

## Some Approximation Properties of Certain $q$ -Baskakov-Beta Operators

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**Abstract.** In this paper, we propose the  $q$  analogue of modified Baskakov-Beta operators. The Voronovskaja type theorem and some direct results for the above operators are discussed. The rate of convergence and weighted approximation by the operators are studied.

**Key Words:**  $q$ -Baskakov-Beta operators, rate of convergence, weighted approximation.

**AMS Subject Classifications:** 41A25, 41A35

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### 1 Introduction

In recent years, the application of  $q$  calculus is the most interesting areas of research in the approximation theory (e.g., [1]). Lupaş [2] and Phillips [3] proposed generalizations of Bernstein polynomials based on the  $q$ -integers. More results on  $q$ -Bernstein operators were investigated (e.g., [4, 5]). Gupta and Aral [6, 7] proposed certain  $q$ -analogues of the Baskakov operators and studied some approximation properties of  $q$ -Baskakov operators.

In approximation theory the Durrmeyer type integral modification of certain discrete operators is also an active area of research. Cai [8] investigated the convergence of modification of Durrmeyer type  $q$ -Baskakov operators. Gupta and collaborators (see [9–13], etc.) introduced several important  $q$  analogues of different Durrmeyer type operators and established interesting approximation results. In [14], Gupta observed that the Baskakov operators by taking weight functions of Beta basis function give better approximation results. Wang [15] also estimated asymptotic formula for Baskakov Beta operators in generalized form.

The aim of this paper is to study the approximation properties of certain generalization of Baskakov Beta operators, based on  $q$ -integer. We first recall some concept of

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$q$ -calculus, which can be found in [16]. In what follows,  $q$  is a real number satisfying  $0 < q < 1$ .

For  $k \in \mathbb{N}$ , the  $q$  integer and  $q$  factorial are given by

$$[k]_q = \frac{1-q^k}{1-q}, \quad [k]_q! = \begin{cases} [k]_q [k-1]_q \cdots [1]_q, & k \geq 1, \\ 1, & k = 0. \end{cases}$$

The  $q$ -binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]_q!}{[k]_q! [n-k]_q!}, \quad 0 \leq k \leq n. \tag{1.1}$$

The  $q$ -Pochhammer symbol is defined as

$$(-x, q)_n = (1+x)(1+qx) \cdots (1+q^{n-1}x) = \prod_{j=0}^{n-1} (1+q^j x).$$

The  $q$ -Jackson integrals and the  $q$ -improper integrals are defined as (see [17, 18])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n, \quad a > 0,$$

and

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0, \tag{1.2}$$

provided the sum converge absolutely.

The  $q$ -Gamma integral (see [19]) is defined by

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx) d_q x, \quad t > 0, \tag{1.3}$$

where

$$E_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q!}.$$

Also  $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$ ,  $\Gamma_q(1) = 1$ .

The  $q$ -Beta integral (see [19]) is defined by

$$B_q(t, s) = K(A, t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}} d_q x, \tag{1.4}$$

where

$$K(x, t) = \frac{1}{x+1} x^t \left(1 + \frac{1}{x}\right)_q^t (1+x)_q^{1-t}.$$

It was observed in that  $K(x, t)$  is a  $q$ -constant, i.e.,  $K(qx, t) = K(x, t)$ . In particular for any positive integer  $n$ , one has

$$K(x, n) = q^{\frac{n(n-1)}{2}}, \quad K(x, 0) = 1 \quad \text{and} \quad B_q(t, s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}. \tag{1.5}$$

The  $q$ -derivative  $\mathcal{D}_q$  is given by

$$\mathcal{D}_q f(x) := \begin{cases} \frac{f(qx) - f(x)}{(q-1)x}, & \text{if } x \neq 0, \\ f'(0), & \text{if } x = 0. \end{cases}$$

And also  $\mathcal{D}_q^0 f := f$ ,  $\mathcal{D}_q^n f := \mathcal{D}_q(\mathcal{D}_q^{n-1} f)$ ,  $n = 1, 2, 3, \dots$ ,

$$\begin{aligned} \mathcal{D}_q(f(x)g(x)) &= g(x)\mathcal{D}_q(f(x)) + f(qx)\mathcal{D}_q(g(x)), \\ \mathcal{D}_q\left(\frac{f(x)}{g(x)}\right) &= \frac{g(x)\mathcal{D}_q(f(x)) - f(x)\mathcal{D}_q(g(x))}{g(x)g(qx)}, \end{aligned}$$

for details (see [20]).

Recently, Aral and Gupta [21] introduced a different  $q$ -generalization of the classical Baskakov operators. For  $f \in C[0, \infty)$ ,  $q > 0$  and each positive integer  $n$ , the operators introduced in [21] are defined as

$$\begin{aligned} \mathcal{B}_{n,q}(f, x) &= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{(qx)^k}{(1+qx)_q^{n+k}} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\ &:= \sum_{k=0}^{\infty} b_{n,k}^q(x) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right). \end{aligned} \tag{1.6}$$

We denote by  $C_B[0, \infty)$  the space of all real valued continuous bounded function  $f$  defined on  $[0, \infty)$ , endowed with the norm

$$\|f\| = \sup_{x \in [0, \infty)} |f(x)|.$$

For every  $n \in \mathbb{N}$ ,  $f(x) \in C_B[0, \infty)$ , the certain  $q$ -Baskakov Beta operators  $D_{n,q}^*(f, x)$  are defined as

$$D_{n,q}^*(f, x) = \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} v_{n,k-1}^q(t) f(t) d_q t + p_{n,0}^q(x) f(0), \tag{1.7}$$

where  $x \in [0, \infty)$ , and

$$\begin{aligned} p_{n,k}^q(x) &= \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2}{2}} \frac{(qx)^k}{(1+qx)_q^{n+k}}, \\ v_{n,k}^q(t) &= \frac{1}{B_q(n, k+1)} q^{\frac{k^2-1}{2}} \frac{t^k}{(1+t)_q^{n+k+1}}. \end{aligned}$$

The above  $q$ -analogues of Baskakov Beta operators are defined on  $q \in (0,1)$ . And these operators are linear and reproduce only the constant functions. The present paper is organized as follows: in the first section, we present the basic notations and the definitions of  $q$  analogues of Baskakov Beta operators. In the second section, we give the moment estimates and establish the recurrence relation for the moments of the operators by the  $q$ -derivatives. In Section three we give the basic convergence theorem and Voronovskaja type theorem. Section four and five, we study the local approximation and the rate of convergence of the operators. And we also estimate the weighted approximation properties.

## 2 Moment estimates

**Lemma 2.1.** For  $\mathcal{B}_{n,q}(t^m, x)$ ,  $m=0,1,2$ , we have

$$\mathcal{B}_{n,q}(1, x) = 1, \quad \mathcal{B}_{n,q}(t, x) = qx, \quad \mathcal{B}_{n,q}(t^2, x) = q^2x^2 + \frac{qx}{[n]_q}(1+x).$$

**Lemma 2.2.** For  $D_{n,q}^*(t^m; x)$ ,  $m=0,1,2$ , one has

- (i)  $D_{n,q}^*(1; x) = 1;$
- (ii)  $D_{n,q}^*(t; x) = \frac{[n]_q}{[n-1]_q}x, \quad \text{for } n > 1;$
- (iii)  $D_{n,q}^*(t^2, x) = \frac{q[n]_q^2 + [n]_q}{q^2[n-1]_q[n-2]_q}x^2 + \frac{(1+q)[n]_q}{q^2[n-1]_q[n-2]_q}x, \quad \text{for } n > 2.$

*Proof.* The operators  $D_{n,q}^*$  are well defined on the function  $1, t, t^2$ . Then for every  $n > 2$  and  $x \in [0, \infty)$ , we obtain

$$\begin{aligned} D_{n,q}^*(1, x) &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} v_{n,k-1}^q(t) d_q t + p_{n,0}^q(x) \\ &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \frac{1}{B_q(n,k)} q^{\frac{(k-1)^2-1}{2}} \int_0^{\infty/A} \frac{t^{k-1}}{(1+t)_q^{n+k}} d_q t + p_{n,0}^q(x). \end{aligned}$$

Using (1.1), (1.4), (1.5) and Lemma 2.1, we may write

$$\begin{aligned} D_{n,q}^*(1, x) &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \frac{1}{B_q(n,k)} q^{\frac{(k-1)^2-1}{2}} \frac{B_q(n,k)}{K(A,k)} + p_{n,0}^q(x) \\ &= \sum_{k=1}^{\infty} p_{n,k}^q(x) q^{\frac{-k}{2}} + p_{n,0}^q(x) \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{(qx)^k}{(1+qx)_q^{n+k}} \\ &= \mathcal{B}_{n,q}(1, x) = 1. \end{aligned}$$

Similarly,

$$\begin{aligned}
 D_{n,q}^*(t,x) &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} v_{n,k-1}^q(t) t d_q t \\
 &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \frac{1}{B_q(n,k)} q^{\frac{(k-1)^2-1}{2}} \int_0^{\infty/A} \frac{t^k}{(1+t)_q^{n+k}} d_q t \\
 &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \frac{1}{B_q(n,k)} q^{\frac{(k-1)^2-1}{2}} \frac{B_q(k+1,n-1)}{K(A,k+1)} \\
 &= \sum_{k=1}^{\infty} p_{n,k}^q(x) q^{\frac{-3k}{2}} \frac{[k]_q}{[n-1]_q} \\
 &= \frac{[n]_q}{q[n-1]_q} \mathcal{B}_{n,q}(t,x).
 \end{aligned}$$

From Lemma 2.1, we get

$$D_{n,q}^*(t,x) = \frac{[n]_q}{[n-1]_q} x.$$

Finally,

$$\begin{aligned}
 D_{n,q}^*(t^2,x) &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} v_{n,k-1}^q(t) t^2 d_q t \\
 &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \frac{1}{B_q(n,k)} q^{\frac{(k-1)^2-1}{2}} \int_0^{\infty/A} \frac{t^{k+1}}{(1+t)_q^{n+k}} d_q t \\
 &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \frac{1}{B_q(n,k)} q^{\frac{(k-1)^2-1}{2}} \frac{B_q(k+2,n-2)}{K(A,k+2)} \\
 &= \sum_{k=1}^{\infty} p_{n,k}^q(x) q^{\frac{-5k-2}{2}} \frac{[k]_q [k+1]_q}{[n-1]_q [n-2]_q}.
 \end{aligned}$$

Using

$$[k+1]_q = [k]_q + q^k,$$

we have

$$\begin{aligned}
 D_{n,q}^*(t^2,x) &= \sum_{k=1}^{\infty} p_{n,k}^q(x) q^{\frac{-5k+2}{2}} \frac{[k]_q ([k]_q + q^k)}{[n-1]_q [n-2]_q} \\
 &= \sum_{k=1}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k^2-5k-2}{2}} \frac{(qx)^k}{(1+qx)_q^{n+k}} \left( \frac{[k]_q^2}{[n-1]_q [n-2]_q} + \frac{q^k [k]_q}{[n-1]_q [n-2]_q} \right) \\
 &= \sum_{k=1}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{(qx)^k}{(1+qx)_q^{n+k}} \left( \frac{[k]_q}{q^{k-1} [n]_q} \right)^2 \frac{[n]_q^2}{q^3 [n-1]_q [n-2]_q}
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{(qx)^k}{(1+qx)_q^{n+k}} \frac{[k]_q}{q^{k-1}[n]_q} \frac{[n]_q}{q^2[n-1]_q[n-2]_q} \\
 & = \frac{[n]_q^2}{q^3[n-1]_q[n-2]_q} \mathcal{B}_{n,q}(t^2, x) + \frac{[n]_q}{q^2[n-1]_q[n-2]_q} \mathcal{B}_{n,q}(t, x).
 \end{aligned}$$

Again using Lemma 2.1, we obtain

$$D_{n,q}^*(t^2, x) = \frac{q[n]_q^2 + [n]_q}{q^2[n-1]_q[n-2]_q} x^2 + \frac{(1+q)[n]_q}{q^2[n-1]_q[n-2]_q} x.$$

Thus, we complete the proof. □

**Remark 2.1.** Let  $n > 2$  be a given number. For every  $0 < q < 1$ , from Lemma 2.1 we have

$$\begin{aligned}
 D_{n,q}^*((t-x), x) & = \frac{q^{n-1}}{[n-1]_q} x, \\
 D_{n,q}^*((t-x)^2, x) & = \left( \frac{q[n]_q^2 + [n]_q}{q^2[n-1]_q[n-2]_q} - \frac{2[n]_q}{[n-1]_q} + 1 \right) x^2 + \frac{(1+q)[n]_q}{q^2[n-1]_q[n-2]_q} x.
 \end{aligned}$$

**Proposition 2.1** (see [7]). For  $n, k \geq 0$ , we have

$$\mathcal{D}_q[x^k(-x, q)_{n+k}^{-1}] = [k]_q x^{k-1} (-x, q)_{n+k}^{-1} - q^k x^k [n+k]_q (-x, q)_{n+k+1}^{-1}.$$

**Remark 2.2.** For  $n, k \in \mathbb{N}$ , from Proposition 2.1, we have

$$\begin{aligned}
 (i) \quad & qx(1+qx) \mathcal{D}_q p_{n,k}^q(x) = \left( \frac{[k]_q}{q^{k-1}[n]_q} - q^2 x \right) \frac{[n]_q}{q} p_{n,k}^q(qx); \\
 (ii) \quad & t(1+t) \mathcal{D}_q v_{n,k}^q(t) = \left( \frac{[k]_q}{q^{k-1}[n+1]_q} - qt \right) \frac{[n+1]_q}{q} v_{n,k}^q(qt); \\
 (iii) \quad & \frac{u}{q} \left( 1 + \frac{u}{q} \right) \mathcal{D}_q v_{n,k}^q\left(\frac{u}{q}\right) = \left( \frac{[k]_q}{q^{k-1}[n+1]_q} - u \right) \frac{[n+1]_q}{q} v_{n,k}^q(u).
 \end{aligned}$$

**Proposition 2.2.** If we define the  $m$ th order ( $m \in \mathbb{N}$ ) moment as

$$T_{n,m}^q(x) = \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t.$$

Then, for  $n > m + 2$  the following recurrence relation holds

$$\begin{aligned}
 q^2 x(1+qx) \mathcal{D}_q T_{n,m}^q(x) & = T_{n,m}^q(qx) - q^2 x [n]_q T_{n,m}^q(qx) + \frac{[n+1]_q}{q} T_{n,m+1}^q(qx) \\
 & \quad - \frac{[m+1]_q}{q} T_{n,m}^q(qx) - \frac{[m+2]_q}{q^2} T_{n,m+1}^q(qx).
 \end{aligned}$$

*Proof.* From Remark 2.2(i), we have

$$\begin{aligned} E &= q^2 x(1+qx) \mathcal{D}_q T_{n,m}^q(x) \\ &= \sum_{k=1}^{\infty} q^2 x(1+qx) \mathcal{D}_q p_{n,k}^q(x) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t \\ &= [n]_q \sum_{k=1}^{\infty} \left( \frac{[k]_q}{q^{k-1}[n]_q} - q^2 x \right) p_{n,k}^q(qx) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t. \end{aligned}$$

Since  $[k]_q = [k-1]_q + q^{k-1}$ , we obtain

$$\begin{aligned} E &= [n]_q \sum_{k=1}^{\infty} \left( \frac{[k-1]_q + q^{k-1}}{q^{k-1}[n]_q} - q^2 x \right) p_{n,k}^q(qx) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t \\ &= \sum_{k=1}^{\infty} \frac{[k-1]_q}{q^{k-1}} p_{n,k}^q(qx) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t \\ &\quad + \sum_{k=1}^{\infty} p_{n,k}^q(qx) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t - q^2 x [n]_q \sum_{k=1}^{\infty} p_{n,k}^q(qx) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t \\ &= \sum_{k=1}^{\infty} \frac{[k-1]_q}{q^{k-1}} p_{n,k}^q(qx) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t + T_{n,m}^q(qx) - q^2 x [n]_q T_{n,m}^q(qx). \end{aligned}$$

Let us consider

$$\begin{aligned} I &= \sum_{k=1}^{\infty} \frac{[k-1]_q}{q^{k-1}} p_{n,k}^q(qx) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t \\ &= \frac{[n]_q}{q} \sum_{k=1}^{\infty} \frac{[k-1]_q}{q^{k-2}[n]_q} p_{n,k}^q(qx) \int_0^{\infty/A} v_{n,k-1}^q(t) t^m d_q t \\ &= \frac{[n]_q}{q} \sum_{k=1}^{\infty} p_{n,k}^q(qx) \int_0^{\infty/A} \left( \frac{[k-1]_q}{q^{k-2}[n]_q} - \frac{[n+1]_q}{[n]_q} t + \frac{[n+1]_q}{[n]_q} t \right) v_{n,k-1}^q(t) t^m d_q t \\ &= \frac{[n+1]_q}{q} T_{n,m+1}^q(qx) + \sum_{k=1}^{\infty} p_{n,k}^q(qx) \int_0^{\infty/A} \left( \frac{[k-1]_q}{q^{k-2}[n+1]_q} - t \right) \frac{[n+1]_q}{q} v_{n,k-1}^q(t) t^m d_q t. \end{aligned}$$

Again using Remark 2.2(iii), we get

$$\begin{aligned} I &= \frac{[n+1]_q}{q} T_{n,m+1}^q(qx) + \frac{1}{q} \sum_{k=1}^{\infty} p_{n,k}^q(qx) \int_0^{\infty/A} t \left( 1 + \frac{t}{q} \right) \mathcal{D}_q v_{n,k-1}^q \left( \frac{t}{q} \right) t^m d_q t \\ &= \frac{[n+1]_q}{q} T_{n,m+1}^q(qx) + \frac{1}{q} \sum_{k=1}^{\infty} p_{n,k}^q(qx) \int_0^{\infty/A} \left( t^{m+1} + \frac{t^{m+2}}{q} \right) \mathcal{D}_q v_{n,k-1}^q \left( \frac{t}{q} \right) d_q t. \end{aligned}$$

Using the  $q$ -integral by parts

$$\int_a^b u(t) \mathcal{D}_q(v(t)) d_q t = u(t)v(t) \Big|_a^b - \int_a^b v(qt) \mathcal{D}_q(u(t)) d_q t.$$

$$\begin{aligned}
 I &= \frac{[n+1]_q}{q} T_{n,m+1}^q(qx) - \frac{1}{q} \sum_{k=1}^{\infty} p_{n,k}^q(qx) \int_0^{\infty/A} \left( [m+1]_q t^m + \frac{[m+2]_q}{q} t^{m+1} \right) v_{n,k-1}^q(t) d_q t \\
 &= \frac{[n+1]_q}{q} T_{n,m+1}^q(qx) - \frac{[m+1]_q}{q} T_{n,m}^q(qx) - \frac{[m+2]_q}{q^2} T_{n,m+1}^q(qx).
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
 E &= T_{n,m}^q(qx) - q^2 x [n]_q T_{n,m}^q(qx) + \frac{[n+1]_q}{q} T_{n,m+1}^q(qx) \\
 &\quad - \frac{[m+1]_q}{q} T_{n,m}^q(qx) - \frac{[m+2]_q}{q^2} T_{n,m+1}^q(qx).
 \end{aligned}$$

This completes the proof of the recurrence relation. □

**Lemma 2.3.** For all  $f \in C_B[0, \infty)$ ,  $n = 1, 2, \dots$ , and  $q \in (0, 1)$ , we have

$$\|D_{n,q}^* f\| \leq \|f\|.$$

*Proof.* For  $x \in [0, \infty)$ , we get

$$\begin{aligned}
 |D_{n,q}^*(f, x)| &\leq \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} v_{n,k-1}^q(t) |f(t)| d_q t + p_{n,0}^q(x) |f(0)| \\
 &\leq \|f\| D_{n,q}^*(1, x) = \|f\|,
 \end{aligned}$$

which completes the proof. □

### 3 Direct theorems

**Theorem 3.1.** Let  $q_n \in (0, 1)$ . Then the sequence  $\{D_{n,q_n}^*(f)\}$  converges to  $f$  uniformly on  $[0, A]$ ,  $A > 0$  for each  $f \in C_B[0, \infty)$  if and only if  $\lim_{n \rightarrow \infty} q_n = 1$ .

**Theorem 3.2** (Voronovskaja type theorem). Assume that  $q_n \in (0, 1)$ ,  $q_n \rightarrow 1$  and  $q_n^n \rightarrow a$  as  $n \rightarrow \infty$ . For any  $f \in C_B[0, \infty)$ , if  $f', f''$  exists on  $[0, \infty)$ , the following equality holds

$$\lim_{n \rightarrow \infty} [n]_{q_n} (D_{n,q_n}^*(f, x) - f(x)) = ax f'(x) + x(2+x) \frac{f''(x)}{2},$$

uniformly on any  $[0, A]$ ,  $A > 0$ .

*Proof.* Let  $x \in [0, \infty)$  be arbitrary but fixed. From the Taylor's theorem, we may write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}f''(x)(t-x)^2 + r(t,x)(t-x)^2, \tag{3.1}$$



where  $r(t, x)$  is the Peano form of the remainder, and  $\lim_{t \rightarrow x} r(t, x) = 0$ . Applying  $D_{n, q_n}^*(f, x)$  on both sides of (3.1), we obtain

$$\begin{aligned} & [n]_{q_n} (D_{n, q_n}^*(f, x) - f(x)) \\ &= [n]_{q_n} D_{n, q_n}^*((t-x), x) f'(x) + [n]_{q_n} D_{n, q_n}^*((t-x)^2, x) \frac{f''(x)}{2} \\ & \quad + [n]_{q_n} D_{n, q_n}^*(r(t, x)(t-x)^2, x). \end{aligned}$$

In view of Remark 2.1, we have

$$\lim_{n \rightarrow \infty} [n]_{q_n} D_{n, q_n}^*((t-x), x) = ax, \tag{3.2a}$$

$$\lim_{n \rightarrow \infty} [n]_{q_n} D_{n, q_n}^*((t-x)^2, x) = x(2+x), \tag{3.2b}$$

uniformly in  $x \in [0, A]$ .

$$[n]_{q_n} D_{n, q_n}^*(r(t, x)(t-x)^2, x) \rightarrow 0$$

uniformly in  $x \in [0, A]$ , when  $n \rightarrow \infty$ . By the Cauchy-Schwartz inequality, we have

$$D_{n, q_n}^*(r(t, x)(t-x)^2, x) \leq D_{n, q_n}^*(r^2(t, x), x)^{\frac{1}{2}} D_{n, q_n}^*((t-x)^4, x)^{\frac{1}{2}}. \tag{3.3}$$

We observe that  $r^2(x, x) = 0$  and  $r^2(\cdot, x) = 0 \in C_B[0, \infty)$ . Then it follows from Theorem 3.1 that

$$\lim_{n \rightarrow \infty} D_{n, q_n}^*(r^2(t, x), x) = r^2(x, x) = 0 \tag{3.4}$$

uniformly in  $x \in [0, A]$ . Now from (3.3), (3.4) we get

$$\lim_{n \rightarrow \infty} D_{n, q_n}^*(r(t, x)(t-x)^2, x) = 0. \tag{3.5}$$

Now combining (3.1)-(3.5), we get the required result. □

Let  $\delta > 0$  and  $W_\infty^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$ . For  $f \in C_B[0, \infty)$ , we consider the following  $K$ -function:

$$K_2(f, \delta) = \inf_{g \in W_\infty^2} \{ \|f - g\| + \delta \|g''\| \}. \tag{3.6}$$

By seeing [22], there exist an absolute constant  $C > 0$  such that

$$K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \tag{3.7}$$

where

$$\omega_2(f, \sqrt{\delta}) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} |f(x+2h) - 2f(x+h) + f(x)| \tag{3.8}$$

is the second order modulus of continuity. The usual modulus of continuity for  $f \in C_B[0, \infty)$  is given by

$$\omega(f, \delta) = \sup_{0 < h \leq \sqrt{\delta}} \sup_{x \in [0, \infty)} |f(x+h) - f(x)|.$$

**Theorem 3.3.** *Let  $f \in C_B[0, \infty)$  and  $q \in (0, 1)$ , then for every  $x \in [0, \infty)$ , there exists an absolute constant  $C > 0$ , such that*

$$|D_{n,q}^*(f, x) - f(x)| \leq C\omega_2(f, \sqrt{\delta_n(x) + \eta_n^2(x)}) + \omega(f, \eta_n(x)),$$

where

$$\delta_n(x) = \left( \frac{q[n]_q^2 + [n]_q}{q^2[n-1]_q[n-2]_q} - \frac{2[n]_q}{[n-1]_q} + 1 \right) x^2 + \frac{(1+q)[n]_q}{q^2[n-1]_q[n-2]_q} x,$$

$$\eta_n^2(x) = \left( \frac{[n]_q x}{[n-1]_q} - x \right)^2.$$

*Proof.* we define

$$\overline{D}_{n,q}^*(f, x) = D_{n,q}^*(f, x) + f(x) - f\left(\frac{[n]_q x}{[n-1]_q}\right). \tag{3.9}$$

From Lemma 2.1, we obtain

$$\overline{D}_{n,q}^*(1, x) = D_{n,q}^*(1, x) = 1, \tag{3.10a}$$

$$\overline{D}_{n,q}^*(t, x) = D_{n,q}^*(t, x) + x - \frac{[n]_q x}{[n-1]_q} = x. \tag{3.10b}$$

Let  $g \in W_{\infty}^2$ , using Taylor's formula, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv,$$

We get

$$\begin{aligned} \overline{D}_{n,q}^*(g, x) - g(x) &= \overline{D}_{n,q}^*\left(\int_x^t (t-v)g''(v)dv, x\right) \\ &= D_{n,q}^*\left(\int_x^t (t-v)g''(v)dv, x\right) - \int_x^{\frac{[n]_q x}{[n-1]_q}} \left(\frac{[n]_q x}{[n-1]_q} - v\right)g''(v)dv. \end{aligned}$$

Which implies that

$$\begin{aligned} |\overline{D}_{n,q}^*(g, x) - g(x)| &\leq D_{n,q}^*\left(\left|\int_x^t (t-v)g''(v)dv\right|, x\right) + \left|\int_x^{\frac{[n]_q x}{[n-1]_q}} \left(\frac{[n]_q x}{[n-1]_q} - v\right)g''(v)dv\right| \\ &\leq D_{n,q}^*((t-x)^2, x)\|g''\| + \left(\frac{[n]_q x}{[n-1]_q} - x\right)^2 \|g''\| \\ &\leq (\delta_n(x) + \eta_n^2(x))\|g''\|. \end{aligned} \tag{3.11}$$

In view of (3.9), we obtain

$$|\overline{D}_{n,q}^*(f,x)| \leq |D_{n,q}^*(f,x)| + |f(x)| + \left| f\left(\frac{[n]_q x}{[n-1]_q}\right) \right|.$$

From Lemma 2.3, we get

$$|\overline{D}_{n,q}^*(f,x)| \leq 3\|f\| \quad \text{for all } f \in C_B[0,\infty). \tag{3.12}$$

Using (3.11), (3.12) in (3.9), we obtain

$$\begin{aligned} & |D_{n,q}^*(f,x) - f(x)| \\ & \leq \left| \overline{D}_{n,q}^*(f,x) - f(x) + f\left(\frac{[n]_q x}{[n-1]_q}\right) - f(x) \right| \\ & \leq |\overline{D}_{n,q}^*(f-g,x)| + |\overline{D}_{n,q}^*(g,x) - g(x)| + |g(x) - f(x)| + \left| f\left(\frac{[n]_q x}{[n-1]_q}\right) - f(x) \right| \\ & \leq 4\|f-g\| + (\delta_n(x) + \eta_n^2(x)\|g''\|) + \left| f\left(\frac{[n]_q x}{[n-1]_q}\right) - f(x) \right| \\ & \leq 4\|f-g\| + (\delta_n(x) + \eta_n^2(x)\|g''\| + \omega\left(f, \left|\frac{[n]_q x}{[n-1]_q} - x\right|\right)). \end{aligned}$$

Taking the infimum on the right hand side over all  $g \in W_\infty^2$  and applying (3.6) we get

$$|D_{n,q}^*(f,x) - f(x)| \leq 4K_2(f, \delta_n(x) + \eta_n^2(x)) + \omega(f, \eta_n(x)).$$

Using (3.7), we have

$$|D_{n,q}^*(f,x) - f(x)| \leq C\omega_2\left(f, \sqrt{\delta_n(x) + \eta_n^2(x)}\right) + \omega(f, \eta_n(x)).$$

This completes the proof. □

### 4 Rate of convergence

Let  $H_{x^2}$  be the set of all functions  $f$  defined on  $[0,\infty)$ , satisfying the condition  $|f(x)| \leq M_f(1+x^2)$ , where  $M_f$  is a constant depending only on  $f$ . By  $C_{x^2}[0,\infty)$ , we denote the subspace of all continuous functions belonging to  $H_{x^2}$ . Also, let  $C_{x^2}^*[0,\infty)$  be the subspace of all functions  $f \in C_{x^2}[0,\infty)$ , for which  $\lim_{x \rightarrow \infty} \frac{f(x)}{1+x^2}$  is finite. The norm on  $C_{x^2}^*[0,\infty)$  is

$$\|f\|_{x^2} = \sup_{x \in [0,\infty)} \frac{f(x)}{1+x^2}.$$

The usual modulus of continuity of  $f$  on the closed interval  $[0, a]$ ,  $a > 0$  is

$$\omega_a(f, \delta) = \sup_{|t-x| \leq \delta} \sup_{x, t \in [0, a]} |f(t) - f(x)|,$$

we know that for a function  $f \in C_{x^2}[0, \infty)$ , the modulus of continuity  $\omega_a(f, \delta)$  tends to zero.

Now we give a rate of convergence theorem for the operators  $D_{n,q}^*$ .

**Theorem 4.1.** *Let  $q \in (0, 1)$  and  $\omega_{a+1}(f, \delta)$  be its modulus of continuity on the finite interval  $[0, a+1] \subset [0, \infty)$ , where  $a > 0$ , then for every  $f \in C_{x^2}[0, \infty)$ , we have*

$$\|D_{n,q}^*(f) - f\|_{C[0, a]} \leq 6M_f(1+a^2)\delta_{n,q}(a) + 2\omega_{a+1}(f, \sqrt{\delta_{n,q}(a)}),$$

where

$$\delta_{n,q}(a) = \left( \frac{q[n]_q^2 + [n]_q}{q^2[n-1]_q[n-2]_q} - \frac{2[n]_q}{[n-1]_q} + 1 \right) a^2 + \frac{(1+q)[n]_q}{q^2[n-1]_q[n-2]_q} a.$$

*Proof.* For  $x \in [0, a]$  and  $t > a+1$ , since  $t-x > 1$ , we have

$$|f(t) - f(x)| \leq M_f(2+x^2+t^2) \leq M_f(2+3x^2+2(t-x)^2) \leq 6M_f(1+a^2)(t-x)^2. \quad (4.1)$$

For  $x \in [0, a]$  and  $t \leq a+1$ , we get

$$|f(t) - f(x)| \leq \omega_{a+1}(f, |t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{a+1}(f, \delta) \quad (4.2)$$

with  $\delta > 0$ .

From (4.1) and (4.2), we can write

$$|f(t) - f(x)| \leq 6M_f(1+a^2)(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right) \omega_{a+1}(f, \delta)$$

for  $x \in [0, a]$  and  $t > 0$ . Hence, by Schwarz's inequality we have

$$\begin{aligned} |D_{n,q}^*(f, x) - f(x)| &\leq D_{n,q}^*(|f(t) - f(x)|, x) \\ &\leq 6M_f(1+a^2)D_{n,q}^*((t-x)^2, x) + \omega_{a+1}(f, \delta) \left(1 + \frac{1}{\delta} D_{n,q}^*((t-x)^2, x)^{\frac{1}{2}}\right). \end{aligned}$$

Using Remark 2.1, for every  $q \in (0, 1)$  and  $x \in [0, a]$ , we obtain

$$\begin{aligned} |D_{n,q}^*(f, x) - f(x)| &\leq 6M_f(1+a^2)\delta_{n,q}(x) + \omega_{a+1}(f, \delta) \left(1 + \frac{1}{\delta} \sqrt{\delta_{n,q}(x)}\right) \\ &\leq 6M_f(1+a^2)\delta_{n,q}(a) + \omega_{a+1}(f, \delta) \left(1 + \frac{1}{\delta} \sqrt{\delta_{n,q}(a)}\right), \end{aligned}$$

where

$$\delta_{n,q}(x) = D_{n,q}^*((t-x)^2, x).$$

By taking

$$\delta_{n,q}(a) = \left( \frac{q[n]_q^2 + [n]_q}{q^2[n-1]_q[n-2]_q} - \frac{2[n]_q}{[n-1]_q} + 1 \right) a^2 + \frac{(1+q)[n]_q}{q^2[n-1]_q[n-2]_q} a$$

and

$$\delta = \sqrt{\delta_{n,q}(a)},$$

we get the assertion of our theorem. □

### 5 Weighted approximation

We shall discuss the weighted approximation theorems in this section, and we assume that  $\{q_n\}$  is a sequence such that  $q_n \in (0,1)$  and  $q_n \rightarrow 1$  as  $n \rightarrow \infty$ .

**Theorem 5.1.** *For each  $f \in C_{x^2}^*[0, \infty)$ , we have*

$$\lim_{n \rightarrow \infty} \|D_{n,q_n}^*(f) - f\|_{x^2} = 0.$$

*Proof.* Using the Korovkin’s theorem in [23], it is sufficient to verify the following three conditions

$$\lim_{n \rightarrow \infty} \|D_{n,q_n}^*(t^m, x) - x^m\|_{x^2} = 0, \quad m = 0, 1, 2. \tag{5.1}$$

Since

$$D_{n,q_n}^*(1, x) = 1,$$

the first condition of (5.1) is fulfilled for  $m = 0$ .

By Lemma 2.2, we have for  $n > 1$

$$\|D_{n,q_n}^*(t, x) - x\|_{x^2} = \sup_{x \in [0, \infty)} \frac{|D_{n,q_n}^*(t, x) - x|}{1+x^2} \leq \frac{q_n^{n-1}}{[n-1]_{q_n}} \sup_{x \in [0, \infty)} \frac{x}{1+x^2} \leq \frac{q_n^{n-1}}{[n-1]_{q_n}}$$

and the second condition of (5.1) holds for  $m = 1$  as  $n \rightarrow \infty$ .

Again using Lemma 2.2, we can write for  $n > 2$

$$\begin{aligned} \|D_{n,q_n}^*(t^2, x) - x^2\|_{x^2} &= \left( \frac{q_n[n]_{q_n}^2 + [n]_{q_n}}{q_n^2[n-1]_{q_n}[n-2]_{q_n}} - 1 \right) \sup_{x \in [0, \infty)} \frac{x^2}{1+x^2} \\ &\quad + \frac{(1+q_n)[n]_{q_n}}{q_n^2[n-1]_{q_n}[n-2]_{q_n}} \sup_{x \in [0, \infty)} \frac{x}{1+x^2} \\ &\leq \left( \frac{q_n[n]_{q_n}^2 + [n]_{q_n}}{q_n^2[n-1]_{q_n}[n-2]_{q_n}} - 1 \right) + \frac{(1+q_n)[n]_{q_n}}{q_n^2[n-1]_{q_n}[n-2]_{q_n}}, \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \|D_{n,q_n}^*(t^2;x) - x^2\|_{x^2} = 0.$$

Thus the proof is completed. □

Next we give the theorem to approximation all function in  $C_{x^2}[0,\infty)$  in the following. This type of result for locally integrable functions is discussed in [24].

**Theorem 5.2.** *For each  $f \in C_{x^2}[0,\infty)$  and  $\alpha > 0$ , we have*

$$\lim_{n \rightarrow \infty} \sup_{x \in [0,\infty)} \frac{|D_{n,q_n}^*(f,x) - f(x)|}{(1+x^2)^{1+\alpha}} = 0.$$

*Proof.* Let  $x_0 \in [0,\infty)$  be arbitrary but fixed, we have

$$\begin{aligned} \sup_{x \in [0,\infty)} \frac{|D_{n,q_n}^*(f,x) - f(x)|}{(1+x^2)^{1+\alpha}} &\leq \sup_{x \leq x_0} \frac{|D_{n,q_n}^*(f,x) - f(x)|}{(1+x^2)^{1+\alpha}} + \sup_{x \geq x_0} \frac{|D_{n,q_n}^*(f,x) - f(x)|}{(1+x^2)^{1+\alpha}} \\ &\leq \|D_{n,q_n}^*(f) - f\|_{C[0,x_0]} + \|f\|_{x^2} \sup_{x \geq x_0} \frac{|D_{n,q_n}^*(1+t^2;x)|}{(1+x^2)^{1+\alpha}} \\ &\quad + \sup_{x \geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}}. \end{aligned} \tag{5.2}$$

Since

$$|f(x)| \leq M_f(1+x^2),$$

we get

$$\sup_{x \geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\alpha}} \leq \sup_{x \geq x_0} \frac{M_f}{(1+x^2)^\alpha} \leq \frac{M_f}{(1+x_0^2)^\alpha}.$$

Let  $\varepsilon > 0$  be arbitrary, we choose  $x_0$  to be so large that

$$\frac{M_f}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}. \tag{5.3}$$

In view of Theorem 3.1, we obtain

$$\|f\|_{x^2} \lim_{n \rightarrow \infty} \frac{|D_{n,q_n}^*(1+t^2;x)|}{(1+x^2)^{1+\alpha}} = \|f\|_{x^2} \frac{1+x^2}{(1+x^2)^{1+\alpha}} = \frac{\|f\|_{x^2}}{(1+x^2)^\alpha} \leq \frac{\|f\|_{x^2}}{(1+x_0^2)^\alpha} < \frac{\varepsilon}{3}. \tag{5.4}$$

Now using Theorem 4.1, the first term of the inequality (5.2) implies that

$$\|D_{n,q_n}^*(f) - f\|_{C[0,x_0]} < \frac{\varepsilon}{3} \quad \text{as } n \rightarrow \infty. \tag{5.5}$$

Combining (5.3)-(5.5), we get the desired result. □

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