$$\omega_{(\mu,\gamma)}(k,\alpha)=\alpha(\log\gamma)^k(\alpha+k\mu)^{k-1}\frac{\gamma^{-(\alpha+k\mu)}}{k!}. \tag{2.1}$$

Then

$$\sum_{k=0}^{\infty}\omega_{(\mu,\gamma)}(k,\alpha)=1. \tag{2.2}$$

Lemma 2.2 (see [1]). *Let $0<\alpha<\infty$, $0\leq\mu<1$ and $1<\gamma\leq e$. Suppose that*

$$S(r,\alpha,\mu,\gamma)=\sum_{k=0}^{\infty}\frac{1}{k!}(\log\gamma)^k(\alpha+k\mu)^{k+r-1}\gamma^{-(\alpha+k\mu)} \tag{2.3}$$

and

$$S(1,\alpha,\mu,\gamma)=1. \tag{2.4}$$

Then

$$S(r,\alpha,\mu,\gamma)=\alpha S(r-1,\alpha,\mu,\gamma)+\mu\log\gamma S(r,\alpha,\mu,\gamma). \tag{2.5}$$

Also,

$$S(r,\alpha,\mu,\gamma)=\sum_{k=0}^{\infty}(\mu\log\gamma)^k(\alpha+k\mu)S(r-1,\alpha+k\mu,\mu,\gamma). \tag{2.6}$$

For (2.5) and (2.6), when $0<\alpha<\infty$ and $|\mu\log\gamma|<1$, we have

$$S(1,\alpha,\mu,\gamma)=\frac{1}{1-\mu\log\gamma}, \tag{2.7a}$$

$$S(2,\alpha,\mu,\gamma)=\frac{\alpha}{(1-\mu\log\gamma)^2}+\frac{\mu^2\log\gamma}{(1-\mu\log\gamma)^3}, \tag{2.7b}$$

$$S(3,\alpha,\mu,\gamma)=\frac{\alpha^2}{(1-\mu\log\gamma)^3}+\frac{3\alpha\mu^2\log\gamma}{(1-\mu\log\gamma)^4}+\frac{(\mu^3+2\mu^4)\log\gamma}{(1-\mu\log\gamma)^5}, \tag{2.7c}$$

$$\begin{aligned}S(4,\alpha,\mu,\gamma)=&\frac{\alpha^3}{(1-\mu\log\gamma)^4}+\frac{6\alpha^2\mu^2\log\gamma}{(1-\mu\log\gamma)^5}+\frac{2\alpha\mu^3(2+\mu)\log\gamma+9\alpha\mu^4(\log\gamma)^2}{(1-\mu\log\gamma)^6}\\&+\frac{\mu^4\log\gamma+2\mu^5(4+\mu)(\log\gamma)^2+4\mu^6(\log\gamma)^3}{(1-\mu\log\gamma)^7}.\end{aligned} \tag{2.7d}$$

In the following lemma, we have computed moments up to fourth order.

Lemma 2.3. *The operators P_n, $n>1$, defined by* (1.1) *satisfy the following relations*

$$\begin{aligned}P_n(1,x)=&1,\\P_n(t,x)=&\frac{a_nx\log\gamma}{b_n(1-\mu\log\gamma)},\\P_n(t^2,x)=&\frac{a_n^2x^2(\log\gamma)^2}{b_n^2(1-\mu\log\gamma)^2}+\frac{a_nx\log\gamma}{b_n^2(1-\mu\log\gamma)^3},\\P_n(t^3,x)=&\frac{a_n^3x^3(\log\gamma)^3}{b_n^3(1-\mu\log\gamma)^3}+\frac{3a_n^2x^2(\log\gamma)^2}{b_n^3(1-\mu\log\gamma)^4}\\&+\frac{xa_n\log\gamma(1+2\mu\log\gamma+2\mu^4(\log\gamma)^3-2\mu^4(\log\gamma)^4)}{b_n^3(1-\mu\log\gamma)^5},\\P_n(t^4,x)=&\frac{a_n^4x^4(\log\gamma)^4}{b_n^4(1-\mu\log\gamma)^4}+\frac{6a_n^3x^3(\log\gamma)^3}{b_n^4(1-\mu\log\gamma)^5}\\&+\frac{a_n^2x^2(\log\gamma)^2\left(7+8\mu\log\gamma+2\mu^4(\log\gamma)^3-2\mu^4(\log\gamma)^4\right)}{b_n^4(1-\mu\log\gamma)^6}\\&+\frac{a_nx\left((\log\gamma)+8\mu(\log\gamma)^2+6\mu^2(\log\gamma)^3\right)}{b_n^4(1-\mu\log\gamma)^7}\\&+\frac{a_nx\left((12\mu^4(\log\gamma)^4-16\mu^5(\log\gamma)^5+6\mu^6(\log\gamma)^6)(1-\log\gamma)\right)}{b_n^4(1-\mu\log\gamma)^7}.\end{aligned}$$

Using equalities (2.2), (2.7a) to (2.7d), one can archived proof of the above lemma.

Lemma 2.4. *Let the operator P_n be defined by relation as* (1.1) *and let $\varphi_x = t - x$ be given by*

$$\begin{aligned}
P_n(\varphi_x,x) &= x\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right),\\
P_n(\varphi_x^2,x) &= x^2\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)^2+\frac{a_nx\log\gamma}{b_n^2(1-\mu\log\gamma)^3},\\
P_n(\varphi_x^3,x) &= x^3\Big(\frac{3a_n^2\mu(\log\gamma)^3}{b_n^2(1-\mu\log\gamma)^3}+\frac{a_n^3(\log\gamma)^3}{b_n^3(1-\mu\log\gamma)^3}-\frac{3a_n^2\log\gamma^2}{b_n^2(1-\mu\log\gamma)^3}\\
&\quad+\frac{3a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\Big)+\frac{3a_nx^2\log\gamma}{b_n^2(1-\mu\log\gamma)^3}\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)\\
&\quad+\frac{a_nx(\log\gamma)\left(1+2\mu(\log\gamma)+2\mu^4(\log\gamma)^3-2\mu^4(\log\gamma)^4\right)}{b_n^3(1-\mu\log\gamma)^5},\\
P_n(\varphi_x^4,x) &= x^4\Big(1-\frac{4a_n^3(\log\gamma)^3}{b_n^3(1-\mu\log\gamma)^5}+\frac{8a_n^3\mu(\log\gamma)^4}{b_n^3(1-\mu\log\gamma)^5}-\frac{4a_n^3\mu^2(\log\gamma)^5}{b_n^3(1-\mu\log\gamma)^5}\\
&\quad+\frac{a_n^4(\log\gamma)^4}{b_n^4(1-\mu\log\gamma)^4}+\frac{6a_n^2(\log\gamma)^2}{b_n^2(1-\mu\log\gamma)^3}-\frac{6a_n^2\mu(\log\gamma)^3}{b_n^2(1-\mu\log\gamma)^3}-\frac{4a_n\log\gamma}{b(1-\mu\log\gamma)}\Big)\\
&\quad+x^3\Big(\frac{6a_n^3(\log\gamma)^3}{b_n^4(1-\mu\log\gamma)^5}-\frac{12a_n^2(\log\gamma)^2}{b_n^3(1-\mu\log\gamma)^5}+\frac{12a_n^2\mu(\log\gamma)^3}{b_n^3(1-\mu\log\gamma)^5}+\frac{6a_n\log\gamma}{b_n^2(1-\mu\log\gamma)^3}\Big)\\
&\quad+x^2\Big(\frac{8a_n\mu(\log\gamma)^2}{b_n^3(1-\mu\log\gamma)^5}-\frac{4a_n\log\gamma}{b_n^3(1-\mu\log\gamma)^5}-\frac{8a_n\mu^4(\log\gamma)^4}{b_n^3(1-\mu\log\gamma)^5}\\
&\quad+\frac{8a_n\mu^4(\log\gamma)^5}{b_n^3(1-\mu\log\gamma)^5}+\frac{a_n^2(\log\gamma)^2\left(7+8\mu(\log\gamma)+2\mu^4(\log\gamma)^3-2\mu^4(\log\gamma)^4\right)}{b_n^4(1-\mu\log\gamma)^6}\Big)\\
&\quad+\frac{a_nx\log\gamma\left(1+8\mu\log\gamma+6\mu^2(\log\gamma)^2\right)}{b_n^4(1-\mu\log\gamma)^7}\\
&\quad+\frac{a_nx\log\gamma\left(2(1-\log\gamma)(\log\gamma)^3\left(6\mu^4-8\mu^5\log\gamma+3\mu^6(\log\gamma)^2\right)\right)}{b_n^4(1-\mu\log\gamma)^7}.
\end{aligned}$$

Proof of the above lemma, follows from the linearity of the operators P_n and Lemma 2.3.

By equality (1.3), $\lim_{n\to\infty}\mu_n=0$ and $\lim_{n\to\infty}\gamma_n=e$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} b_nP_n(\varphi_x,x)=0,\\
&\lim_{n\to\infty} b_nP_n(\varphi_x^2,x)=x,\\
&\lim_{n\to\infty} b_nP_n(\varphi_x^3,x)=0,\\
&\lim_{n\to\infty} b_n^2P_n(\varphi_x^4,x)=3x^2.
\end{aligned}$$

The above equality can be verified for the case $\gamma_n=e$ in [17, page 5].

3 Direct theorem

Consider the Banach space $B_r([0,\infty))=\{f:[0,\infty)\to\mathbb{R}:|f(x)|\le M(1+x^r)\}$, for some $M>0$ and $r>0$. By $C_r([0,\infty))$, we denote the subspace of all continuous functions to $B_r([0,\infty))$. Also, $C_r^*([0,\infty))$ is a subspace of all functions $f\in C_r([0,\infty))$, for which $\lim_{n\to\infty}\frac{f(x)}{1+x^r}$ is finite. The norm on $C_r^*([0,\infty))$ is $\|f\|=\sup_{x\in[0,\infty)}\frac{f(x)}{1+x^r}$.

The convergence property of the operator (1.1) is proved in the following theorem:

Theorem 3.1. *If $f\in C_r([0,\infty))$ and $\mu_n\to 0$, $\gamma_n\to e$ as $n\to\infty$, then the sequence P_n converges uniformly to f(x) in $[c,d]$, where $0\le c<d<\infty$.*

Proof. Since P_n is a positive linear operator for $0\le\mu_n<1$ and $1<\gamma_n\le e$, it is sufficient, by Korovkin's result [18], to verify the uniform convergence for test functions $f(t)=1,t$ and t^2.

It is clear that

$$P_n(1,x)=1.$$

Going to $f(t)=t$,

$$\lim_{n\to\infty}P_n(t,x)=\lim_{n\to\infty}\frac{a_n x\log(\gamma_n)}{b_n(1-\mu_n\log(\gamma_n))}=x\quad\text{as}\quad \mu_n\to 0\quad\text{and}\quad\gamma_n\to e.$$

Proceeding to the function $f(t)=t^2$, it can easily be shown that

$$\begin{aligned}\lim_{n\to\infty}P_n(t^2,x)&=\lim_{n\to\infty}\left(\frac{a_n^2x^2(\log(\gamma_n))^2}{b_n^2(1-\mu_n\log(\gamma_n))^2}+\frac{a_n x\log(\gamma_n)}{b_n^2(1-\mu\log(\gamma_n))^3}\right)\\&=x^2\quad as\ \ \mu_n\to 0\quad\text{and}\quad\gamma_n\to e,\end{aligned}$$

and hence by Korovkin's theorem the proof of theorem is complete. □

Let the space $C_B([0,\infty))$ of all continuous and bounded functions be endowed with the norm $\|f\|=\sup\{|f(x)|:x\in[0,\infty)\}$. Further let us consider the following K-functional:

$$K_2(f,\delta)=\inf_{g\in W^2}\{\|f-g\|+\delta\|g''\|\},\tag{3.1}$$

where $\delta>0$ and $W^2=\{g\in C_B([0,\infty)):g',g''\in C_B([0,\infty))\}$. By the method as given in [19, pp. 177, Theorem 2.4], there exists an absolute constant $C>0$ such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta}),\tag{3.2}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h<\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|\tag{3.3}$$

is the second order modulus of smoothness of $f\in C_B([0,\infty))$. Also, we set

$$\omega(f,\sqrt{\delta})=\sup_{0<h<\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+h)-f(x)|.\tag{3.4}$$

Theorem 3.2. *For $f\in C_B([0,\infty))$, we have*

$$|P_n(f,x)-f(x)|\le C\omega_2\Big(f,\sqrt{P_n(\varphi_x^2,x)+\Big(x\Big(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\Big)\Big)^2}\Big)$$
$$+\omega_1\Big(f,x\Big|\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\Big|\Big),$$

where C is positive constant.

Proof. We are introducing the auxiliary operators as follows

$$\widehat{P}_n(f,x)=P_n(f,x)-f\left(\frac{a_nx\log\gamma}{b_n(1-\mu\log\gamma)}\right)+f(x)$$

for every $x\in[0,\infty)$. The operators $\widehat{P}_n$ are linear and preserves the linear functions, therefore

$$\widehat{P}_n(t-x,x)=0. \tag{3.5}$$

Let $g\in W_\infty^2$ and $x,t\in[0,\infty)$. By Taylor's expansion, we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)du.$$

Applying $\widehat{P}_n$, we get

$$\widehat{P}_n(g,x)-g(x)=g'(x)\widehat{P}_{n,\gamma}^{\alpha,\beta}(t-x,x)+\widehat{P}_n\left(\int_x^t(t-u)g''(u)du,x\right),$$

and applying (3.5), we get

$$\widehat{P}_n(g,x)-g(x)=\widehat{P}_n\left(\int_x^t(t-u)g''(u)du,x\right).$$

Hence,

$$\begin{aligned}|\widehat{P}_n(g,x)-g(x)|\le&\left|P_n\left(\int_x^t(t-u)g''(u)du,x\right)\right|\\&+\left|\int_x^{\frac{a_nx\log\gamma}{b_n(1-\mu\log\gamma)}}\left(\frac{a_nx\log\gamma}{b_n(1-\mu\log\gamma)}-u\right)g''(u)du\right|\\\le&P_n(\varphi_x^2,x)\|g''\|+\int_x^{\frac{a_nx\log\gamma}{b_n(1-\mu\log\gamma)}}\left|\left(\frac{a_nx\log\gamma}{b_n(1-\mu\log\gamma)}-u\right)g''(u)\right|du\\\le&\left[P_n(\varphi_x^2,x)+\left(x\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)\right)^2\right]\|g''\|.\end{aligned}$$

Also, we have $|P_n(f,x)| \le \|f\|$. Using these, we get

$$\begin{aligned}|P_n(f,x)-f(x)| \le & |\widehat{P}_n(f-g,x)-(f-g)(x)|+|\widehat{P}_n(g,x)-g(x)| \\ & +\left|\left(\frac{a_n x\log\gamma}{b_n(1-\mu\log\gamma)}\right)-f(x)\right| \\ \le & 4\|f-g\|+\left[P_n(\varphi_x^2,x)+\left(x\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)\right)^2\right]\|g''\| \\ & +\omega_1\left(f,x\left|\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right|\right).\end{aligned}$$

Hence, taking infimum on the right hand side over all $g\in W^2$, we get

$$\begin{aligned}|P_n(f,x)-f(x)| \le & K\left(f,P_n(\varphi_x^2,x)+\left(x\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)\right)^2\right) \\ & +\omega_1\left(f,x\left|\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right|\right).\end{aligned}$$

In view of (3.2), we get

$$\begin{aligned}|P_n(f,x)-f(x)| \le & C\omega_2\left(f,\sqrt{P_n(\varphi_x^2,x)+\left(x\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)\right)^2}\right) \\ & +\omega_1\left(f,x\left|\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right|\right).\end{aligned}$$

This completes the proof of the theorem. □

We know that a continuous function f defined on I satisfies the condition

$$|f(x)-f(y)| \le M_f|x-y|^\eta, \quad (x,y)\in I\times E,$$

it called locally $Lip\eta$ on E $(0<\eta\le 1,\ E\subset I)$, where M_f is a constant depending only on f.

Theorem 3.3. *Let E be any subset of $[0,\infty)$. If f is locally Lip η on E, then we have*

$$|P_n(f,x)-f(x)| \le M_f C(\eta,\mu,\gamma,a_n,b_n)\max\{x^{\eta/2},x^\eta\}+2M_f d^\eta(x,E),$$

where

$$C(\eta,\mu,\gamma,a_n,b_n)=\left(\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)^2+\frac{a_n\log\gamma}{b_n^2(1-\mu\log\gamma)^3}\right)^\eta$$

and $d(x,E)$ is the distance between x and E defines as

$$d(x,E)=\inf\{|x-y| : y\in E\}.$$

Proof. Since f is continuous,

$$|f(x)-f(y)|\leq M_f|x-y|^\eta$$

holds for any $x\geq 0$ and $y\in\overline{E}$, $\overline{E}$ is closure of $E\subset(-\infty,\infty)$. Let $(x,x_0)\in[0,\infty)\times\overline{E}$ be such that $|x-x_0|=d(x,E)$.

Using linear properties of P_n, inequality $(A+B)^\eta\leq A^\eta+B^\eta$ $(A\geq 0,B\geq 0,0<\alpha\leq 1)$ and Hölder's inequality, we get

$$\begin{aligned}|P_n(f,x)-f(x)|&\leq P_n(|f-f(x_0)|,x)+|f(x_0)-f(x)|\\&\leq M_fP_n(|(t-x)+(x-x_0)|^\eta,x)+M_f|x_0-x|^\eta\\&\leq M_f(P_n(|t-x|^\eta,x)+|x-x_0|^\eta)+M_f|x_0-x|^\eta\\&\leq M_f\Big(\big(P_n(\phi_x^2,x)\big)^{\eta/2}+2|x-x_0|^\eta\Big)\\&=M_f\left(\left(x^2\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)^2+\frac{a_nx\log\gamma}{b_n^2(1-\mu\log\gamma)^3}\right)^{\eta/2}+2|x-x_0|^\eta\right)\\&\leq M_f\Big(C(\eta,\mu,\gamma,a_n,b_n)\max\{x^{\eta/2},x^\eta\}+2|x_0-x|^\eta\Big),\end{aligned}$$

which is required results. $\square$

Now, we establish the Voronovskaja type asymptotic formula for the operators P_n. In this section, we denoted $C^r([a,b])$ as the set of all real-valued r-times continuously differentiable functions on the interval $[a,b]$, $(r\in\mathbb{N})$ and it is a subspace of $C([a,b])$. The norm on the space $C^r([a,b])$ can be defined as

$$\|f\|_{C^r([a,b])}=\|f\|_{C([a,b])}+\|f^{(1)}\|_{C([a,b])}+\cdots+\|f^{(k)}\|_{C([a,b])},f\in C^r([a,b]).$$

$\|h\|_{C([a,b])}$ represents the sup-norm of the function $h|_{[a,b]}$.

Theorem 3.4. *Let $f,f',f''\in C([0,\infty))$ and let the operator P_n be defined as in* (1.2). *If $\mu_n\to 0$ and $\gamma_n\to e$ as $n\to\infty$ holds, then*

$$\lim_{n\to\infty}b_n(P_n(f,x)-f(x))=\frac{x}{2}f''(x),\quad\forall\ x>0.$$

Proof. Let $f,f',f''\in C([0,\infty))$ and $x\in[0,\infty)$ be fixed. By the Taylor's formula, we have

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t;x)(t-x)^2,\tag{3.6}$$

where $r(t;x)$ is the Peano form of the remainder, $r(\cdot;x)\in C^2([0,\infty))$ and $\lim_{t\to x}r(t,x)=0$.

We apply P_n to Eq. (3.6), we get

$$\begin{aligned}P_n(f,x)-f(x)=&f'(x)P_n((t-x),x)+\frac{1}{2}f''(x)P_n((t-x)^2,x)+P_n(r(t;x)(t-x)^2,x)\\=&f'(x)\left[x\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)\right]+P_n(r(t;x)(t-x)^2,x)\\&+\frac{f''(x)}{2}\left[x^2\left(\frac{a_n\log\gamma}{b_n(1-\mu\log\gamma)}-1\right)^2+\frac{a_nx\log\gamma}{b_n^2(1-\mu\log\gamma)^3}\right].\end{aligned}$$

In the second term $P_n(r(t;x)(t-x)^2,x)$ applying the Cauchy-Schwartz inequality, we have

$$0\leq|P_n(r(t;x)(t-x)^2,x)|\leq\sqrt{P_n((t-x)^4,x)}\sqrt{P_n(r(t;x),x)}. \tag{3.7}$$

We have marked that $\lim_{t\to x}r(t,x)=0$. In harmony with $\mu_n\to0$ and $\gamma_n\to e$ as $n\to\infty$, we have

$$\lim_{n\to\infty}P_n(r(t,x),x)=0. \tag{3.8}$$

On the basis of (3.7) and (3.8), we get that

$$\lim_{n\to\infty}b_n(P_n(f,x)-f(x))=\frac{x}{2}f''(x),\quad\forall x>0.$$

Hence, the proof is completed. □

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