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Some Results on the Upper Convex Densities of the Self-Similar Sets at the Contracting-Similarity Fixed Points

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Abstract. In this paper, some results on the upper convex densities of self-similar sets at the contracting-similarity fixed points are discussed. Firstly, a characterization of the upper convex densities of self-similar sets at the contracting-similarity fixed points is given. Next, under the strong separation open set condition, the existence of the best shape for the upper convex densities of self-similar sets at the contracting-similarity fixed points is proven. As consequences, an open problem and a conjecture, which were posed by Zhou and Xu, are answered.

Key Words: Self-similar set, upper convex density, Hausdorff measure and Hausdorff dimension, contracting-similarity fixed point.

AMS Subject Classifications: 28A78, 28A80

1 Introduction and preliminaries

It is well known that the theory of Hausdorff measure is the basis of fractal geometry and Hausdorff measure is an important notion in the study of fractals (see [1,2]). But unfortunately, it is usually difficult to calculate the exact value of the Hausdorff measures of fractal sets. Since Hutchinson [3] first introduced the self-similar set satisfying the open set condition (OSC), many authors have studied this class of fractals and obtained a number of meaningful results (see [1–10]). Among them, Zhou and Feng's paper [5] has attracted widespread attention since it was published in 2004. In [5], Zhou and Feng thought the reason for the difficulty in calculating Hausdorff measures of fractals is neither computational trickiness nor computational capacity, but a lack of full understanding of the essence of Hausdorff measure. Some authors recently studied self-similar sets

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by means of upper convex density and best covering, which are very important to the study of Hausdorff measure (see [5–10]). In [5], Zhou and Feng posed eight open problems and six conjectures on Hausdorff measure of similar sets. Among them, a problem and a conjecture are as follows.

Let $E\subset R^n$ be a self-similar set satisfying OSC, the Hausdorff dimension of E be s, i.e., $\dim_H E=s$, and $x\in R^n$. Denote by $\overline{D}_C^s(E,x)$ the upper convex density of E at the point x.

Problem 1.1 (see [5]). Under what conditions is there a subset U_x in R^n with $|U_x|>0$ such that

$$\overline{D}_C^s(E,x)=\frac{H^s(E\cap U_x)}{|U_x|^s}?$$

Such a set U_x is called a best shape for the upper convex density of E at the point x.

Conjecture 1.1 (see [5]). $s=\dim_H E>1 \Rightarrow$ there is an $x\in E$ such that $\overline{D}_C^s(E,x)<1$. Furthermore, $\overline{D}_C^s(C\times C,A)<1$, where $C\times C$ denotes the Cartesian product of the middle third Cantor set with itself and A denotes any vertex of $C\times C$ (see [5, Fig. 4]).

Recently, in order to study Conjecture 1.1 above, Xu [6] and Xu and Zhou [7] introduced the notion "contracting-similarity fixed point", and obtained a sufficient and necessary condition for the upper convex density of the self-similar s-set at the simple-contracting-similarity fixed point less than 1. In [7], a conjecture was posed as follows.

Conjecture 1.2 (see [7]). Let $E\subset R^n$ be a self-similar s-set satisfying OSC. Suppose that x is a contracting-similarity fixed point of E. Then $\overline{D}_C^s(E,x)<1$ if and only if $H^s(E\cap U)<|U|^s$ for each compact subset U in R^n with $x\in U$ and $|U|>0$.

This is an important conjecture connecting Hausdorff measure and upper convex density. In Xu [6], it was shown that Conjecture 1.2 would be true if we only considered the upper convex density at the simple-contracting-similarity fixed point of a self-similar s-set satisfying the strong separation set condition (SSC), instead of the one at the contracting-similarity fixed point of a self-similar s-set satisfying OSC. In the subsequent section (i.e., Section 2), we will set up a characterization of the upper convex densities of self-similar set at the contracting-similarity fixed points. Then, under the strong separation condition (SSC), we show that the existence of the best shape for the upper convex densities of the self-similar sets at the contracting-similarity fixed points. As applications, we answer an open problem (i.e., Problem 1.1 above), which was posed by Zhou and Feng in 2004. As consequences, we prove Conjecture 1.2 does hold true in the case that SSC is satisfied, thus generalizing the corresponding the result in Xu [6]. Some definitions, notations and known results are from references [1–4].

Let d be the standard distance function on R^n, where R^n is Euclidian n-space. Denote $d(x,y)$ by $|x-y|$, $\forall x,y\in R^n$. If U is a nonempty subset of R^n, we define the diameter of U as $|U|=\sup\{|x-y|:x,y\in U\}$. Let δ be a positive number. If $E\subset\bigcup_i U_i$ and $0<|U_i|\le\delta$ for each i, we say that $\{U_i\}$ is a δ-covering of E.

Let $E\subset R^n$ and $s\geq 0$. For $\delta>0$, define

$$H^s_\delta(E)=\inf\Big\{\sum_i |U_i|^s:\bigcup_i U_i\supset E,\ 0<|U_i|\leq\delta\Big\}. \tag{1.1}$$

Letting $\delta\to 0$, we call the limit

$$H^s(E)=\lim_{\delta\to 0}H^s_\delta(E)$$

the s-dimensional Hausdorff measure of E. Note that the Hausdorff dimension of E is defined as

$$\dim_H E=\inf\{s\geq 0: H^s(E)=0\}=\sup\{s\geq 0: H^s(E)=\infty\}.$$

An H^s-measurable set $E\subset R^n$ with $0<H^s(E)<\infty$ is termed an s-set.

Let E be an s-set in R^n, and $x\in R^n$. The upper convex density of E at x is defined as (see [1])

$$\overline{D}^s_C(E,x)=\limsup_{\rho\to 0}\Big\{\frac{H^s(E\cap U)}{|U|^s}: U \text{ convex in } R^n,\ x\in U,\ 0<|U|\leq\rho\Big\}. \tag{1.2}$$

Note that the above supremum may just be taken over all subsets U in R^n with $x\in U$ and $0<|U|\leq\rho$.

Now we review the self-similar s-set satisfying the open set condition. Let $D\subset R^n$ be closed. A map $S:D\to D$ is called a contracting similarity, if there is a number c with $0<c<1$ such that

$$|S(x)-S(y)|=c|x-y|,\quad \forall x,y\in D,$$

where c is called the similar ratio. It was proved by Hutchinson (see [1]) that given $m\geq 2$ and contracting similarities $S_i:D\to D$, $(i=1,2,\cdots,m)$ with similarity ratios c_i there exists a unique nonempty compact set $E\subset R^n$ satisfying

$$E=\bigcup_{i=1}^m S_i(E).$$

The set E is called the self-similar s-set for the iterated function system (IFS) $\{S_1,\cdots,S_m\}$, here we assume that there is a bounded nonempty open set V such that

$$\bigcup_{i=1}^m S_i(V)\subset V,$$

and

$$S_i(V)\cap S_j(V)=\emptyset,\quad i\neq j,\quad j=1,2,\cdots,m,$$

which is often referred to as the open set condition (OSC). In this case, we know that $0<H^s(E)<\infty$ and so E is an s-set. Furthermore, we call E satisfying the strong separation condition (SSC), if E meets with OSC and satisfies

$$S_i(E)\cap S_j(E)=\emptyset,\quad i\neq j,\quad j=1,2,\cdots,m.$$

Denote by J_k the set of all k-sequences $(i_1,\cdots,i_k)$, where $1\le i_1,\cdots,i_k\le m$, $k\ge 1$ and put $E_{i_1\cdots i_k}=S_{i_1}\circ\cdots\circ S_{i_k}(E)$, which is referred to as k-contracting-copy of E. Obviously, $\forall(i_1,\cdots,i_k)\in J_k$, we have

$$|E_{i_1\cdots i_k}|^s=|S_{i_1}\circ\cdots\circ S_{i_k}(E)|^s=c_{i_1}\cdots c_{i_k}|E|^s.$$

It is not hard to see that for each $k\ge 1$,

$$E=\bigcup_{J_k}E_{i_1\cdots i_k}. \tag{1.3}$$

Definition 1.1 (see [6,7]). Let $E\subset R^n$ be self-similar s-set for the IFS $\{S_1,\cdots,S_m\}$ satisfying OSC and $x\in E$. x is called a contracting-similarity fixed point of E, if there are a positive integer k and $(i_1,\cdots,i_k)\in J_k$ such that

$$S_{i_1}\circ\cdots\circ S_{i_k}(x)=x.$$

Note that here x is alternatively called the contracting-similarity fixed point of E associated with $S_{i_1}\circ\cdots\circ S_{i_k}$.

In particular, x is called a simple-contracting-similarity fixed point of E, if there is $i\in\{1,\cdots,m\}$ such that $S_i(x)=x$.

Definition 1.2 (see [5–7]). Let E be an s-set in R^n. A sequence $\{U_i\}$ of subsets in R^n is called an H^s-a.e. covering of E, if there is an H^s-measurable subset E_0 in R^n with $E_0\subset E$ and $H^s(E_0)=0$ such that $\{U_i\}$ is a covering of $E-E_0$ (i.e., $E-E_0\subset\bigcup_i U_i$). A sequence $\{U_i\}$ of subsets in R^n is called a best H^s-a.e. covering of E if it is an H^s-a.e. covering of E such that

$$H^s(E)=\sum_i|U_i|^s.$$

Note that a best H^s-a.e. covering may be alternatively called a best almost covering or an optimal almost covering (see [10]).

A family $\{U_i\}_{i\ge 1}$ of subsets in R^n is called a best covering of E (see [5]), if it is a covering of E with $|U_i|>0$ $(\forall i\ge 1)$ such that

$$H^s(E)=\sum_{i\ge 1}|U_i|^s.$$

Note that a sequence $\{U_i\}$ of H^s-measurable subsets in R^n is an H^s-a.e. covering of E if and only if $H^s(E-\bigcup_i U_i)=0$.

Let M and N be both nonempty subsets in R^n. Then the Euclidian metric between M and N is defined as follows:

$$\mathrm{dist}(M,N)=\inf\{|x-y|:x\in M,\ y\in N\}.$$

Let A and B be both nonempty compact sets in R^n. Recall that the definition of Hausdorff metric is as follows:

$$h(A,B)=\inf\{\varepsilon : A\subset N_\varepsilon(B),\ B\subset N_\varepsilon(A)\},$$

where

$$N_\varepsilon(A)=\{x\in R^n : \exists\, y\in A, \text{ such that } |y-x|<\varepsilon\},$$

which is called the open ε-neighborhood of A.

2 Main results

In this section, we first give a characterization of the upper convex densities of the self-similar sets at the contracting-similarity fixed points. As applications, we prove that under the strong separation open set condition, there exists a best shape for the upper convex density of the self-similar set at any contracting-similarity fixed points.

Proposition 2.1. Let $E\subset R^n$ be an self-similar s-set satisfying OSC and $x\in R^n$, then the upper convex density of E at x is equivalently as

$$\overline{D}_C^s(E,x)=\limsup_{\rho\to 0}\Big\{\frac{H^s(E\cap U)}{|U|^s} : U \text{ closed in } R^n,\ x\in U,\ 0<|U|\le\rho\Big\}. \tag{2.1}$$

Proof. It is easily shown that (2.1) is true by means of the definition of $\overline{D}_C^s(E,x)$ as (1.2). So, the proof is completed. □

Theorem 2.1. *Let $E\subset R^n$ be an self-similar s-set satisfying OSC. Suppose that $x\in E$ is any contracting-similarity fixed point of E, then*

$$\overline{D}_C^s(E,x)=\sup\Big\{\frac{H^s(E\cap U)}{|U|^s} : U \textit{ closed in } R^n,\ x\in U,\ |U|>0\Big\}. \tag{2.2}$$

Proof. Write

$$D=\sup\Big\{\frac{H^s(E\cap U)}{|U|^s} : U \text{ closed in } R^n,\ x\in U,\ |U|>0\Big\}. \tag{2.3}$$

It suffices to show that $\overline{D}_C^s(E,x)\ge D$, since we can easily see that $\overline{D}_C^s(E,x)\le D$ by (2.1) and (2.3). In fact, suppose that there are a positive integer k and $(i_1,\cdots,i_k)\in J_k$ such that

$$S_{i_1}\circ\cdots\circ S_{i_k}(x)=x.$$

Then, for any closed subset U in R^n with $x\in U$ and $|U|>0$, there exists a sequence $\{V_j\}_{j=1}^\infty$ defined by

$$V_k=(S_{i_1}\circ\cdots\circ S_{i_k})^j(U),$$

such that $x\in V_j$, $|V_j|>0$ and

$$\begin{aligned}\frac{H^s(E\cap V_j)}{|V_j|^s}&\geq\frac{H^s((S_{i_1}\circ\cdots\circ S_{i_k})^j(E))\cap H^s((S_{i_1}\circ\cdots\circ S_{i_k})^j(U))}{|(S_{i_1}\circ\cdots\circ S_{i_k})^j(U)|^s}\\&=\frac{(c_{i_1}\cdots c_{i_k})^{js}H^s(E\cap U)}{(c_{i_1}\cdots c_{i_k})^{js}|U|^s}=\frac{H^s(E\cap U)}{|U|^s}\end{aligned}$$

for all $j\geq 1$. Moreover, we also see $|V_j|\to 0$ as $j\to\infty$ since

$$0\leq|V_j|=|(S_{i_1}\circ\cdots\circ S_{i_k})^j(U)|=(c_{i_1}\cdots c_{i_k})^j|U|\leq c^j|E|\to 0,\quad j\to\infty,$$

where $c=\max\{c_{i_1},\cdots,c_{i_k}\}<1$. Thus, by the definition of upper convex density and (2.1) and (2.3), we have

$$\overline{D}_C^s(E,x)\geq\limsup_{j\to\infty}\frac{H^s(E\cap V_j)}{|V_j|^s}\geq D,\tag{2.4}$$

which is the desired result. □

The following theorem is obvious according to the definitions of upper convex density and contracting-similarity fixed point. So we omit its proof.

Theorem 2.2. *Let $E\subset R^n$ be an self-similar s-set satisfying OSC. Suppose that $x\in E$ is any contracting-similarity fixed point of E, then*

$$\overline{D}_C^s(E,x)=\sup\Big\{\frac{H^s(U)}{|U|^s}:U\text{ compact in }R^n,\ x\in U,\ U\subset E,\ |U|>0\Big\}.\tag{2.5}$$

By using Theorem 2.1, we can easily deduce the following corollary. We omit its proof.

Corollary 2.1. Let $E\subset R^n$ be an self-similar s-set satisfying OSC. Suppose that $x\in E$ is any contracting-similarity fixed point of E, then

$$\overline{D}_C^s(E,x)\geq\frac{H^s(E\cap U)}{|U|^s}\tag{2.6}$$

for any subset U in R^n with $x\in U$ and $|U|>0$.

Remark 2.1. If we take $E=C\times C$ and $x=A=(0,1)$ in Corollary 2.1, then we see that Corollary 2.1 is reduced to [13, Lemma 5.1].

Now we will use the main results to deduce the existence of the best shape for the upper convex densities of the self-similar sets satisfying SSC at the contracting-similarity fixed points.

Lemma 2.1. *Let $E \subset R^n$ be a self-similar s-set yielded by contracting similarities $S_1,\cdots,S_m$, satisfying SSC and $x\in E$. Suppose that x is a contracting-similarity fixed point of E, then there is a real number ε with $0<\varepsilon<|E|$ such that*

$$\overline{D}^s_C(E,x)=\sup\Big\{\frac{H^s(U)}{|U|^s}: U \text{ compact in } R^n,\ U\subset E,\ x\in U,\ |U|\geq\varepsilon\Big\}. \tag{2.7}$$

Proof. Since $x\in E$ is a contracting-similarity fixed point of E, by Definition 1.1, there are a positive integer k and $(i_1,\cdots,i_k)\in J_k$ such that

$$S_{i_1}\circ\cdots\circ S_{i_k}(x)=x. \tag{2.8}$$

Set

$$\begin{aligned}\varepsilon=\min\{&\mathrm{dist}(S_{j_1}\circ\cdots\circ S_{j_k}(E),S_{j_1}\circ\cdots\circ S_{j_k}(E)):(j_1,\cdots,j_k),(l_1,\cdots,l_k)\in J_k,\\ &(j_1,\cdots,j_k)\neq(l_1,\cdots,l_k)\},\end{aligned} \tag{2.9a}$$

$$l=\sup\Big\{\frac{H^s(U)}{|U|^s}: U \text{ compact in } R^n,\ U\subset E,\ x\in U,\ |U|\geq\varepsilon\Big\}, \tag{2.9b}$$

$$L=\sup\Big\{\frac{H^s(U)}{|U|^s}: U \text{ compact in } R^n,\ U\subset E,\ x\in U,\ |U|>0\Big\}. \tag{2.9c}$$

By Theorem 2.2, it suffices to prove $l\geq L$, since $l\leq L$ is obviously seen by (2.9b) and (2.9c). By the property of SSC of E, for any compact set V in R^n with $V\subset E$, if $|V|>0$, $|V|<\varepsilon$ and $x\in V$, then $V\subset S_{i_1}\circ\cdots\circ S_{i_k}(E)$. Set

$$p=\max\{t: V\subset(S_{i_1}\circ\cdots\circ S_{i_k})^t(E)\}.$$

It is easy to see that $p\geq 1$. Thus, we get

$$V\subset(S_{i_1}\circ\cdots\circ S_{i_k})^p(E),$$

but the following

$$V\subset(S_{i_1}\circ\cdots\circ S_{i_k})^{p+1}(E)$$

does not hold. Now write $g=(S_{i_1}\circ\cdots\circ S_{i_k})^p$, then we have

$$\frac{H^s(V)}{|V|^s}=\frac{(c_{i_1}\cdots c_{i_k})^{-ps}}{(c_{i_1}\cdots c_{i_k})^{-ps}}\cdot\frac{H^s(V)}{|V|^s}=\frac{H^s(g^{-1}(V))}{|g^{-1}(V)|^s}, \tag{2.10}$$

where c_{i_j} is the contracting ratio of S_{i_j} $(1\leq i_j\leq m,\ j=1,2,\cdots,k)$. Since $x\in g^{-1}(V)\cap S_{i_1}\circ\cdots\circ S_{i_k}(E)$, $g^{-1}(V)$ is compact in R^n, $g^{-1}(V)\subset E$, and $g^{-1}(V)\subset S_{i_1}\circ\cdots\circ S_{i_k}(E)$ does not hold, the inequality $|g^{-1}(V)|\geq\varepsilon$ does hold and therefore it follows from (2.9b), (2.9c) and (2.10) that $l\geq L$, as required. ∎

Lemma 2.2 (see [7]). *Let $\{A_n\}$ be a sequence of nonempty compact subsets in R^n and $A\subset R^n$. Suppose that $\{A_n\}$ converges to A in Hausdorff metric. Then we have*

(i) $\lim_{n\to\infty}|A_n|=|A|$.

(ii) if there is a compact subset F in R^n such that $A_n\subset F$ for all $n\geq 1$, then $A\subset F$.

(iii) $\limsup_{n\to\infty}H^s(A_n)\leq H^s(A)$ *if* $0<H^s(A_n)<+\infty$.

(iv) if $x\in A_n$ for all $n\geq 1$, then $x\in A$.

Now we begin to show the main result in this section.

Theorem 2.3. *Let $E\subset R^n$ be a self-similar s-set satisfying SSC. Suppose that x is a contracting-similarity fixed point of E. Then there is a compact set U_x in R^n with $x\in U_x$, $|U_x|>0$ and $U_x\subset E$ such that*

$$\overline{D}_C^s(E,x)=\frac{H^s(U_x)}{|U_x|^s}. \tag{2.11}$$

That is, there exists a best shape U_x for the upper convex density $\overline{D}_C^s(E,x)$.

Proof. By Lemma 2.1, there exist a real positive number $\varepsilon>0$ and a sequence $\{U_i\}$ of compact subsets in R^n with $x\in U_i$, $U_i\subset E$ and $|U_i|\geq\varepsilon>0$ ($\forall i\geq 1$) such that

$$\frac{H^s(U_i)}{|U_i|^s}\to\overline{D}_C^s(E,x)\quad\text{as } i\to\infty. \tag{2.12}$$

Thus, there is a subsequence $\{U_{i_k}\}\subset\{U_i\}$ such that $\{U_{i_k}\}$ converges to a compact set U_x in R^n in Hausdorff metric (see [11]). Without loss of generality, assume that $\{U_i\}$ converges to the compact set U_x in Hausdorff metric. Hence, by Lemma 2.2, we see $U_x\subset E$, $|U_x|\geq\varepsilon>0$, $x\in U_x$ and

$$\limsup_{i\to\infty}H^s(U_i)\leq H^s(U_x).$$

So, by Corollary 2.1, we obtain the chain of inequalities

$$\overline{D}_C^s(E,x)\geq\frac{H^s(U_x\cap E)}{|U_x|^s}=\frac{H^s(U_x)}{|U_x|^s}\geq\frac{\limsup_{i\to\infty}H^s(U_i)}{\lim_{i\to\infty}|U_i|^s}=\lim_{j\to\infty}\frac{H^s(U_{i_j})}{|U_{i_j}|^s}=\overline{D}_C^s(E,x),$$

where $\{U_{i_j}\}$ is some subsequence of $\{U_i\}$. This shows that $H^s(U_x)=|U_x|^s$ as required. ☐

By Theorem 2.3, we immediately obtain the following two corollaries, so we omit their proofs.

Corollary 2.2. Let $E\subset R^n$ be a self-similar s-set satisfying SSC. Suppose that x is a contracting-similarity fixed point x of E. Then $\overline{D}_C^s(E,x)=1$ if and only if there is a compact subset U in R^n with $x\in U$, $|U|>0$ and $U\subset E$ such that $H^s(U)=|U|^s$.

Corollary 2.3. Let $E\subset R^n$ be a self-similar s-set satisfying SSC. Suppose that x is a contracting-similarity fixed point x of E. Then $\overline{D}^s_C(E,x)<1$ if and only if $H^s(U)<|U|^s$ for each compact set U in R^n with $x\in U$, $|U|>0$ and $U\subset E$.

Remark 2.2. Theorem 2.3 gives an answer to Problem 1.1, which was posed by Zhou and Feng in 2004.

Remark 2.3. Corollary 2.2 and Corollary 2.3 generalize Theorem 4.1 and Corollary 4.2 in Xu [6], respectively. In addition, Corollary 2.3 shows that Conjecture 1.2, which was posed by Xu and Zhou in 2005, is true in the case that $E\subset R^n$ is a self-similar s-set satisfying SSC.

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References

[1] K. J. Falconer, The Geometry of Fractal Set, Cambridge University Press, 1985.
[2] K. J. Falconer, Fractal Geometry, Wiley, New York, 1990.
[3] T. E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J., 30 (1981), 713–747.
[4] Z. Zhou, The Hausdorff measure of the self-similar sets-The Koch curve, Scientia Sinica Ser. A, 41 (1998), 723–728.
[5] Z. Zhou and L. Feng, Twelve open problems on the exact value of the Hausdorff measure and on topological entropy: a brief survey of recent results, Nonlinearity, 17 (2004), 493–502.
[6] S. Xu, Connecting Hausdorff measure and upper convex density or H^s-a.e. covering, J. Math. Anal. Appl., 311 (2005), 324–337.
[7] S. Xu and Z. Zhou, On the Hausdorff measure of the self-similar sets satisfying the strong separation open set condition, Adv. Math., 35 (2005), 545–552.
[8] B. Jia and S. Xu, An answer to a conjecture on self-similar sets, Anal. Theory Appl., 23 (2007), 9–15.
[9] J. Yin, An negative answer to a conjecture on self-similar sets with open set condition, Anal. Theory Appl., 25 (2009), 182–200.
[10] E. Ayer and R. S. Strichartz, Exact Hausdorff measure and intervals of maximum density for Cantor set, Trans. Amer. Math. Soc., 351 (1999), 3725–3741.
[11] G. T. Whyburn, Topological Analys, Princeton University Press, 1958.
[12] M. Llorente and M. Moran, Self-similar sets with optimal coverings and packings, J. Math. Anal. Appl., 334 (2007), 1088–1095.
[13] Z. Zhou and L. Feng, The minimym of the upper convex density of the product of the Cantor set with itself, Nonlinear Anal. Anal., 68 (2008), 3439–3444.