Analysis in Theory and Applications
Anal. Theory Appl., Vol. **31**, No. 1 (2015), pp. 13-24

DOI: 10.4208/ata.2015.v31.n1.2

Parameterized Littlewood-Paley Operators and Their Commutators on Lebesgue Spaces with Variable Exponent

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Received 23 May 2014; Accepted (in revised version) 22 January 2015

Abstract. In this paper, by applying the technique of the sharp maximal function and the equivalent representation of the norm in the Lebesgue spaces with variable exponent, the boundedness of the parameterized Littlewood-Paley operators, including the parameterized Lusin area integrals and the parameterized Littlewood-Paley g_λ^*-functions, is established on the Lebesgue spaces with variable exponent. Furthermore, the boundedness of their commutators generated respectively by BMO functions and Lipschitz functions are also obtained.

Key Words: Parameterized Littlewood-Paley operators, commutators, Lebesgue spaces with variable exponent.

AMS Subject Classifications: 42B20, 42B25, 42B35

1 Introduction and main results

The Littlewood-Paley operators, including Lusin area integrals, Littlewood-Paley g-functions and g_λ^*-functions, play very important roles in harmonic analysis and PDE (see [1–4]). In [5], Lu and Yang investigated the behavior of Littlewood-Paley operators in the space $CBMO_p(\mathbb{R}^n)$. In 2009, Xue and Ding gave weighted estimates for Littlewood-Paley operators and their commutators (see [6]). In 2013, Wei and Tao proved Litttleood-Paley operators with rough kernels are bounded on weighted $(L^q,L^p)^\alpha(\mathbb{R}^n)$ spaces (see [7]).

In 1960, the parameterized Littlewood-Paley operators were discussed by Hörmander (see [8]) for the first time. Now, let us review the definitions of the parameterized Lusin area integral and the parameterized Littlewood-Paley g_λ^*-function.

Let S^{n-1} denote the unit sphere of $\mathbb{R}^n$ equipped with Lebesgue measure $d\sigma(x')$ and $\psi^\rho(x)=\Omega(x)|x|^{-n+\rho}\chi_{\{|x|\le 1\}}$, where $0<\rho<n$ and Ω satisfies the following conditions:

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(a) $\Omega(\lambda x)=\Omega(x)$ for all $\lambda>0$;

(b) $\int_{\mathbb{R}^n}\Omega(x')d\sigma(x')=0$;

(c) $\Omega\in L^1(S^{n-1})$.

Then the parameterized Lusin area integral S^ρ and the parameterized Littlewood-Paley g_λ^*-function $g_\lambda^{*,\rho}$ are defined respectively by

$$S^\rho(f)(x)=\Big(\iint_{\Gamma_a(x)}|\psi_t^\rho*f(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}$$

and

$$g_\lambda^{*,\rho}(f)(x)=\Big(\iint_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}|\psi_t^\rho*f(y)|^2\frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\Gamma(x)=\{(t,y)\in\mathbb{R}_+^{n+1}:|y-x|<t\}$, $\lambda>1$.

In [9], Torchinsky and Wang studied the boundedness of the operators S^ρ and $g_\lambda^{*,\rho}$ on weighted $L^2(\mathbb{R}^n)$ for $\rho=1$ and $\Omega(x)\in Lip_\alpha(S^{n-1})$ $(0<\alpha\le 1)$. For general ρ, Sakamoto and Yabuta considered the L^p boundedness of S^ρ and $g_\lambda^{*,\rho}$ in [10]; Wei and Tao given the boundedness of parameterized Litttlewood-Paley operators with rough kernels on weighted weak Hardy spaces in [11].

Now let us turn to the introduction of the corresponding m-order commutators of the parameterized Littlewood-Paley operators above. Let $b\in L^1_{\rm loc}(\mathbb{R}^n)$, $m\in\mathbb{N}$, the commutators $[b^m,S^\rho]$ and $[b^m,g_\lambda^{*,\rho}]$ are defined respectively by

$$[b^m,S^\rho](f)(x)=\Big(\iint_{\Gamma_a(x)}\Big|\frac{1}{t^\rho}\int_{|y-x|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac{1}{2}}$$

and

$$\begin{aligned}&[b^m,g_\lambda^{*,\rho}](f)(x)\\ =&\Big(\iint_{\mathbb{R}_+^{n+1}}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\frac{1}{t^\rho}\int_{|y-x|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{\frac{1}{2}}.\end{aligned}$$

In 2007, Ding and Xue established the weak $L\log L$ estimates of the commutators $[b^m,S^\rho]$ and $[b^m,g_\lambda^{*,\rho}]$ for $b\in BMO(\mathbb{R}^n)$ (see [12]). In 2009, Chen and Ding investigated the characterization of the commutators for the parameterized Littlewood-Paley operators (see [13,14]).

On the other hand, Lebesgue spaces with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ become one of the important class function spaces due to the seminal paper [15] by Kováčik and Rákosník. In the past twenty years, the theory of these spaces was made progress rapidly, and the study of which was widely applied in some fields such as fluid dynamics, elasticity dynamics, calculus of variations and differential equations with non-standard growth conditions (see [16–20]). In [21], Cruz-Uribe, Fiorenza, Martell and Pérez studied the extrapolation theorem which leads the boundedness of some classical operators including the commutators on $L^{p(\cdot)}(\mathbb{R}^n)$. Meanwhile Karlovich and Lerner also independently proved the boundedness for the commutators of singular integrals in [22]. In 2012, Wang, Fu and Liu stated that higher-order commutators of Marcinkiewicz integrals are bounded on spaces $L^{p(\cdot)}(\mathbb{R}^n)$ (see [23]).

Inspired by the results mentioned previously, it is natural to ask whether the parameterized Littlewood-Paley operators S^ρ and $g_\lambda^{*,\rho}$ and their commutators $[b^m,S^\rho]$ and $[b^m,g_\lambda^{*,\rho}]$ are bounded on Lebesgue spaces with variable exponent or not. The purpose of this paper is to give an affirmative answer to this question. Before stating our main results, we need to recall some relevant definitions and notations. Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with $|E|>0$.

Definition 1.1 (see [15])**.** Let $p(\cdot):E\to[1,\infty)$ be a measureable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E)=\Big\{f \text{ is measurable}: \int_E\Big(\frac{|f(x)|}{\eta}\Big)^{p(x)}dx<\infty \text{ for some constant } \eta>0\Big\}.$$

And the space $L^{p(\cdot)}_{loc}(E)$ is defined by

$$L^{p(\cdot)}_{loc}(E)=\big\{f \text{ is measurable}: f\in L^{p(\cdot)}(K) \text{ for all compact subsets } K\subset E\big\}.$$

It is easy to see that the Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach space with the following Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(E)}=\inf\Big\{\eta>0: \int_E\Big(\frac{|f(x)|}{\eta}\Big)^{p(x)}dx\le 1\Big\}.$$

Remark 1.1. (1) Noting that if the function $p(x)=p_0$ is a constant function, then $L^{p(\cdot)}(\mathbb{R}^n)$ equals $L^{p_0}(\mathbb{R}^n)$. This implies that the Lebesgue spaces with variable exponent generalize the usual Lebesgue spaces. And they have many properties in common with the usual Lebesgue spaces.

(2) Denote $p_-:=\text{essinf}\{p(x):x\in E\}$, $p_+:=\text{esssup}\{p(x):x\in E\}$. Then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_->1$ and $p_+<\infty$.

(3) The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)|dy.$$

Denote $\mathcal{B}(E)$ to be the set of all functions $p(\cdot)\in\mathcal{P}(E)$ satisfying the condition that M is bounded on $L^{p(\cdot)}(E)$.

Definition 1.2 (see [24])**.** Let $\Omega\in L^q(S^{n-1})$ for $q\ge 1$. Then the integral modulus $\omega_q(\delta)$ of L^q continuity of Ω is defined by

$$\omega_q(\delta)=\sup_{\|\rho\|<\delta}\Big(\int_{S^{n-1}}|\Omega(\rho x')-\Omega(x')|^q d\sigma(x')\Big)^{1/q},\quad 1\le q<\infty,$$

and

$$\omega_\infty(\delta)=\sup_{\|\rho\|<\delta}|\Omega(\rho x')-\Omega(x')|,$$

where $0<\delta\le 1$, ρ denotes the rotation on $\mathbb{R}^n$ and $\|\rho\|=\sup_{x'\in S^{n-1}}|\rho x'-x'|$.

Definition 1.3 (see [25]). For $0<\beta\le 1$, the Lipschitz spaces $Lip_\beta(\mathbb{R}^n)$ is defined by

$$Lip_\beta(\mathbb{R}^n)=\Big\{f:\|f\|_{Lip_\beta}=\sup_{x,y\in\mathbb{R}^n;x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\beta}<\infty\Big\}.$$

Our main results in this paper are formulated as follows.

Theorem 1.1. *Suppose that $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\rho>n/2$ and $\Omega\in L^2(S^{n-1})$ satisfying*

$$\int_0^1\frac{\omega_2(\delta)}{\delta}(1+|\log\delta|)^\sigma d\delta<\infty,\quad \sigma>2. \tag{1.1}$$

Then there exists a constant $C>0$ independent of f such that

$$\|S^\rho(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.2. *Suppose that $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\rho>n/2$, $\lambda>2$ and $\Omega\in L^2(S^{n-1})$ satisfying* (1.1). *Then there exists a constant $C>0$ independent of f such that*

$$\|g_\lambda^{*,\rho}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.3. *Let $b\in BMO(\mathbb{R}^n)$ and $m\in\mathbb{N}$. Suppose that $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\rho>n/2$ and $\Omega\in L^2(S^{n-1})$ satisfying* (1.1). *Then there exists a constant $C>0$ independent of f such that*

$$\|[b^m,S^\rho](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_*^m\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.4. *Let $b\in BMO(\mathbb{R}^n)$ and $m\in\mathbb{N}$. Suppose that $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $\rho>n/2$, $\lambda>2$ and $\Omega\in L^2(S^{n-1})$ satisfying* (1.1). *Then there exists a constant $C>0$ independent of f such that*

$$\|[b^m,g_\lambda^{*,\rho}](f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_*^m\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.5. *Let $b\in Lip_\beta(\mathbb{R}^n)$, $m\in\mathbb{N}$ and $\Omega\in L^2(S^{n-1})$. Suppose that $\rho>n/2$, $0<\beta<\min\{1,n/m\}$ and $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ be such that $p_+<n/m\beta$. Define $q(\cdot)$ by*

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{m\beta}{n}.$$

If $q(\cdot)(n-m\beta)/n\in\mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C>0$ independent of f such that

$$\|[b^m,S^\rho](f)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|b\|_{Lip_\beta(\mathbb{R}^n)}^m\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 1.6. *Let $b\in Lip_\beta(\mathbb{R}^n)$, $m\in\mathbb{N}$ and $\Omega\in L^2(S^{n-1})$. Suppose that $\rho>n/2$, $\lambda>2$, $0<\beta<\min\{1,n/m\}$ and $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ be such that $p_+<n/m\beta$. Define $q(\cdot)$ by*

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{m\beta}{n}.$$

If $q(\cdot)(n-m\beta)/n\in\mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C>0$ independent of f such that

$$\|[b^m,g_\lambda^{*,\rho}](f)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

We end this section by introducing some conventional notations which will be used later. Throughout this paper, $|E|$ denotes the Lebesgue measure of $E\subset\mathbb{R}^n$. χ_E denotes the characterization function of E. $p'(\cdot)$ means the conjugate exponent of $p(\cdot)$, namely $1/p(x)+1/p'(x)=1$ holds. C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2 Preliminary lemmas

In this section, we need some conclusions which will be used in the proofs of our main results.

Lemma 2.1 (see [15]). *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. If endowing the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with the following Orlicz type norm:*

$$\|f\|^0_{L^{p(\cdot)}(\mathbb{R}^n)}=\sup\Big\{\int_{\mathbb{R}^n}|f(x)g(x)|dx:\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le 1\Big\},$$

then the norm $\|\cdot\|^0_{L^{p(\cdot)}(\mathbb{R}^n)}$ above is equivalent to the Luxemburg-Nakano norm $\|\cdot\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ in Definition 1.1.

Lemma 2.2 (Generalized Hölder Inequality, see [15]). *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then for any $f\in L^{p(\cdot)}(\mathbb{R}^n)$ and $g\in L^{p'(\cdot)}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx\le C_p\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $C_p=1+1/p_--1/p_+$.

Lemma 2.3. *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then the following conditions are equivalent*

(1) $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$.

(2) $p'(\cdot)\in\mathcal{B}(\mathbb{R}^n)$.

(3) $p(\cdot)/q\in\mathcal{B}(\mathbb{R}^n)$ for some $1<q<p_-$.

(4) $(p(\cdot)/q)'\in\mathcal{B}(\mathbb{R}^n)$ for some $1<q<p_-$.

Lemma 2.4. *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. Then $C_c^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$, where $C_c^\infty(\mathbb{R}^n)$ denotes the infinity times differentiable functions on $\mathbb{R}^n$ with compact support set.*

Since $C_c^\infty(\mathbb{R}^n)$ is L^∞-norm dense in $C_0(\mathbb{R}^n)$ (see [24]), and $C_0^\infty(\mathbb{R}^n)$ is dense in $L^{p(\cdot)}(\mathbb{R}^n)$ (see [15]), it is easy to know Lemma 2.4 holds. Here

$$C_0(\mathbb{R}^n)=\Big\{f \text{ is continuous on } \mathbb{R}^n: \lim_{|x|\to\infty} f(x)=0\Big\}$$

and

$$C_0^\infty(\mathbb{R}^n)=\{f: f\in C_0(\mathbb{R}^n) \text{ and } f \text{ is infinity times differentiable}\}.$$

For $\delta>0$, $f\in L^\delta_{\rm loc}(\mathbb{R}^n)$, let

$$M_\delta(f)(x)=\sup_{Q\ni x}\Big(\frac{1}{|Q|}\int_Q |f(y)|^\delta dy\Big)^{1/\delta}$$

and

$$f^\sharp_\delta(x)=\sup_{Q\ni x}\inf_{c\in\mathbb{R}}\Big(\frac{1}{|Q|}\int_Q |f(y)-c|^\delta dy\Big)^{1/\delta}.$$

The non-increasing rearrangement of a measurable function f on $\mathbb{R}^n$ is defined by [26]

$$f^*(t)=\inf\{\lambda>0: |\{x\in\mathbb{R}^n: |f(x)|>\lambda\}|\le t\},\quad (0<t<\infty).$$

Furthermore, for $\tau\in(0,1)$ and a measurable function f on $\mathbb{R}^n$, the local sharp maximal operator $M^\sharp_\tau$ is defined by [22]

$$M^\sharp_\tau(f)(x)=\sup_{Q\ni x}\inf_{c\in\mathbb{R}}((f-c)\chi_Q)^*(\tau|Q|).$$

Lemma 2.5 (see [22])**.** *Let $\delta>0$, $\tau\in(0,1)$ and $f\in L^\delta_{\rm loc}(\mathbb{R}^n)$. Then for any $x\in\mathbb{R}^n$*

$$M^\sharp_\tau(f)(x)\le(1/\tau)^{1/\delta}f^\sharp_\delta(x).$$

Lemma 2.6 (see [17])**.** *Let $g\in L^1_{\rm loc}(\mathbb{R}^n)$, $\tau\in(0,1)$ and a measurable function f satisfying*

$$|\{x: |f(x)|>\alpha\}|<\infty\quad \textit{for all}\ \ \alpha>0. \tag{2.1}$$

Then

$$\int_{\mathbb{R}^n}|f(x)g(x)|dx\le C\int_{\mathbb{R}^n}M^\sharp_\tau(f)(x)M(g)(x)dx.$$

Lemma 2.7 (see [12])**.** *Suppose that $\rho>n/2$, $\lambda>2$, $\Omega\in L^2(S^{n-1})$ satisfies* (1.1)*. Then for all smooth functions f with compact support, there exists a positive constant $0<C=C_\delta$ such that*

(1) if $0<\delta<1$, then

$$(S^\rho(f))^\sharp_\delta(x)\le CM(f)(x)\quad \textit{and}\quad (g^{*,\rho}_\lambda(f))^\sharp_\delta(x)\le CM(f)(x).$$

(2) if $0<\delta<l<1$, $m\in\mathbb{N}$ and $b\in BMO(\mathbb{R}^n)$, then

$$([b^m,S^\rho](f))^\sharp_\delta(x)\le C\sum_{j=0}^{m-1}\|b\|_*^{m-j}M_l([b^j,S^\rho](f))(x)+C\|b\|_*^m M^{m+1}(f)(x),$$

and

$$([b^m,g_\lambda^{*,\rho}](f))^\sharp_\delta(x)\le C\sum_{j=0}^{m-1}\|b\|_*^{m-j}M_l([b^j,g_\lambda^{*,\rho}](f))(x)+C\|b\|_*^m M^{m+1}(f)(x).$$

Where M^m denotes m times iteration of M.

Given $0<\alpha<n$, define the fractional integral operator I_α by

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Then we have the following conclusion:

Lemma 2.8 (see [21]). *Let $p(\cdot),q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ be such that $p_+<n/\alpha$ and*

$$\frac{1}{p(x)}-\frac{1}{q(x)}=\frac{m\beta}{n}.$$

If $q(\cdot)(n-\alpha)/n\in\mathcal{B}(\mathbb{R}^n)$, then

$$\|I_\alpha(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

3 Proof of main theorems

It is easy to check that

$$S^\rho(f)(x)\le 2^{n\lambda}g_\lambda^{*,\rho}(f)(x),\quad [b^m,S^\rho](f)(x)\le 2^{n\lambda}[b^m,g_\lambda^{*,\rho}](f)(x)\quad \text{for } m\in\mathbb{N}.$$

Therefore, it is enough to consider the operators $g_\lambda^{*,\rho}$ and $[b^m,g_\lambda^{*,\rho}]$ in the proofs of our results. That is to say, we only need to prove Theorems 1.2, 1.4 and 1.6 respectively.

Proof of Theorem 1.2. Let $f\in C_c^\infty(\mathbb{R}^n)$. Then by Lemma 2.4, we have $f\in L^{p(\cdot)}(\mathbb{R}^n)$. For any $g\in L^{p'(\cdot)}(\mathbb{R}^n)\subset L^1_{\rm loc}(\mathbb{R}^n)$ to be $\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le 1$, since $g_\lambda^{*,\rho}$ is weak (1,1) type (see [28]), it satisfies (2.1) in Lemma 2.6. Thus, applying Lemma 2.5 and 2.6, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|g_\lambda^{*,\rho}(f)(x)g(x)|dx&\le C\int_{\mathbb{R}^n}M^\sharp_\tau(g_\lambda^{*,\rho})(f)(x)M(g)(x)dx\\&\le C\int_{\mathbb{R}^n}(1/\tau)^{1/\delta}(g_\lambda^{*,\rho}(f))^\sharp_\delta(x)M(g)(x)dx,\end{aligned}$$

where $\delta,\tau\in(0,1)$.

Noting that if $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, then $p'(\cdot)\in\mathcal{B}(\mathbb{R}^n)$ (Lemma 2.3). Together with generalized Hölder inequality (Lemma 2.2 and (1) in Lemma 2.7, we get

$$\begin{aligned}\int_{\mathbb{R}^n} g_\lambda^{*,\rho}(f)(x)g(x)dx &\le C\int_{\mathbb{R}^n} M(f)(x)M(g)(x)dx\\ &\le C\|M(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|M(g)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ &\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, using Lemma 2.1, we have

$$\|g_\lambda^{*,\rho}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le\|g_\lambda^{*,\rho}(f)\|^0_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Furthermore, by Lemma 2.4, we know that for any $f\in L^{p(\cdot)}(\mathbb{R}^n)$

$$\|g_\lambda^{*,\rho}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.4. Let $b\in BMO(\mathbb{R}^n)$, $f\in C_c^\infty(\mathbb{R}^n)$. Then by Lemma 2.4, we have $f\in L^{p(\cdot)}(\mathbb{R}^n)$. For any $g\in L^{p'(\cdot)}(\mathbb{R}^n)\subset L^1_{\rm loc}(\mathbb{R}^n)$ to be $\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\le 1$, noting that $[b^m,g_\lambda^{*,\rho}]$ is bounded on usual Lebesgue spaces $L^p(\mathbb{R}^n)$ (see [12]), so it satisfies (2.1) in Lemma 2.6. Thus, applying Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|[b^m,g_\lambda^{*,\rho}](f)(x)g(x)|dx &\le C\int_{\mathbb{R}^n} M^\sharp_\tau([b^m,g_\lambda^{*,\rho}])(f)(x)M(g)(x)dx\\ &\le C\int_{\mathbb{R}^n}(1/\tau)^{1/\delta}([b^m,g_\lambda^{*,\rho}](f))^\sharp_\delta(x)M(g)(x)dx,\end{aligned}$$

where $\delta,\tau\in(0,1)$.

By (2) in Lemma 2.7, for $0<\delta<l<1$

$$\begin{aligned}\int_{\mathbb{R}^n}|[b^m,g_\lambda^{*,\rho}](f)(x)g(x)|dx &\le C\sum_{j=0}^{m-1}\|b\|_*^{m-j}\int_{\mathbb{R}^n} M_l([b^j,g_\lambda^{*,\rho}](f))(x)M(g)(x)dx\\ &\quad +C\|b\|_*\int_{\mathbb{R}^n} M^{m+1}(f)(x)M(g)(x)dx\\ &\triangleq I_1+I_2.\end{aligned}$$

Observing that for $0<l<1$, $j\in\mathbb{N}$, we have

$$M_l([b^j,g_\lambda^{*,\rho}](f))(x)\le M([b^j,g_\lambda^{*,\rho}](f))(x)\ \text{ a.e. } x\in\mathbb{R}^n.$$

Meanwhile, as $m=0$, by Theorem 1.2, we know that $[b^0,g_\lambda^{*,\rho}]=g_\lambda^{*,\rho}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Thus, if $m=1$, by the generalized Hölder inequality (Lemma 2.2) and Lemma 2.3, for $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}I_1=&C\|b\|_*\int_{\mathbb{R}^n}M_l(g_\lambda^{*,\rho}(f))(x)M(g)(x)dx\\ \le&C\|b\|_*\int_{\mathbb{R}^n}M(g_\lambda^{*,\rho}(f))(x)M(g)(x)dx\\ \le&C\|b\|_*\|M(g_\lambda^{*,\rho}(f))\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|M(g)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ \le&C\|b\|_*\|g_\lambda^{*,\rho}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ \le&C\|b\|_*\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

On the other hand, also using generalized Hölder inequality (Lemma 2.2) and Lemma 2.3, for $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$,

$$\begin{aligned}I_2\le&C\|b\|_*^m\|M^{m+1}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|M(g)\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ \le&C\|b\|_*^m\|M^{m}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\ \le&C\|b\|_*^m\|M^{m-1}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ \le&C\|b\|_*^m\|M^{m-2}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le\cdots\cdots\\ \le&C\|b\|_*^m\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

According to the estimates of I_1 and I_2 above and Lemma 2.1, we can obtain

$$\int_{\mathbb{R}^n}|[b,g_\lambda^{*,\rho}](f)(x)g(x)|dx\le I_1+I_2\le C\|b\|_*\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

and

$$\|[b,g_\lambda^{*,\rho}]\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le\|[b,g_\lambda^{*,\rho}]\|^0_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_*\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Thus, as $m=1$, the conclusion of Theorem 1.4 holds. Take the similar steps as $[b^m,g_\lambda^{*,\rho}]$ with $m=1$, we shall successively get

$$\|[b^m,g_\lambda^{*,\rho}]\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_*^m\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\quad\text{for } m=2,3,\cdots.$$

Hence, by Lemma 2.4, for any $f\in L^{p(\cdot)}(\mathbb{R}^n)$, we have

$$\|[b^m,g_\lambda^{*,\rho}]\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\|b\|_*^m\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 1.4. □

Proof of Theorem 1.6. Let $b\in Lip_\beta(\mathbb{R}^n)$, $0<\beta<1$. Then by Definition 1.3, we shall get

$$|b(x)-b(y)|\le|x-y|^\beta\|b\|_{Lip_\beta(\mathbb{R}^n)}.$$

Thus, for any $f\in L^{p(\cdot)}(\mathbb{R}^n)$

$$
\begin{aligned}
&[b^m,g_\lambda^{*,\rho}](f)(x)\\
=&\Big(\int_0^\infty\int_{\mathbb{R}^n}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\Big|\frac{1}{t^\rho}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}[b(x)-b(z)]^m f(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}\\
\le& C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\Big(\int_0^\infty\int_{\mathbb{R}^n}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\times\Big|\frac{1}{t^\rho}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}|x-z|^{m\beta}f(z)dz\Big|^2\frac{dydt}{t^{n+1}}\Big)^{1/2}.
\end{aligned}
$$

Using the Minkowski's inequality, we have

$$
\begin{aligned}
&[b^m,g_\lambda^{*,\rho}](f)(x)\\
\le& C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|f(z)|}{|x-z|^{-m\beta}}\times\Big(\int_0^\infty\int_{|y-z|\le t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\Big)^{1/2}dz\\
\le& C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|f(z)|}{|x-z|^{-m\beta}}\times\Big(\int_0^{|x-z|}\int_{|y-z|\le t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\Big)^{1/2}dz\\
&+C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|f(z)|}{|x-z|^{-m\beta}}\times\Big(\int_{|x-z|}^\infty\int_{|y-z|\le t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\Big)^{1/2}dz.
\end{aligned}
$$

Since $z\in\mathbb{R}^n$, $|y-z|\le t$, then $|y-z|\sim|y|$. Thus, for $\Omega\in L^2(S^{n-1})$ and $\rho>n/2$, the following inequality holds:

$$
\begin{aligned}
&\int_{|y-z|\le t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\le\int_{|y|\le t}\frac{|\Omega(y)|^2}{|y|^{2n-2\rho}}dy\\
\le&\int_0^t r^{2\rho-n-1}dt\int_{S^{n-1}}|\Omega(y')|^2d\sigma(y')\le t^{2\rho-n}\|\Omega\|^2_{L^2(S^{n-1})}.
\end{aligned}
$$

Noticing that $|x-z|\le|x-y|+|y-z|\le|x-y|+t$ and $\Omega\in L^2(S^{n-1})$, therefore, by the inequality above, for $\lambda>2$, there exists $\varepsilon:0<\varepsilon<(\lambda-2)n$, such that

$$
\begin{aligned}
&\int_0^{|x-z|}\int_{|y-z|\le t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\\
\le&\int_0^{|x-z|}\int_{|y-z|\le t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n-2n-\varepsilon}\frac{1}{|x-z|^{2n+\varepsilon}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho-n-\varepsilon+1}}\\
\le&\frac{1}{|x-z|^{2n+\varepsilon}}\int_0^{|x-z|}\int_{|y-z|\le t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho-n-\varepsilon+1}}\\
\le&\frac{\|\Omega\|^2_{L^2(S^{n-1})}}{|x-z|^{2n+\varepsilon}}\int_0^{|x-z|}t^{\varepsilon-1}dt\le C|x-z|^{-2n}
\end{aligned}
$$

and

$$\begin{aligned}
&\int_{|x-z|}^{\infty}\int_{|y-z|\le t}\Big(\frac{t}{t+|x-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\\
\le&\int_{|x-z|}^{\infty}\int_{|y-z|\le t}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{2\rho+n+1}}\\
\le&\|\Omega\|^2_{L^2(S^{n-1})}\int_{|x-z|}^{\infty}t^{-2n-1}dt\le C|x-z|^{-2n}.
\end{aligned}$$

Combined with the above estimates, we obtain

$$[b^m,g^{*,\rho}_{\lambda}](f)(x)\le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|f(z)|}{|x-z|^{n-m\beta}}dz\le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}I_{m\beta}(|f|)(x).$$

Applying Lemma 2.8, take $\alpha=m\beta<n$, we get

$$\|[b^m,g^{*,\rho}_{\lambda}](f)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|I_{m\beta}(|f|)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

The proof of Theorem 1.6 is finished. □

Acknowledgments

Shuangping Tao is supported by National Natural Foundation of China (Grant Nos. 11161042 and 11071250).

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