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Construction Theory of Function on Local Fields

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Abstract. We establish the construction theory of function based upon a local field K_p as underlying space. By virture of the concept of pseudo-differential operator, we introduce "fractal calculus" (or, p-type calculus, or, Gibbs-Butzer calculus). Then, show the Jackson direct approximation theorems, Bermstein inverse approximation theorems and the equivalent approximation theorems for compact group $D(\subset K_p)$ and locally compact group $K_p^+(=K_p)$, so that the foundation of construction theory of function on local fields is established. Moreover, the Jackson type, Bernstein type, and equivalent approximation theorems on the Hölder-type space $C^\sigma(K_p)$, $\sigma>0$, are proved; then the equivalent approximation theorem on Sobolev-type space $W_\sigma^r(K_p)$, $\sigma\geq 0$, $1\leq r<+\infty$, is shown.

Key Words: Construction theory of function, local field, fractal calculus, approximation theorem, Hölder-type space.

AMS Subject Classifications: 41A65, 28A20

1 Concept and notation

A local field K_p is a locally compact, non-trivial, totally disconnected, non-Archimedean norm valued, T_2-type, complete topological field [1]. It can be a p-series field, or its finite algebraic extension field (with addition $+$, multiplication $\times$, term by term, mod p, and no carrying); or can be a p-adic filed, or its finite algebraic extension field (with $+$, $\times$, term by term, mod p, carrying from left to right), with $p\geq 2$ prime.

This kind of fields has important theoretical and applied meaning, for example, the dyadic system in the computer science, and switch functions in physics science, they are special cases of local fields at $p=2$.

We concern the cases of p-series field and p-adic field, denoted by $K_p\equiv(K_p,+,\times)$, and call them local fields. For the algebraic extension of K_p, denoted by K_q, $q=p^c$, $c\in\mathbb{N}$, we refer to [1].

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1.1 Haar measure and Haar integral on a local field K_p

The addition group $K_p^+ \equiv K_p$ of a local field $K_p \equiv (K_p,+,\times)$ is a locally compact group, there exist the Haar measure and Haar integral with invariance of translation. The Haar measure of a Haar measurable subset $A \subset K_p$ is denoted by $|A|$; the Haar integral of a Haar measurable function $f: K_p \to \mathbb{C}$ is denoted by $\int_{K_p} f(x)dx$.

1.2 Non-Archimedean valued norm $|x|$ on K_p

If a mapping $|x|: K_p \to [0,+\infty)$ satisfies: (1) $|x| \geq 0$, $|x| = 0 \Leftrightarrow x = 0$; (2) $|x \times y| = |x||y|$; (3) $|x+y| \leq \max\{|x|,|y|\}$; Then $|x|$ is said to be a non-Archimedean valued norm of $x \in K_p$.

There exists an element $\beta \in K_p$ with $|\beta| = p^{-1}$ in K_p, called prime element. $\forall x \in K_p$ can be expressed as

$$x = x_{-l}\beta^{-l} + x_{-l+1}\beta^{-l+1} + \cdots + x_{-1}\beta^{-1} + x_0\beta^0 + x_1\beta^1 \cdots, \tag{1.1}$$

where $x_j \in \{0,1,\cdots,p-1\}$, $j = -l,-l+1,\cdots$, $l \in \mathbb{Z}$.

Each $x \in K_p$ in p-series field or in p-aidc field can be expressed as the form in (1.1), the difference is: the operations in p-series field are term by term, mod p, no carrying; whereas, the operations in p-aidc field are term by term, mod p, carrying from left to right.

The range of non-Archimedean valued norm is $|x| \in \{p^{-k} : k \in \mathbb{Z}\}$.

1.3 Important subsets in K_p

(1) Compact group in K_p (ring of integers): $D = \{x \in K_p : |x| \leq 1\}$, it is a unique maximal compact subring in K_p, and is an open, closed, compact subset with Haar measure $|D| = 1$.

(2) Unit open ball in K_p (prime ideal): $B = \{x \in K_p : |x| < 1\}$, it is a unique maximal ideal in D, also principle ideal, prime ideal; and is open, closed, compact subset with Haar measure $|B| = p^{-1}$.

(3) Ball in K_p (fractional ideal): $B^k = \{x \in K_p : |x| \leq p^{-k}\}$, $k \in \mathbb{Z}$, it is a ball in K_p with center $0 \in K_p$ and radius p^{-k}; and is open, closed, compact subset with Haar measure $|B^k| = p^{-k}$, $k \in \mathbb{Z}$.

(4) Base for neighborhood system of $0 \in K_p : \{B^k \subset K_p : k \in \mathbb{Z}\}$ satisfies $B^{k+1} \subset B^k$, $k \in \mathbb{Z}$; $K_p = \bigcup_{k=-\infty}^{+\infty} B^k$, $\{0\} = \bigcap_{k=-\infty}^{+\infty} B^k$; the set of all p-coset representatives of B^1 in D is $D/B^1 = \{0 \times \beta^0 + B^1,\ 1 \times \beta^0 + B, \cdots, (p-1) \times \beta^0 + B^1\}$, it is isomorphic with the finite Galois field $D/B^1 \overset{iso.}{\longleftrightarrow} GF(p)$.

(5) Character group of $K_p : \Gamma_p = \{\chi : K_p \to \mathbb{C},\ \chi(x_1+x_2) = \chi(x_1)\chi(x_2);\ |\chi(x)| = 1\}$ is the character group of K_p, it is a locally compact group, and $\Gamma_p \overset{iso.}{\longleftrightarrow} K_p$.

The annihilators in Γ_p are $\Gamma^k=\{\chi\in\Gamma_p:\forall x\in B^k\Rightarrow\chi(x)=1\}$, $k\in\mathbb{Z}$, with $|\Gamma^k|=p^k$.

The base for neighborhood system of unit $I\in\Gamma_p$ is $\{\Gamma^k\subset\Gamma_p:k\in\mathbb{Z}\}$, satisfies $\Gamma^{k+1}\supset\Gamma^k,k\in\mathbb{Z}$; $\Gamma_p=\bigcup_{k=-\infty}^{+\infty}\Gamma^k$, $\{I\}=\bigcap_{k=-\infty}^{+\infty}\Gamma^k$; $\Gamma^k\overset{iso.}{\longleftrightarrow}B^{-k}$, $k\in\mathbb{Z}$.

Theorem 1.1 (see [1,2]). *(i) Let S,T be two balls in local field K_p, then either S and T are disjoint, or one ball contains the other one; (ii) Any ball in K_p has multi-centers; (iii) Any ball in K_p is open, closed, and compact.*

2 Test function class and distribution space on local fields

We introduce the test function class (Schwartz type space) and the distribution space of Schwartz type space on a local field K_p.

2.1 Test function class (Schwartz type space) $\mathbb{S}(K_p)$

The space

$$\mathbb{S}(K_p)=\Big\{\varphi:K_p\to\mathbb{C},\ \varphi(x)=\sum_{j=1}^{n}c_j\tau_{h_j}\Phi_{B^{k_j}}(x),\ c_j\in\mathbb{C},\ h_j\in K_p,\ k_j\in\mathbb{Z}\Big\}$$

is said to be a test function class, or Schwartz type space, where

$$\Phi_{B^{k_j}}(x)=\begin{cases}1, & x\in B^{k_j},\\ 0, & x\notin B^{k_j},\end{cases}$$

is the characteristic function of B^{k_j}; $\tau_{h_j}\Phi_{B^{k_j}}(x)$ is translation (for h_j) of $\Phi_{B^{k_j}}(x)$, $j=1,2,\cdots,n$.

2.1.1 The topology of $\mathbb{S}(K_p)$

The null sequence $\{\varphi_n(x))\}_{n=1}^{+\infty}\subset\mathbb{S}(K_p)$ is defined by

1. $\forall\varphi_n$ exists the same index pair (the index pair of a $\varphi\in\mathbb{S}(K_p):\exists(k,l)\in\mathbb{Z}\times\mathbb{Z}$, st. (i) φ is constant on the coset of B^k; (ii) supp $\varphi=B^l$; denoted by the index $\varphi=(k,l)$).
2. $\lim_{n\to+\infty}\varphi_n(x)=0$, $x\in K_p$, uniformly.

With the above topology, $\mathbb{S}(K_p)$ becomes a complete, separated, T_2-type, semi-normed topological linear space.

2.1.2 The important properties of $\mathbb{S}(K_p)$

1. $\mathbb{S}(K_p)$ is an algebra of continuous functions with compact support, consisted of finite linear combinations of translations of characteristic fucntions of balls, which distinguishes points, and $\overline{\mathbb{S}(K_p)}=C(K_p)$.

2. The Fourier transformation
$$\varphi^{\wedge}(\xi)=\int_{K_p}\varphi(x)\overline{\chi}_{\xi}(x)dx,\quad \xi\in\Gamma_p,$$
is topological isomorphic from $\mathbb{S}(K_p)$ onto $\mathbb{S}(\Gamma_p)$. The inverse Fourier transformation
$$\psi^{\vee}(x)=\int_{K_p}\psi(\xi)\chi_x(\xi)d\xi,\quad x\in K_p,$$
is topological isomorphic from $\mathbb{S}(\Gamma_p)$ onto $\mathbb{S}(K_p)$. And $(\varphi^{\wedge})^{\vee}(x)=\varphi(x)$, $\forall\varphi\in\mathbb{S}(K_p)$, $x\in K_p$ holds; as well as $(\psi^{\vee})^{\wedge}(\xi)=\psi(\xi)$, $\forall\psi\in\mathbb{S}(\Gamma_p)$, $\xi\in\Gamma_p$ holds.

2.2 Schwartz distribution space $\mathbb{S}^*(K_p)$

The space $\mathbb{S}^*(K_p)=\{u:\text{ continuous linear functional on }\mathbb{S}(K_p)\}$, endowed with w^*-topology, it becomes a complete, separated, T_2-type, topological linear space.

The Fourier transformation of $T\in\mathbb{S}^*(K_p)$ (in distribution sense) is defined as a distribution $T^{\wedge}\in\mathbb{S}^*(\Gamma_p)$ satisfies
$$\langle T^{\wedge},\varphi\rangle=\langle T,\varphi^{\wedge}\rangle,\quad \forall\varphi\in\mathbb{S}(K_p).$$
The inverse Fourier transformation of $S\in\mathbb{S}^*(\Gamma_p)$ (in distribution sense) is defined as a distribution $S^{\vee}\in\mathbb{S}^*(K_p)$ satisfies
$$\langle S^{\vee},\psi\rangle=\langle S,\psi^{\vee}\rangle,\quad \forall\psi\in\mathbb{S}(\Gamma_p).$$
Thus, the Fourier transformation in the distribution sense $\wedge:T\to T^{\wedge}$ is a bounded linear operator; and it is isomorphic from $\mathbb{S}^*(K_p)$ onto $\mathbb{S}^*(\Gamma_p)$. And hold $(T^{\wedge})^{\vee}=T$, $\forall T\in\mathbb{S}^*(K_p)$; $(S^{\vee})^{\wedge}=S$, $\forall S\in\mathbb{S}^*(\Gamma_p)$.

3 Fractal calculus on local fields

By virtue of the pseudo-differential operator, we define a kind of new calculus on local fields, called fractal calculus (or p-type calculus, since the p of "pseudo"; or Gibbs-Butzer calculus) [2–6].

3.1 Symbol class $S^{\alpha}_{\rho\delta}(K_p)\equiv S^{\alpha}_{\rho\delta}(K_p\times\Gamma_p)$, $\alpha\in\mathbb{R}$, $\rho\geq 0$, $\delta\geq 0$ on K_p

The space $S^{\alpha}_{\rho\delta}(K_p)=\{\sigma(x,\xi):K_p\times\Gamma_p\to\mathbb{C}$, with $(i),(ii)\}$, $\alpha\in\mathbb{R}$, $\rho\geq 0$, $\delta\geq 0$, where (i) $\exists c>0$, c is a constant, st. $|\sigma(x,\xi)|\leq c\langle\xi\rangle^{\alpha}$, $x\in K_p$, $0\neq\xi\in\Gamma_p$, $\langle\xi\rangle=\max\{1,|\xi|\}$; (ii) $\forall(\mu,\nu)\in\mathbb{P}\times\mathbb{P}$ holds: $\exists c_{\mu\nu}>0$, st. $|\Delta^x_h\Delta^{\xi}_{\zeta}\sigma(x,\xi)|\leq c_{\mu\nu}|h|^{\mu}|\zeta|^{\nu}\langle\xi\rangle^{\alpha+\delta\mu-\rho\nu}$, $x,h\in K_p$, $\xi,\zeta\in\Gamma_p$, $\xi\neq 0$; $\exists c_{\mu}>0$, st. $|\Delta^x_h\sigma(x,\xi)|\leq c_{\mu\nu}|h|^{\mu}\langle\xi\rangle^{\alpha+\delta\mu}$, $x,h\in K_p$, $\xi\neq 0$; $\exists c_{\nu}>0$, st. $|\Delta^{\xi}_{\zeta}\sigma(x,\xi)|\leq c_{\nu}|\zeta|^{\nu}\langle\xi\rangle^{\alpha-\rho\nu}$, $\xi,\zeta\in\Gamma_p$, $|\zeta|<\langle\xi\rangle$, $\xi\neq 0$, here $\Delta^x_h\Delta^{\xi}_{\zeta}\sigma(x,\xi)$ is the second order difference, $\Delta^x_h\sigma(x,\xi)$, $\Delta^{\xi}_{\zeta}\sigma(x,\xi)$ are first order differences. The set $S^{\alpha}_{\rho\delta}(K_p)\equiv S^{\alpha}_{\rho\delta}(K_p\times\Gamma_p)$ defined on $K_p\times\Gamma_p$ is said to be symbol class on K_p; $\sigma\in S^{\alpha}_{\rho\delta}(K_p)$ is said to be a symbol.

3.2 Pseudo-differential operator $T_{\sigma(x,D)}$ with symbol $\sigma\in S^{\alpha}_{\rho\delta}(K_p)$ on K_p

Let $f:K_p\to\mathbb{C}$ be a Haar measurable function on K_p. Denote by

$$T_{\sigma(x,D)}f(x)\equiv\int_{\Gamma_p}\int_{K_p}\sigma(x,\xi)f(t)\overline{\chi}_{\xi}(t-x)dtd\xi,\quad x\in K_p,$$

we call $T_{\sigma(x,D)}$ a pseudo-differential operator on K_p with the symbol $\sigma\in S^{\alpha}_{\rho\delta}(K_p)$.

For a function $\varphi\in\mathbb{S}(K_p)$ in Schwartz-type space $\mathbb{S}(K_p)$, the action of pseudo-differential operator $T_{\sigma(x,D)}$ on it can be denoted by

$$T_{\sigma(x,D)}\varphi(x)=\int_{\Gamma_p}\sigma(x,\xi)\varphi^{\wedge}(\xi)\chi_x(\xi)d\xi=[\sigma(x,\cdot)\varphi^{\wedge}(\cdot)]^{\vee}(x),\quad x\in K_p,$$

since the Fourier transformation $\wedge:\mathbb{S}(K_p)\to(\Gamma_p)$ is isomorphic from $\mathbb{S}(K_p)$ onto $\mathbb{S}(\Gamma_p)$.

3.3 Fractal calculus on K_p

The pseudo-differential operator with symbol $\sigma=\langle\xi\rangle^{\alpha}\in S^{\alpha}_{\rho\delta}(K_p)$ acting on Haar measurable function $f:K_p\to\mathbb{C}$ is denoted by

$$T_{\langle\cdot\rangle^{\alpha}}f(x)=\int_{\Gamma_p}\int_{K_p}\langle\xi\rangle^{\alpha}f(t)\overline{\chi}_{\xi}(t-x)dtd\xi,\quad x\in K_p.$$

Fractal derivative — for $\alpha>0$, if the integral

$$T_{\langle\cdot\rangle^{\alpha}}f(x)=\int_{\Gamma_p}\int_{K_p}\langle\xi\rangle^{\alpha}f(t)\overline{\chi}_{\xi}(t-x)dtd\xi,\quad x\in K_p,$$

exists, then $T_{\langle\cdot\rangle^{\alpha}}f(x)$ is said to be an α-order point-wise fractal derivative of $f(x)$ at x, denoted by $f^{\langle\alpha\rangle}(x)=T_{\langle\cdot\rangle^{\alpha}}f(x)$.

Fractal integral — for $\alpha>0$, if the integral

$$T_{\langle\cdot\rangle^{-\alpha}}f(x)=\int_{\Gamma_p}\int_{K_p}\langle\xi\rangle^{-\alpha}f(t)\overline{\chi}_{\xi}(t,x)dtd\xi,\quad x\in K_p,$$

exists, then $T_{\langle\cdot\rangle^{-\alpha}}f(x)$ is said to be an α-order point-wise fractal integral of $f(x)$ at x, denoted by $f_{\langle\alpha\rangle}(x)=T_{\langle\cdot\rangle^{-\alpha}}f(x)$.

Similarly, the α-order L^r-strong fractal derivative of $f(x)$ and the α-order L^r-strong fractal integral of $f(x)$ can be defined, i.e., the strong limits of $T_{\langle\cdot\rangle^{\alpha}}f(x)$ and $T_{\langle\cdot\rangle^{-\alpha}}f(x)$ in $L^r(K_p)$ sense, denoted by $D^{\langle\alpha\rangle}f(x)$ and $I_{\langle\alpha\rangle}f(x)$, respectively.

3.4 Properties of fractal calculus on $\mathbb{S}(K_p)$

For $\varphi \in \mathbb{S}(K_p)$ and $\alpha \in \mathbb{R}$, the derivative operator ($\alpha > 0$) and integral operator ($\alpha < 0$)

$$T_{\langle\cdot\rangle^\alpha}\varphi(x) = \int_{\Gamma_p}\int_{K_p} \langle\xi\rangle^\alpha \varphi(t)\overline{\chi}_\xi(t-x)dtd\xi$$
$$= \begin{cases} \varphi^{\langle\alpha\rangle}(x), & \alpha > 0, \\ \varphi(x), & \alpha = 0, \qquad x \in K_p, \\ \varphi_{\langle -\alpha\rangle}(x), & \alpha < 0, \end{cases}$$

has the following properties.

Theorem 3.1. *For Schwartz-type function $\varphi \in \mathbb{S}(K_p)$, it holds in the fractal calculus sense*

(i) φ is any order point-wise fractal derivable, and any order point-wise fractal integrable; and

$$\varphi^{\langle\alpha\rangle}(x) = T_{\langle\cdot\rangle^\alpha}\varphi(x) \in \mathbb{S}(K_p), \quad \varphi_{\langle\alpha\rangle}(x) = T_{\langle\cdot\rangle^{-\alpha}}\varphi(x) \in \mathbb{S}(K_p), \quad x \in K_p, \quad \alpha > 0;$$

i.e., point-wise fractal derivative operation and point-wise fractal integral operation are closed on $\mathbb{S}(K_p)$.

Moreover, φ is any order L^r-strong fractal derivable, and any order L^r-strong fractal integrable; and

$$D^{\langle\alpha\rangle}\varphi \in \mathbb{S}(K_p), \quad I_{\langle\alpha\rangle}\varphi \in \mathbb{S}(K_p), \quad \alpha > 0;$$

i.e., L^r-strong fractal derivative operation and L^r-strong fractal integral operation are closed on $\mathbb{S}(K_p)$.

(ii) For $\alpha > 0$, holds $D^{\langle\alpha\rangle}\varphi(x) = \varphi^{\langle\alpha\rangle}(x)$, $x \in K_p$; i.e., the point-wise fractal derivative equals the L^r -strong fractal derivative; And so does the fractal integral, $I_{\langle\alpha\rangle}\varphi(x) = \varphi_{\langle\alpha\rangle}(x)$, $x \in K_p$.

Thus, it is no necessary to distinguish between "point-wise derivability" and "L^r-strong derivability"; Also neither for "point-wise integrability" and "L^r-strong integrability".

(iii) The fractal derivative operator and fractal integral operator are isomorphic linear mappings from $\mathbb{S}(K_p)$ onto $\mathbb{S}(K_p)$ (linear, one-one, continuous).

(iv) The fractal derivative operator and fractal integral operator are inverse each other, i.e., for $\alpha > 0$, holds

$$(\varphi^{\langle\alpha\rangle}(\cdot))_{\langle\alpha\rangle}(x) = \varphi(x) = (\varphi_{\langle\alpha\rangle}(\cdot))^{\langle\alpha\rangle}(x), \quad x \in K_p.$$

3.5 Definitions and properties of fractal calculus on $\mathbb{S}^*(K_p)$

By Theorem 3.1, we may generalize the fractal calculus to the distribution space $\mathbb{S}^*(K_p)$.

Let $T \in \mathbb{S}^*(K_p)$ be a Schwartz-type distribution. The fractal derivative and fractal integral of T are defined as follows.

Fractal derivative — for $\alpha > 0$, an α-order fractal derivative $T^{\langle\alpha\rangle}$ of $T \in \mathbb{S}^*(K_p)$ is defined as a Schwartz-type distribution satisfying

$$\langle T^{\langle\alpha\rangle}, \varphi\rangle = \langle T, \varphi^{\langle\alpha\rangle}\rangle, \quad \forall \varphi \in \mathbb{S}(K_p).$$

Fractal integral — for $\alpha>0$, an α-order fractal integral $T_{\langle\alpha\rangle}$ of $T\in\mathbb{S}^*(K_p)$ is defined as a Schwartz-type distribution satisfying

$$\langle T_{\langle\alpha\rangle},\varphi\rangle=\langle T,\varphi_{\langle\alpha\rangle}\rangle,\quad \forall\varphi\in\mathbb{S}(K_p).$$

Corresponding to Theorem 3.1, we have

Theorem 3.2. *For Schwartz-type distribution $T\in\mathbb{S}^*(K_p)$, hold*

(i) T is any order fractal derivable, and any order fractal integrable, and $T^{\langle\alpha\rangle},T_{\langle\alpha\rangle}\in\mathbb{S}^*(K_p)$, $\alpha\ge 0$; i.e., the fractal derivative operation and fractal integral operation are closed on $\mathbb{S}^*(K_p)$.

(ii) The fractal derivative operator and fractal integral operator are isomorphic linear mappings from $\mathbb{S}^*(K_p)$ onto $\mathbb{S}^*(K_p)$ (linear, one-one, continuous).

(iii) The fractal derivative operator and fractal integral operator are inverse each other, i.e., for $\alpha>0$, holds $(T^{\langle\alpha\rangle})_{\langle\alpha\rangle}=T=(T_{\langle\alpha\rangle})^{\langle\alpha\rangle}$.

3.6 Principle for establish new calculus

The principle to establish some new calculus is suggested in [5], we now verify the fractal calculus on local fields satisfies the principle.

(1) Fractal derivative operator and fractal integral operator are inverse each other (in the point of view of mathematical analysis and operator theory)

$$\forall\alpha\ge 0,\quad \forall\varphi\in\mathbb{S}(K_p)\Longrightarrow(\varphi^{\langle\alpha\rangle}(\cdot))_{\langle\alpha\rangle}(x)=\varphi(x)=(\varphi_{\langle\alpha\rangle}(\cdot))^{\langle\alpha\rangle}(x),\quad x\in K_p;$$
$$\forall\alpha\ge 0,\quad \forall T\in\mathbb{S}^*(K_p)\Longrightarrow(T^{\langle\alpha\rangle})_{\langle\alpha\rangle}=T=(T_{\langle\alpha\rangle})^{\langle\alpha\rangle}.$$

(2) Fourier transformation formulas (in the point of view of spectrum theory)

$$\forall\alpha\ge 0,\quad \forall\varphi\in\mathbb{S}(K_p)\Longrightarrow(\varphi^{\langle\alpha\rangle}(\cdot))^\wedge(\xi)=\langle\xi\rangle^\alpha\varphi^\wedge(\xi),\quad \xi\in\Gamma_p;$$
$$\forall\alpha\ge 0,\quad \forall T\in\mathbb{S}^*(K_p)\Longrightarrow(T^{\langle\alpha\rangle})^\wedge=\langle\xi\rangle^\alpha T^\wedge,\quad \xi\in\Gamma_p.$$

(3) Equivalent theorems (in the point of view of construction theory of function)
On locally compact group K_p, for $\forall s\ge 0$,

$$\forall f\in X(K_p)=\begin{cases} C(K_p),\\ L^r(K_p),\quad 1\le r<+\infty,\end{cases}$$

holds

$$f^{\langle s\rangle}\in \mathrm{Lip}(X(K_p),\alpha),\quad \alpha>0\Longleftrightarrow E_{p^n}(X(K_p),f)=\mathcal{O}(p^{-n(s+\alpha)}),\quad \alpha>0,\quad n\in\mathbb{N},$$

with Lip class $\mathrm{Lip}(X(K_p),\alpha)$, $\alpha>0$; the best approximation

$$E_{p^n}(X(K(p),f)=\inf_{p_n\in\mathbb{S}_n(K_p)}\|f-p_n\|_{X(K_p)},\quad n\in\mathbb{N},$$

and $\mathbb{S}_n(K_p)=\{\varphi\in\mathbb{S}(K_p):\text{index }\varphi=(n,l),\ l\in\mathbb{Z}\}$ for a fixed $n\in\mathbb{Z}$.

(4) Relationship between characters of local fields with eigen-functions of Newton mechanics (in the point of views of group theory and physics science).

For locally compact group K_p, the character group is $\Gamma_{K_p}=\{\chi_\xi(x):\xi\in K_p\}\overset{iso.}{\longleftrightarrow}\in K_p$.

Character function $y(k)=\chi_k(x),\quad x\in K_p \longleftrightarrow$ eigen-function in Newton mechanics,

Character equation $y^{\langle 1\rangle}=\lambda y \longleftrightarrow$ eigen-equation in Newton mechanics,

Character value $\lambda=\langle\xi\rangle,\quad \xi\in K_p \longleftrightarrow$ eigen-value in Newton mechanics.

The character values are the numbers for which the character equations have non-zero solutions.

4 The construction theory of function in Hölder-type spaces

The classical approximation theory of functions, also called construction theory of function, has had its bright era in the 40's to 90's of last century. Starting from the Weierstrass approximation theory of trigonometric functions and polynomial functions, as well as the Fourier series theory, it has created and developed successfully the idea and mentality of construction theory of function, and kept back lots of valuable wealth for mathematical science.

It is worth to mention the two important contributions of classical construction theory of function [7]

1. lots of approximation identity kernels and approximation identity operators with theoretical and applied senses are constructed.
2. the direct (Jackson) and inverse (Bernstein) approximation theorems, and equivalent theorem on function spaces, such as, on $C([a,b])$ and $L^p([a,b])$, $1\le p<+\infty$, are proved. These theorems reveal an essential property of functions: the smoother the functions, the faster to zero the best approximations; and vice versa.

In this section, we summarize the foundation of construction theory of function on the spaces $X(D)$ and $X(K_p)$; and prove the Jackson type, Bernstein type approximation theorems, and equivalent theorems on the Hölder-type space and Sobolev-type space.

4.1 Foundation of construction theory of function on K_p

Since the 70's of last century, mathematicians in all of the world have contributed lots of excellent work for studying construction theory of function over local fields, such as, the jobs of Chinese mathematicians for compact group D, locally compact group K_p, see [8–25], or the citations in [2]. Results of foreign mathematicians have listed in references [25–30], or the citations in [29,30].

We now show the fundamental theorems of construction theory of function on the function spaces $X(D)$ and $X(K_p)$ over local fields: Jackson theorems, Bernstein theorems, equivalent theorems.

4.1.1 Approximation theorems on function spaces $X(D)$ over compact group D

Let $D=\{x\in K_p:|x|\le 1\}\subset K_p$ be the compact group of a local field K_p, and $X(D)$ be the function spaces

$$X(D)=\begin{cases} C(D)=\{f:\text{bounded and continuous function on } D\},\\ L^r(D)=\left\{f:\int_D |f(x)|^r dx<+\infty\right\}, \quad 1\le r<+\infty,\end{cases}$$

with norms

$$\|f\|_{X(D)}=\begin{cases} \max\{|f(x)|:f\in C(D),\, x\in D\}, & X(D)=C(D),\\ \left\{\int_D |f(x)|^r dx\right\}^{\frac{1}{r}}, \quad 1\le r<+\infty, & X(D)=L^r(D).\end{cases}$$

Modulus of continuity, Lipschitz function class, the best approximation on $X(D)$

Modulus of continuity

$$\omega(X(D),f,\delta)=\sup_{h\in D,|h|\le\delta}\|f(\cdot+h)-f(\cdot)\|_{X(D)}, \quad \delta>0.$$

Lipschitz class

$$\mathrm{Lip}(X(D),\alpha)=\{f\in X(D):\|f(\cdot+h)-f(\cdot)\|_{X(D)}=\mathcal{O}(|h|^\alpha),\, h\in D\}, \quad \alpha>0.$$

The best approximation

$E_{p^n}(X(D),f)=\inf_{q_n\in\mathbb{P}_n}\|f-q_n\|_{X(D)}$, $n\in\mathbb{N}$, where $\mathbb{P}_n$ is the set of all k-degree character polynomials, $\mathbb{P}_n=\{q_k(x):x\in D,\, 0\le k\le n\}$, $0\le k\le n$, and $q_k(x)=a_k\chi_k(x)+a_{k-1}\chi_{k-1}(x)+\cdots+a_1\chi_1(x)+a_0$, $a_j\in\mathbb{C}$, $0\le j\le k$.

Remark 4.1. (i) each k-degree character polynomial $q_k(x)$, is any $s(s>0)$-order fractal derivable and $(q_k)^{\langle s\rangle}\in\mathbb{P}_n$; and is infinitely order fractal integrable; (ii) the best approximation character polynomial $q_n^*\in\mathbb{P}_n$ exists and unique,

$$E_{p^n}(X(D),f)=\|f-q_n^*\|_{X(D)}=\inf_{q_n\in\mathbb{P}_n}\|f-q_n\|_{X(D)}.$$

Partial sun of Fourier series

$$S_n(X(D),f,x)=\sum_{k=0}^{n}c_k\chi_k(x)\equiv\int_D f(x-t)D_n(t)dt, \quad n\in\mathbb{N},$$

with the Dirichlet kernel

$$D_n(t)=\begin{cases}\sum_{k=0}^{n-1}\chi_k(t), & n\ge 1,\\ 1, & n=0.\end{cases}$$

Direct (Jackson) and inverse (Bernstein) approximation theorems on $X(D)$

The direct theorem on $X(D)$

Lemma 4.1 (Jackson type theorem). *For function spaces $X(D)$, if $s\geq 0$, then*

$$f^{\langle s\rangle}\in X(D)\Rightarrow E_{p^n}(X(D),f)=\mathcal{O}(p^{-ns}\omega(X(D),f^{\langle s\rangle},p^{-n})),\quad n\in\mathbb{N}.$$

Specially,

$$f^{\langle s\rangle}\in \mathrm{Lip}(X(D),\alpha),\ \alpha>0\Rightarrow E_{p^n}(X(D),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\in\mathbb{N},\quad \alpha>0.$$

The inverse theorem on $X(D)$

Lemma 4.2 (Bernstein type theoem). *For function spaces $X(D)$, if $s\geq 0$, $\alpha>0$, then*

$$E_{p^n}(X(D),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\to+\infty\Rightarrow\omega(X(D),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^{\alpha}),\quad \delta\to 0.$$

Specially,

$$E_{p^n}(X(D),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad \alpha>0,\quad n\in\mathbb{N}\Rightarrow f^{\langle s\rangle}\in\mathrm{Lip}(X(D),\alpha),\quad \alpha>0.$$

Equivalent theorem on $X(D)$

Theorem 4.1. *For function spaces $X(D)$, if $s\geq 0$, then the following statements are equivalent*

(i) $f^{\langle s\rangle}\in\mathrm{Lip}(X(D),\alpha)$, $\alpha>0$.
(ii) $\omega(X(D),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^{\alpha})$, $\delta\to 0$.
(iii) $E_{p^n}(X(D),f)=\mathcal{O}(p^{-n(\alpha+s)})$, $n\to+\infty$.
(iv) $\|S_{p^n}(X(D),f,\cdot)-f(\cdot)\|_{X(D)}=\mathcal{O}(p^{-n(\alpha+s)})$, $n\to+\infty$.

4.1.2 Approximation theorems on function spaces $X(K_p)$ over locally compact group K_p

Let $K_p^+=K_p$ be the locally compact group of a local field K_p, and $X(K_p)$ be function spaces

$$X(K_p)=\begin{cases} C(K_p)=\{f:\text{bounded and continuous function on }K_p\},\\ L^r(K_p)=\Big\{f:\int_{K_p}|f(x)|^r dx<+\infty\Big\},\quad 1\leq r<+\infty,\end{cases}$$

with norms

$$\|f\|_{x(K_p)}=\begin{cases}\max\{|f(x)|:f\in C(K_p),\ x\in K_p\},\quad X(K_p)=C(K_p),\\ \Big\{\int_{K_p}|f(x)|^r dx\Big\}^{\frac{1}{r}},\quad 1\leq r<+\infty,\quad X(K_p)=L^r(K_p).\end{cases}$$

Modulus of continuity, Lipschitz function class, the best approximation on $X(K_p)$
Modulus of continuity

$$\omega(X(K_p),f,\delta)=\sup_{h\in K_p,|h|\leq\delta}\|f(\cdot+h)-f(\cdot)\|_{X(K_p)},\quad \delta>0.$$

Lipschitz class

$$\mathrm{Lip}(X(K_p),\alpha)=\{f\in X(K_p): \|f(\cdot+h)-f(\cdot)\|_{X(K_p)}=\mathcal{O}(|h|^\alpha),\, h\in K_p\},\quad \alpha>0.$$

The best approximation

$E_{p^n}(X(K_p),f)=\inf_{\varphi\in\mathbb{S}_n(K_p)}\|f-\varphi\|_{X(K_p)}$, $n\in\mathbb{N}$, where the set $\mathbb{S}_n(K_p)$ is a subset of $\mathbb{S}(K_p)$, in which $\forall\varphi\in S_n(K_p)=\{\varphi\in\mathbb{S}(K_p):\text{ index }\varphi=(n,l),\ l\in\mathbb{Z}\}$ is constant on each coset of B^n for the same $n\in\mathbb{Z}$, and $\mathrm{supp}\varphi=B^l$.

Remark 4.2. $\exists\varphi_n^*\in\mathbb{S}_n(K_p)$ as the best approximation function, such that

$$E_{p^n}(X(K_p),f)=\|f-\varphi_n^*\|_{X(K_p)}=\inf_{\varphi\in\mathbb{S}_n(K_p)}\|f-\varphi\|_{X(K_p)},$$

also

$$\overline{\mathbb{S}_n}(K_p)\subset\overline{\mathbb{S}(K_p)}\subset\overline{C(K_p)}=L^r(K_p).$$

Direct (Jackson) and inverse (Bernstein) approximation theorems on $X(K_p)$

The direct theorem on $X(K_p)$

Lemma 4.3 (Jackson type theorem)**.** *For function spaces $X(K_p)$, if $s\geq 0$, then*

$$f^{\langle s\rangle}\in X(K_p)\Rightarrow E_{p^n}(X(K_p),f)=\mathcal{O}(p^{-ns}\omega(X(K_p),f^{\langle s\rangle},p^{-n})),\quad n\in\mathbb{N}.$$

Specially,

$$f^{\langle s\rangle}\in\mathrm{Lip}(X(K_p),\alpha),\quad \alpha>0\Rightarrow E_{p^n}(X(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\in\mathbb{N},\quad \alpha>0.$$

The inverse type theorems on $X(K_p)$

Lemma 4.4 (Bernstein type theorem)**.** *For function spaces $X(K_p)$, if $s\geq 0$, $\alpha>0$, then*

$$E_{p^n}(X(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\to+\infty\Rightarrow\omega(X(K_p),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^\alpha),\quad \delta\to 0.$$

Specially,

$$E_{p^n}(X(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad \alpha>0,\quad n\in\mathbb{N}\Rightarrow f^{\langle s\rangle}\in\mathrm{Lip}(X(K_p),\alpha),\quad \alpha>0.$$

Equivalent theorem on $X(K_p)$

Theorem 4.2. *For function spaces $X(K_p)$, if $s\geq 0$, then the following statements are equivalent*

(i) $f^{\langle s\rangle}\in\mathrm{Lip}(X(K_p),\alpha)$, $\alpha>0$.

(ii) $\omega(X(K_p),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^\alpha)$, $\delta\to 0$.

(iii) $E_{p^n}(X(K_p),f)=\mathcal{O}(p^{-n(\alpha+s)})$, $n\to+\infty$.

4.2 Hölder-type space $C^\sigma(K_p),\sigma\in\mathbb{R}$

4.2.1 The definition of Hölder-type space

1) $\sigma=0\Rightarrow C^0(K_p)\equiv C(K_p)=\{f:K_p\to\mathbb{C}$, continuous and bounded on $K_p\}$, with the norm $\|f\|_{C(K_p)}=\sup_{x\in K_p}|f(x)|$.

2) $\sigma>0\Rightarrow C^\sigma(K_p)\equiv\Big\{u\in\mathbb{S}^*(K_p):u=\sum_{j=0}^{+\infty}u_j\Big\}$, where $u\in\mathbb{S}^*(K_p)$ has Littlewood-Paley decomposition $u=\sum_{j=0}^{+\infty}u_j$ with (i) $\operatorname{supp}u_0^\wedge\subset\Gamma_0$, $\operatorname{supp}u_j^\wedge\subset\Gamma_j\setminus\Gamma_{j-1}$, $j>0$; (ii) $\|u_j\|_{L^\infty(K_p)}\le cq^{-j\sigma}$, $j\ge 0$. The norm of $u\in C^\sigma(K_p)$ is $\|u\|_{C^\sigma(K_p)}=\sup_j\{p^{j\sigma}\|u_j\|_{L^\infty(K_p)}\}$, $\sigma>0$.

3) $\sigma<0\Rightarrow C^\sigma(K_p)=B^\sigma_{\infty\infty}(K_p)$. The norm of $u\in C^\sigma(K_p)$ is $\|u\|_{C^\sigma(K_p)}=\|u\|_{B^\sigma_{\infty\infty(K_p)}}$, where

$$B^s_{rt}(K_p)=\{f\in\mathbb{S}^*(K_p):\|f\|_{B^s_{rt}(K_p)}<+\infty\},\quad s\in\mathbb{R},\quad 0<r,\quad t\le+\infty,$$

is the B-type space of Teribel with norm

$$\|f\|_{B^s_{rt(K_p)}}=\|p^{sj}|\varphi_j(\cdot)f^\wedge|^\vee\|_{l_t(L^r)}=\Big\{\sum_{j=0}^{+\infty}\|p^{sj}(\varphi_j(\cdot)f^\wedge)^\vee(\cdot)\|^t_{L^r(K_p)}\Big\}^{\frac{1}{t}}$$

see [33,34], and sequence as $\{\varphi_j(\xi)\}_{j=0}^{+\infty}=\{\Phi_{\Gamma^0}(\xi),\Phi_{\Gamma^j\setminus\Gamma^{j-1}}(\xi)\}_{j=1}^{+\infty}$.

4.2.2 The important properties of Hölder-type space

Theorem 4.3 (Equivalent definition, see [32,33]). *For the Hölder-type space, hold*

(i) $C^\sigma(K_p)=B^\sigma_{\infty\infty}(K_p)$, $\sigma\in\mathbb{R}$.

(ii) $C^\sigma(K_p)=\mathrm{Lip}(K_p,\sigma)$, $\sigma>0$.

(iii) $\overline{\mathbb{S}(K_p)}=C^\sigma(K_p)$, $\sigma\in\mathbb{R}$.

Theorem 4.4 (see [32]). *For the Hölder-type space $C^\sigma(K_p)$, if $\sigma\in[0,+\infty)$, then*

(i) $f\in C^\sigma(K_p)$, $\forall\lambda\in[0,\sigma]\Rightarrow f$ has any λ-order fractal derivatives $f^{\langle\lambda\rangle}\equiv T_{\langle\cdot\rangle^\lambda}f$, and $f^{\langle\lambda\rangle}\in C^{\sigma-\lambda}(K_p)$; specially, $f\in C^\sigma(K_p)\Rightarrow f^{\langle\sigma\rangle}\in C(K_p)$.

(ii) $f^{\langle\sigma\rangle}=T_{\langle\cdot\rangle^\sigma}f\in C(K_p)$, $\forall\lambda\in[0,\sigma]\Rightarrow f$ has any λ -order fractal derivatives $f^{\langle\lambda\rangle}\equiv T_{\langle\cdot\rangle^\lambda}f$, and $f^{\langle\lambda\rangle}\in C^{\sigma-\lambda}(K_p)$; specially, $f^{\langle\sigma\rangle}\in C(K_p)\Rightarrow f\in C^\sigma(K_p)$.

Thus, $f\in C^\sigma(K_p)\Leftrightarrow f^{\langle\sigma\rangle}\in C(K_p)$, $\forall\sigma\in[0,+\infty)$, i.e., the Hölder-type space $C^\sigma(K_p)$ is the space in which fractals live.

As we know, the Newton k-order continuous derivable function space is $C^k(\mathbb{R}^n)$, $k\in\mathbb{N}$; and for the Hölder space $C^\sigma(\mathbb{R}^n)$, $\sigma\in\mathbb{R}^+\setminus\mathbb{N}$, the parameter σ has a "gap" at natural numbers, $\sigma\notin\mathbb{N}$. However, in the case of local field, the parameter $\sigma\in\mathbb{R}$ has "no gap" of the Hölder type space $C^\sigma(K_p)$. On the other hand, the Lipschitz function classes $\mathrm{Lip}(C([a,b]),\alpha)$, $\mathrm{Lip}(C(\mathbb{R}),\alpha)$ on Euclidean spaces exist just for $0<\alpha\le 1$, of parameter α, but the $\mathrm{Lip}(C(D),\alpha)$, $\mathrm{Lip}(C(K_p),\alpha)$ on local fields exist for $\alpha>0$, without restriction of $0<\alpha\le 1$. These differences show that the smoothness of functions defined on $\mathbb{R}$ and on K_p are quite different.

4.3 Approximation theorems on Hölder-type space $C^\sigma(K_p)$

We study approximation theorems on the Hölder-type $C^\sigma(K_p)$, $\sigma>0$, over a local field K_p.

4.3.1 Modulus of continuity, Lipschitz function class, the best approximation on $C^\sigma(K_p)$

Modulus of continuity

$\omega(C^\sigma(K_p),f,\delta)$, $\delta>0$, defined as

$$\omega(C^\sigma(K_p),f,\delta)=\sup_{h\in K_p,|h|\le\delta}\|f(\cdot+h)-f(\cdot)\|_{C^\sigma(K_p)},\quad \delta>0.$$

Lipschitz class

$\mathrm{Lip}(C^\sigma(K_p),\alpha)$, $\alpha>0$, defined as

$$\mathrm{Lip}(C^\sigma(K_p),\alpha)=\big\{f\in C^\sigma(K_p):\|f(\cdot+h)-f(\cdot)\|_{C^\sigma(K_p)}=\mathcal{O}(|h|^\alpha),\ h\in K_p\big\},\quad \alpha>0.$$

The best approximation

$E_{p^n}(C^\sigma(K_p),f)$, $n\in\mathbb{N}$, defined as

$$E_{p^n}(C^\sigma(K_p),f)=\inf_{\varphi\in\mathcal{S}_n}\|f-\varphi\|_{C^\sigma(K_p)},\quad n\in\mathbb{N},$$

where $\mathcal{S}_n=\{\varphi\in\mathcal{S}(K_p):\text{index }\varphi=(n,l),\ l\in\mathbb{Z}\}$, $n\in\mathbb{N}$.

4.3.2 Direct (Jackson) and inverse (Bernstein) approximation theorems on $C^\sigma(K_p)$, $\sigma>0$

The direct type theorem on $C^\sigma(K_p)$

Lemma 4.5 (Jackson type theorem)**.** *For Hölder-type space $C^\sigma(K_p)$, $\sigma>0$, if $s\ge0$, then*

$$f^{\langle s\rangle}\in C^\sigma(K_p)\Rightarrow E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-ns}\omega(C^\sigma(K_p),f^{\langle s\rangle},p^{-n})),\quad n\in\mathbb{N}. \tag{4.1}$$

Specially,

$$f^{\langle s\rangle}\in\mathrm{Lip}(C^\sigma(K_p),\alpha),\quad \alpha>0\Rightarrow E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\in\mathbb{N},\quad \alpha>0.$$

Proof. To prove the assertion (4.1), we only need prove the following (4.2) for $\alpha>0$

$$\varphi^{\langle s\rangle}\in\mathcal{S}(K_p)\Rightarrow E_{p^n}(C^\sigma(K_p),\varphi)=\mathcal{O}(p^{-ns}\omega(C^\sigma(K_p),\varphi^{\langle s\rangle},p^{-n})),\quad n\in\mathbb{N}, \tag{4.2}$$

since $\overline{\mathcal{S}(K_p)}\subset C^\sigma(K_p)$, and $\|\varphi\|_{\mathcal{S}(K_p)}=\|\varphi\|_{C^\sigma(K_p)}$. Then for $f\in C^\sigma(K_p)$, $\exists\varphi_j\in\mathcal{S}(K_p)$, st. hold $\|\varphi_j(\cdot)-f(\cdot)\|_{C^\sigma(K_p)}\to0$, $j\to+\infty$, and

$$\|(\varphi_j)^{\langle s\rangle}(\cdot)-f^{\langle s\rangle}(\cdot)\|_{C^\sigma(K_p)}\to0,\quad j\to+\infty.$$

By [2, Theorem 3.3.1], $\varphi\in\mathcal{S}(K_p)\Rightarrow\varphi^{\langle s\rangle}\in\mathcal{S}(K_p)$, $s\in\mathbb{Z}$, and

$$\varphi^{\langle s\rangle}(x)=T_{\langle\cdot\rangle^s}\varphi(x)=\kappa_s*\varphi(x)=\begin{cases}\text{fractal derivative}, & s>0,\\ \varphi(x), & s=0,\\ \text{fractal integer}, & s<0,\end{cases}$$

where $\kappa_s\in\mathcal{S}^*(K_p)$ is a bounded linear functional on $\mathcal{S}(K_p)$. We estimate $\|\varphi^{\langle s\rangle}\|_{C^\sigma(K_p)}$. And also only need to estimate for $\Phi_{B^k}\in\mathcal{S}(K_p)$. By $(\Phi_{B^k})^\wedge(\xi)=p^{-k}\Phi_{\Gamma^k}(\xi)$, $\xi\in\Gamma_p$, $k\in\mathbb{Z}$, it holds for $s\geq 0$

$$\begin{aligned}
T_{\langle\cdot\rangle}\Phi_{B^k}(x)=&\int_{\Gamma_p}\langle\xi\rangle^s(\Phi_{B^k})^\wedge(\xi)\chi_x(\xi)d\xi=\int_{\Gamma_p}\langle\xi\rangle^s p^{-k}\Phi_{\Gamma^k}(\xi)\chi_x(\xi)d\xi\\
=&\int_{\Gamma^k}\langle\xi\rangle^s p^{-k}\chi_x(\xi)d\xi=\sum_{j=-\infty}^{k}\int_{\Gamma^j\setminus\Gamma^{j-1}}\langle\xi\rangle^s p^{-k}\chi_x(\xi)d\xi\\
=&p^{-k}\sum_{j=-\infty}^{k}\int_{\Gamma^j\setminus\Gamma^{j-1}}|\xi|^s\chi_x(\xi)d\xi\\
=&p^{-k}\sum_{j=-\infty}^{k}p^{sj}\int_{\Gamma^j\setminus\Gamma^{j-1}}\chi_x(\xi)d\xi=p^{-k}\sum_{j=-\infty}^{k}p^{sj}\int_{|\xi|=p^j}\chi_x(\xi)d\xi\\
=&p^{-k}\sum_{j=-\infty}^{k}p^{sj}\begin{cases}p^j(1-p^{-1}), & j\leq -l,\\ -p^{-l}, & j=-l+1,\\ 0, & j>-l+1,\end{cases}\quad |x|=p^l,\quad l\in\mathbb{Z},\\
&(\text{for } x\in B^k\Rightarrow|x|=p^{-k})\\
=&p^{-k}\sum_{j=-\infty}^{k}p^{sj}\begin{cases}p^j(1-p^{-1}), & j\leq k,\\ -p^{-l}, & j=k+1,\\ 0, & j>k+1,\end{cases}\\
&(\text{for }\ x\in B^k\Rightarrow|x|=p^{-k}\Rightarrow l=-k)\\
=&p^{-k}\Big\{\sum_{j=-\infty}^{k}p^{sj}p^j(1-p^{-1})\Big\}=(1-p^{-1})p^{-k}\Big\{\sum_{j=-\infty}^{k}p^{sj+j}\Big\}\\
=&(1-p^{-1})p^{-k}\frac{p^{(s+1)k}-1}{p^{s+1}-1}\\
=&\frac{p-1}{p(p^{s+1}-1)}(p^{sk+k}p^{-k}-1)=\frac{p-1}{p(p^{s+1}-1)}(p^{sk}-1)\\
=&\frac{p-1}{p(p^{s+1}-1)}(p^{sk}-1)\Phi_{B^k}(x).
\end{aligned}$$

Thus,

$$\forall\varphi\in\mathcal{S}(K_p)\Rightarrow\|\varphi^{\langle s\rangle}\|_{C^\sigma(K_p)}=\mathcal{O}(p^{-sn}\|\varphi\|_{C^\sigma(K_p)}),\quad s\geq 0.$$

By

$$\varphi(x+h)-\varphi(x)=T_{\langle\cdot\rangle^{-s}}T_{\langle\cdot\rangle^{s}}\{\varphi(x+h)-\varphi(x)\}=T_{\langle\cdot\rangle^{-s}}\{\varphi^{\langle s\rangle}(x+h)-\varphi^{\langle s\rangle}(x)\},$$

then for $s\geq 0$

$$\begin{aligned}\|\varphi(\cdot+h)-\varphi(\cdot)\|_{C^\sigma(K_p)}&=\|T_{\langle\cdot\rangle^{-s}}\{\varphi^{\langle s\rangle}(\cdot+h)-\varphi^{\langle s\rangle}(\cdot)\}\|_{C^\sigma(K_p)}\\&=\mathcal{O}\big(p^{-sn}\|\varphi^{\langle s\rangle}(\cdot+h)-\varphi^{\langle s\rangle}(\cdot)\|_{C^\sigma(K_p)}\big).\end{aligned}$$

This implies

$$\omega(C^\sigma(K_p),\varphi,p^{-n})=\mathcal{O}(p^{-sn}\omega(C^\sigma(K_p),\varphi^{\langle s\rangle},p^{-n})),\quad n\in\mathbb{N},\quad s\geq 0, \tag{4.3}$$

(generally, it holds $\omega(C^\sigma(K_p),\varphi,\delta)=\mathcal{O}(\delta^s\omega(C^\sigma(K_p),\varphi^{\langle s\rangle},\delta))$, $\delta\to 0$, $s\geq 0$).

Since $E_{p^n}(C^\sigma(K_p),\varphi)\leq\omega(C^\sigma(K_p),\varphi,p^{-n})$, combining (4.3), it holds for $s\geq 0$

$$E_{p^n}(C^\sigma(K_p),\varphi)\leq\omega(C^\sigma(K_p),\varphi,p^{-n})\leq\mathcal{O}(p^{-sn}\omega(C^\sigma(K_p),\varphi^{\langle s\rangle},p^{-n}),\quad n\in\mathbb{N},$$

this implies (4.2). Moreover, by $\overline{\mathcal{S}(K_p)}\subset C^\sigma(K_p)$, it follows for $s\geq 0$

$$f^{\langle s\rangle}\in C^\sigma(K_k)\Rightarrow E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-sn}\omega(C^\sigma(K_p),f^{\langle s\rangle},p^{-n})),\quad n\in\mathbb{N}.$$

Thus, (4.1) is proved.

For the special case,

$$f^{\langle s\rangle}\in\mathrm{Lip}(C^\sigma(K_p),\alpha),\quad \alpha>0\Rightarrow E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\in\mathbb{N},\quad \alpha>0,$$

since (4.2) holds for $f\in C^\sigma(K_p)$, thus

$$\begin{aligned}&\omega(C^\sigma(K_p),f,p^{-n})=\mathcal{O}(p^{-sn}\omega(C^\sigma(K_p),f^{\langle s\rangle},p^{-n}))\Rightarrow f^{\langle s\rangle}\in C^\sigma(K_p)\\&\Rightarrow\omega(C^\sigma(K_p),f,p^{-n})=\mathcal{O}(p^{-\alpha n}\|f^{\langle s\rangle}\|_{C^\sigma(K_p)}),\end{aligned}$$

then

$$\begin{aligned}f^{\langle s\rangle}\in\mathrm{Lip}(C^\sigma(K_p),\alpha)&\Rightarrow\|f^{\langle s\rangle}(\cdot+h)-f^{\langle s\rangle}(\cdot)\|_{C^\sigma(K_p)}=\mathcal{O}(|h|^\alpha)\\&\Rightarrow\|f^{\langle s\rangle}\|_{C^\sigma(K_p)}=\mathcal{O}(|h|^\alpha)\Rightarrow\omega(C^\sigma(K_p),f,p^{-n})\\&\quad=\mathcal{O}(p^{-ns}\cdot p^{-n\alpha})=\mathcal{O}(p^{-(\alpha+s)n})\\&\Rightarrow E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\in\mathbb{N},\quad\alpha>0.\end{aligned}$$

The Lemma 4.5 is proved. □

The inverse type theorem on $C^\sigma(K_p)$

Lemma 4.6 (Bernstein type theorem). *For Hölder-type space $C^\sigma(K_p)$, $\sigma>0$, if $s\geq 0$, $\alpha>0$, then*

$$E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\to+\infty\Rightarrow\omega(C^\sigma(K_p),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^\alpha),\quad \delta\to 0.$$

Specially,

$$E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\in\mathbb{N},\quad \alpha>0\Rightarrow f^{\langle s\rangle}\in\mathrm{Lip}(C^\sigma(K_p),\alpha),\quad \alpha>0.$$

Proof. We prove Bernstein type inequality:

$$\forall q_n\in\mathbb{S}_n(K_p)\Rightarrow (q_n)^{\langle s\rangle}\in\mathbb{S}_n(K_p)\quad\text{and}\quad \|(q_n)^{\langle s\rangle}\|_{C^\sigma(K_p)}\leq cp^{ns}\|q_n\|_{C^\sigma(K_p)}.$$

Since $\forall q_n\in\mathbb{S}_n(K_p)\subset\mathbb{S}(K_p)$, index $q_n=(n,l)\Rightarrow (q_n)^{\langle s\rangle}\in\mathbb{S}(K_p)$, index $(q_n)^{\langle s\rangle}=(n,l)$; by [2], Theorem 3.2.3, Theorem 3.1.7, and $(q_n)^{\langle s\rangle}(x)=(\langle\cdot\rangle^s(q_n)^\wedge(\cdot))^\vee(x)$, we have

$$\text{index } q_n=(n,l)\Rightarrow\text{index } (q_n)^\wedge=(l,n)\Rightarrow\text{index } (\langle\xi\rangle^s(q_n)^\wedge)^\vee=(n,l).$$

Thus, $\forall q_n\in\mathbb{S}_n(K_p)\Rightarrow (q_n)^{\langle s\rangle}\in\mathbb{S}_n(K_p)$.

Then by

$$\forall\varphi\in\mathbb{S}_n(K_p)\Rightarrow\|\varphi^{\langle s\rangle}\|_{C^\sigma(K_p)}=\mathcal{O}(p^{-sn}\|\varphi\|_{C^\sigma(K_p)}),$$

we get Bernstein type inequality.

We prove theorem for $s=0$:

$$E_{p^n}(C^\sigma(K_p),f)=O(p^{-\alpha n}),\quad n\to+\infty\Rightarrow\omega(C^\sigma(K_p),f,\delta)=\mathcal{O}(\delta^\alpha),\quad\delta\to 0.$$

By the assumption, $E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-\alpha n})$, $n\to+\infty$, then

$$\begin{aligned}\|f(\cdot+h)-f(\cdot)\|_{C^\sigma(K_p)}&\leq\|f(\cdot+h)-q_n^*(\cdot)\|_{C^\sigma(K_p)}+\|q_n^*(\cdot+h)-f(\cdot)\|_{C^\sigma(K_p)}\\&\leq 2E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-\alpha n})\\&\Rightarrow\|f(\cdot+h)-f(\cdot)\|_{C^\sigma(K_p)}=\mathcal{O}(\delta^\alpha)(p^{-n-1}\leq\delta<p^{-n})\\&\Rightarrow\omega(C^\sigma(K_p),f,\delta)=\mathcal{O}(\delta^\alpha).\end{aligned}$$

Specially, this implies $f\in\mathrm{Lip}(C^\sigma(K_p),\alpha)$.

Next, we will prove theorem for $s>0$:

$$E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\to+\infty\Rightarrow\omega(C^\sigma(K_p),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^\alpha),\quad\delta\to 0.$$

By the assumption, $E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n})$, $n\to+\infty$, then $\Rightarrow$ let $\sum_{n=1}^{+\infty}\{q_n^*(x)-q_{n-1}^*(x)\}$, $q_0^*=0$, then the sum is $q_n^*(x)$, with $\|f(x)-q_n^*(x)\|_{C^\sigma(K_p)}\to 0$, and

$$\begin{aligned}\|q_n^*(x)-q_{n-1}^*(x)\|_{C^\sigma(K_p)}&\leq\|q_n^*(x)-f(x)\|_{C^\sigma(K_p)}+\|f(x)-q_{n-1}^*(x)\|_{C^\sigma(K_p)}\\&=\mathcal{O}(p^{-(\alpha+s)n}),\end{aligned}\tag{4.4}$$

$\Rightarrow \sum_{n=1}^{+\infty} p^{-(\alpha+s)n}$ converges implies $f(x)=\sum_{n=1}^{+\infty}\{q_n^* - q_{n-1}^*(x)\}$ and the s-order fractal derivative is

$$f^{\langle s\rangle}(x)=\sum_{n=1}^{+\infty}\{(q_n^*)^{\langle s\rangle}(x)-(q_{n-1}^*)^{\langle s\rangle}(x)\},$$

$\Rightarrow$ the Bernstein inequality implies

$$\begin{aligned}
&\|(q_n^*)^{\langle s\rangle}(\cdot)-(q_{n-1}^*(\cdot)\|_{C^\sigma(K_p)}\le cp^{ns}\|q_n^*(\cdot)-q_{n-1}^*(\cdot)\|_{C^\sigma(K_p)}\\
\Rightarrow&\|(q_n^*)^{\langle s\rangle}(\cdot)-(q_{n-1}^*)^{\langle s\rangle}(\cdot)\|_{C^\sigma(K_p)}\le cp^{ns}\|q_n^*(\cdot)-q_{n-1}^*(\cdot)\|_{C^\sigma(K_p)}\\
&=cp^{ns}\mathcal{O}(p^{-(\alpha+s)n})=c\mathcal{O}(p^{-n\alpha})\\
\Rightarrow&E_{p^n}(C^\sigma(K_p),f^{\langle s\rangle})\le\sum_{j=n+1}^{+\infty}\|(q_n^*)^{\langle s\rangle}(\cdot)-(q_{j-1}^*)^{\langle s\rangle}(\cdot)\|_{C^\sigma(K_p)}\le\sum_{j=n+1}^{+\infty}\mathcal{O}(p^{-\alpha j})=\mathcal{O}(p^{-\alpha n})\\
\Rightarrow&\omega(C^\sigma(K_p),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^\alpha),\quad \delta\to 0.
\end{aligned}$$

Specially, this implies $f^{\langle s\rangle}\in\mathrm{Lip}(C^\sigma(K_p),\alpha)$. The Lemma 4.6 is proved. □

Equivalent theorem on $C^\sigma(K_p)$

Theorem 4.5. *For Hölder-type space $C^\sigma(K_p)$, $\sigma>0$, if $s\ge 0$, then the following statements are equivalent*

(i) $f^{\langle s\rangle}\in\mathrm{Lip}(C^\sigma(K_p),\alpha)$, $\alpha>0$;

(ii) $\omega(C^\sigma(K_p),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^\alpha)$, $\delta\to 0$;

(iii) $E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n})$, $n\to+\infty$.

Proof. (i) $\Rightarrow$ (iii) By the Jackson type theorem,

$$f^{\langle s\rangle}\in\mathrm{Lip}(C^\sigma(K_p),\alpha),\quad \alpha>0\Rightarrow E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\in\mathbb{N},\quad \alpha>0.$$

(iii) $\Rightarrow$ (ii) By the Bernstein type theorem,

$$E_{p^n}(C^\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n}),\quad n\to+\infty\Rightarrow\omega(C^\sigma(K_p),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^\alpha),\quad \delta\to 0.$$

(ii) $\Rightarrow$ (i) By the definition,

$$\omega(C^\sigma(K_p),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^\alpha),\quad \delta\to 0\Rightarrow f^{\langle s\rangle}\in\mathrm{Lip}(C^\sigma(K_p),\alpha),\quad \alpha>0.$$

Then Theorem 4.5 is proved. □

For the applications of approximation theorems on Hölder-type space $C^\sigma(K_p)$, $\sigma>0$, we refer to [5].

5 Approximation theorems on Sobolev-type space $L^r_\sigma(K_p)$

5.1 Definiitons of Lebesgue-type spaces $L^r_\sigma(K_p)$

Lebesgue-type space $L^r_\sigma(K_p)$, $\sigma\in\mathbb{R}$, $1\le r\le+\infty$, is defined as [34]

$$L^r_\sigma(K_p)\equiv\big\{f\in\mathbb{S}^*(K_p):\|(\langle\cdot\rangle^\sigma f^\wedge)^\vee(\cdot)\|_{L^r(K_p)}<+\infty\big\},\quad \sigma\in\mathbb{R},\quad 1\le r\le+\infty,$$

with norm $\|f\|_{L^r_\sigma(K_p)}=\|(\langle\cdot\rangle^\sigma f^\wedge)^\vee(\cdot)\|_{L^r(K_p)}$.

Sobolev-type space $W^r_\sigma(K_p)\equiv L^r_\sigma(K_p)$, $\sigma\in[0,+\infty)$, $1\le r<+\infty$ is defined as

$$W^r_\sigma(K_p)\equiv\big\{f\in\mathbb{S}^*(K_p):\|(\langle\cdot\rangle^\sigma f^\wedge)^\vee(\cdot)\|_{L^r(K_p)}<+\infty\big\},\quad \sigma\in[0,+\infty,)\quad 1\le r\le+\infty,$$

with norm $\|f\|_{W^r_\sigma(K_p)}=\|(\langle\cdot\rangle^\sigma f^\wedge)^\vee(\cdot)\|_{L^r(K_p)}$.

The classical Sobolev-type space $W^\sigma(K_p)\equiv L^\sigma_2(K_p)$, $\sigma\in[0,+\infty)$, is defined as

$$W^\sigma(K_p)\equiv\big\{f\in\mathbb{S}^*(K_p):\|(\langle\cdot\rangle^\sigma f^\wedge)^\vee(\cdot)\|_{L^2(K_p)}<+\infty\big\},\quad \sigma\in[0,+\infty),$$

with norm $\|f\|_{W^\sigma(K_p)}=\|(\langle\cdot\rangle^\sigma f^\wedge)^\vee(\cdot)\|_{L^2(K_p)}$.

It is clear that

$$L^2(K_p)\equiv\big\{f\in\mathbb{S}^*(K_p):\|(f^\wedge)^\vee(x)\|_{L^2(K_p)}<+\infty\big\}\quad \text{with}\quad \|f\|_{L^2(K_p)}=\|(f^\wedge)^\vee\|_{L^2(K_p)}.$$

5.2 Modulus of continuity, Lipschitz class, the best approximation on $W^r_\sigma(K_p)$

Modulus of continuity $\omega(W^r_\sigma(K_p),f,\delta)$, $\delta>0$, is defined as

$$\omega(W^r_\sigma(K_p),f,\delta)=\sup_{h\in K_p,|h|\le\delta}\|f(\cdot+h)-f(\cdot)\|_{W^r_\sigma(K_p)},\quad \delta>0.$$

Lipschitz class $\mathrm{Lip}(W^r_\sigma(K_p),\alpha)$, $\alpha>0$, is defined as

$$\mathrm{Lip}(W^r_\sigma(K_p),\alpha)=\big\{f\in W^r_\sigma(K_p):\|f(\cdot+h)-f(\cdot)\|_{W^r_\sigma(K_p)}=\mathcal{O}(|h|^\alpha),\ h\in K_p\big\},\quad \alpha>0.$$

The best approximation $E_{p^n}(W^r_\sigma(K_p),f)$, $n\in\mathbb{N}$, is defined by

$$E_{p^n}(W^r_\sigma(K_p),f)=\inf_{\varphi\in\mathbb{S}_n}\|f-\varphi\|_{W^r_\sigma(K_p)},\quad n\in\mathbb{N},$$

with $\mathbb{S}_n(K_p)=\{\varphi\in\mathbb{S}(K_p):\mathrm{index}\varphi=(n,l),\ l\in\mathbb{Z}\}$, $n\in\mathbb{N}$.

5.3 Equivalent theorem on Sobolev-type spaces $W^r_\sigma(K_p)$

Theorem 5.1. *For Sobolev-type spaces $W^r_\sigma(K_p)$, $\sigma\ge 0$, $1\le r<+\infty$, if $s\ge 0$, then the following statements are equivalent*

(i) $f^{\langle s\rangle}\in \mathrm{Lip}(W^r_\sigma(K_p),\alpha)$, $\alpha>0$.

(ii) $\omega(W^r_\sigma(K_p),f^{\langle s\rangle},\delta)=\mathcal{O}(\delta^\alpha)$, $\delta\to 0$.

(iii) $E_{p^n}(W^r_\sigma(K_p),f)=\mathcal{O}(p^{-(\alpha+s)n})$, $n\to+\infty$.

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