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The Strong Approximation of Functions by Fourier-Vilenkin Series in Uniform and Hölder Metrics

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Abstract. We will study the strong approximation by Fourier-Vilenkin series using matrices with some general monotone condition. The strong Vallee-Poussin, which means of Fourier-Vilenkin series are also investigated.

Key Words: Vilenkin systems, strong approximation, generalized monotonicity.

AMS Subject Classifications: 40F05, 42C10, 43A55, 43A75

1 Introduction

Let $\mathbf{P}=\{p_n\}_{n=1}^{\infty}$ be a sequence of natural numbers such that $2\le p_i\le N$, $i\in\mathbb{N}=\{1,2,\cdots\}$. By definition $\mathbb{Z}(p_j)=\{0,1,\cdots,p_j-1\}$, $m_0=1$, $m_n=p_1p_2\cdots p_n$ for $n\in\mathbb{N}$. Then every $x\in[0,1)$ has an expansion

$$x=\sum_{n=1}^{\infty}\frac{x_n}{m_n},\quad x_n\in\mathbb{Z}(p_n),\quad n\in\mathbb{N}. \tag{1.1}$$

For $x=k/m_l$, $0<k<m_l$, $k,l\in\mathbb{N}$, we take the expansion with a finite number of $x_n\neq 0$. Let $G(\mathbf{P})$ be the Abel group of sequences $\mathbf{x}=(x_1,x_2,\cdots)$, $x_n\in\mathbb{Z}(p_n)$, with addition $\mathbf{x}\oplus\mathbf{y}=\mathbf{z}=(z_1,z_2,\cdots)$, where $z_n\in\mathbb{Z}(p_n)$ and $z_n=x_n+y_n \pmod{p_n}$, $n\in\mathbb{N}$. We define maps $g:[0,1)\to G(\mathbf{P})$ and $\lambda:G(\mathbf{P})\to[0,1)$ by formulas $g(x)=(x_1,x_2,\cdots)$, where x is in the form (1.1) and $\lambda(\mathbf{x})=\sum_{i=1}^{\infty}x_i/m_i$, where $\mathbf{x}\in G(\mathbf{P})$. Then for $x,y\in[0,1)$, we can introduce $x\oplus y:=\lambda(g(x)\oplus g(y))$, if $\mathbf{z}=g(x)\oplus g(y)$ does not satisfy equality $z_i=p_i-1$ for all $i\ge i_0$. In a similar way, we introduce $x\ominus y$ and for all $x,y\in[0,1)$ generalized distance

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$\rho(x,y)=\lambda(g(x)\ominus g(y))$. Every $k\in\mathbb{Z}_+=\{0,1,2,\cdots\}$ can be expressed uniquely in the form of

$$k=\sum_{n=1}^{\infty}k_n m_{n-1},\quad k_n\in\mathbb{Z}_n,\quad n\in\mathbb{N}. \tag{1.2}$$

For a given $x\in[0,1)$ with expansion (1.1) and $k\in\mathbb{Z}_+$ with expansion (1.2), we set $\chi_k(x)=\exp(2\pi i\sum_{j=1}^{\infty}x_jk_j/p_j)$. The system $\{\chi_k\}_{k=0}^{\infty}$ is called a multiplicative or Vilenkin system. It is orthonormal and complete in $L[0,1)$ and we have

$$\chi_k(x\oplus y)=\chi_k(x)\chi_k(y),\quad \chi_k(x\ominus y)=\chi_k(x)\overline{\chi_k(y)},$$

for a.e. y, whenever $x\in[0,1)$ is fixed (see [8, Section 1.5]).

The Fourier-Vilenkin coefficients and partial Fourier-Vilenkin sums for $f\in L^1[0,1)$ are defined by

$$\hat{f}(k)=\int_0^1 f(x)\overline{\chi_k(x)}dx,\quad k\in\mathbb{Z}_+,\quad S_n(f)(x)=\sum_{k=0}^{n-1}\hat{f}(k)\chi_k(x),\quad n\in\mathbb{N}.$$

If $f,g\in L^1[0,1)$, then $f*g(x)=\int_0^1 f(x\ominus t)g(t)dt=\int_0^1 f(t)g(x\ominus t)dt$. For Dirichlet kernel $D_n(t)=\sum_{k=0}^{n-1}\chi_k(t)$, $n\in\mathbb{N}$, we have an equality $S_n(f)(x)=\int_0^1 f(x\ominus t)D_n(t)dt$. The space $L^p[0,1)$, $1\le p<\infty$ consists of all measurable functions f on $[0,1)$ with finite norm $\|f\|_p=(\int_0^1|f(t)|^pdt)^{1/p}$. If $\omega^*(f,\delta)_\infty:=\sup\{|f(x)-f(y)|:x,y\in[0,1),\ \rho(x,y)<\delta\}$, $\delta\in[0,1]$, then $C^*[0,1)$ contains all functions f with property $\lim_{h\to0}\omega^*(f,h)_\infty=0$ and finite norm $\|f\|_\infty=\sup_{x\in[0,1)}|f(x)|$.

Let us introduce a modulus of continuity $\omega^*(f,\delta)_p=\sup_{0<h<\delta}\|f(x\ominus h)-f(x)\|_p$ in $L^p[0,1)$, $1\le p<\infty$. If $\mathcal{P}_n=\{f\in L^1[0,1):\hat{f}(k)=0,k\ge n\}$, then $E_n(f)_p=\inf\{\|f-t_n\|_p,t_n\in\mathcal{P}_n\}$, $1\le p\le\infty$. Let $\omega(\delta)$ be a function of modulus of continuity type ($\omega(\delta)\in\Omega$), i.e., $\omega(\delta)$ is continuous and increasing on $[0,1)$ and $\omega(0)=0$. Then the space $H_p^\omega[0,1)$ consists of $f\in L^p[0,1)$ $(1\le p<\infty)$ or $f\in C^*[0,1)$ $(p=\infty)$ such that $\omega^*(f,\delta)_p\le C\omega(\delta)$, where C depends only on f. Denote by h_p^ω the subspace of H_p^ω consioting of all functions f such that $\lim_{h\to0}\omega^*(f,h)_p/\omega(h)=0$. The spaces $h_p^\omega[0,1)$ and $H_p^\omega[0,1)$, $1\le p\le\infty$, with the norm $\|f\|_{p,\omega}=\|f\|_p+\sup_{0<h<1}\omega^*(f,h)_p/\omega(h)$ are Banach ones. In $h_p^\omega[0,1)$ we can consider $E_n(f)_{p,\omega}=\inf\{\|f-t_n\|_{p,\omega},t_n\in\mathcal{P}_n\}$, $n\in\mathbb{N}$.

Let $A=\{a_{nk}\}_{n,k=1}^{\infty}$ be a lower triangle matrix such that

$$a_{n,k}\ge0,\quad n,k\in\mathbb{N},\quad \sum_{k=1}^{n}a_{n,k}=1. \tag{1.3}$$

Using matrix A, we can define a summation method by formula

$$T_n(f)(x)=\sum_{k=1}^{n}a_{n,k}S_k(f)(x).$$

In the case of trigonometric system and monotone by k sequence $\{a_{nk}\}_{n,k=0}^{\infty}$, the estimates of $\|f-T_n(f)\|_\infty$ were obtained by P. Chandra [4] in terms of modulus of continuity. Later L. Leindler [10] generalized these results to the cases

$$\sum_{k=m}^{n-1}|a_{n,k}-a_{n,k+1}|\le Ca_{n,m},\quad 1\le m\le n-1,\quad n\in\mathbb{N}, \tag{1.4}$$

and

$$\sum_{k=1}^{m-1}|a_{n,k}-a_{n,k+1}|\le Ca_{n,m},\quad 1\le m\le n,\quad n\in\mathbb{N}. \tag{1.5}$$

Here C doesn't depend on m, n. For Vilenkin system $\{\chi_k\}_{k=0}^{\infty}$ the estimates of $\|f-T_n(f)\|_p$, $1\le p\le\infty$, and $\|f-T_n(f)\|_{p,v}$ for $f\in H_p^\omega$, where $v(t)=t^\beta$, $\omega(t)=t^\alpha$, $\beta<\alpha$, are obtained in [9]. Further we shall consider

$$R_n(f,r)(x)=\Big(\sum_{k=1}^{n}a_{n,k}|S_k(f)(x)-f(x)|^r\Big)^{1/r}.$$

The estimates of $\|R_n(f,r)\|_\infty$ for monotone by k sequence $\{a_{nk}\}_{n,k=0}^{\infty}$ with additional restrictions on their oscillations were proved by T. Xie and X. Sun in [19]. For matrices satisfying (1.4) and (1.5), similar results are established by B. Szal [16]. In [17], some estimates close to ones of P. Chandra [3] and L. Leindler [8] are obtained.

In the present paper, we study the rate of $\|R_n(f,r)\|_p$, $1<p\le\infty$, where a matrix A satisfies one of the following conditions:

$$\sum_{k=m}^{2m-1}|a_{n,k}-a_{n,k+1}|\le Ka_{n,m},\quad 1\le m\le\frac{(n-1)}{2}, \tag{1.6}$$

or

$$\sum_{k=[m/2]}^{m-1}|a_{n,k}-a_{n,k+1}|\le Ka_{n,m},\quad 2\le m\le n. \tag{1.7}$$

In both cases K does not depend on n,m. The class GM of real non-negative sequences $\{a_i\}_{i=0}^{\infty}$, satisfying inequality $\sum_{k=m}^{2m-1}|a_k-a_{k+1}|\le Ca_m$, $m\in\mathbb{N}$, was introduced by S. Tikhonov [18]. In particular, in [18] it is established that GM contains the class of quasi monotone sequences QM (with property $a_n n^{-\tau}\downarrow 0$ for some $\tau\ge 0$ and $n\in\mathbb{N}$). Further, we assume that $\omega(t)\in\Omega$ satisfies Δ_2-condition, i.e., $\omega(t)\le C\omega(t/2)$, $t\in[0,1)$.

Some results are devoted to the strong Fejer and de la Valle-Poussin means (Lemmas 2.7, 2.8, Theorem 3.5, Corollaries 3.1, 3.2).

2 Auxiliary propositions

Lemma 2.1. *For $f \in L^p[0,1)$, $1<p<\infty$, we have $\|S_n(f)\|_p \le C\|f\|_p$, $n\in\mathbb{N}$, where C does not depend on f and n. As a corollary, we obtain inequality*

$$\|S_n(f)-f\|_p \le (C+1)E_n(f)_p, \quad n\in\mathbb{N}.$$

For arbitrary sequence $\{p_n\}_{n=1}^{\infty}$, Lemma 2.1 is established by W.-S. Young [20], F. Schipp [14] and P. Simon [15].

Let $\mathbf{g}=(g_1,g_2,\cdots,g_j,\cdots)$, where g_j are measurable on $[0,1)$ functions. Let us define

$$\|\mathbf{g}\|_{L^p(l^r)} = \Big\| \Big(\sum_{j=1}^{\infty}|g_j|^r\Big)^{1/r}\Big\|_p, \quad \|\mathbf{g}\|_{l^r(L^p)} = \Big(\sum_{j=1}^{\infty}\|g_j\|_p^r\Big)^{1/r}.$$

Lemma 2.2. *If $1\le r\le p<\infty$, then $\|\mathbf{g}\|_{L^p(l^r)} \le \|\mathbf{g}\|_{l^r(L^p)}$.*

The proof of Lemma 2.2 is similar to the case $r=2$, studied by S. Fridli [5].

Lemma 2.3. *Let $\{a_n\}_{n=1}^{\infty}\in\mathbb{C}$. Then for $q\in(1,\infty)$ a Sidon-type inequality*

$$\Big\|\sum_{k=1}^{n}a_kD_k\Big\|_1 \le C(q)n^{1-1/q}\Big(\sum_{k=1}^{n}|a_k|^q\Big)^{1/q} \tag{2.1}$$

holds. For $q=\infty$, we also have

$$\Big\|\sum_{k=1}^{n}a_kD_k\Big\|_1 \le Cn\sup_{1\le k\le n}|a_k|.$$

In an implicit form, inequality (2.1) is proved in [3] for bounded sequences $\{p_n\}_{n=1}^{\infty}$. M. Avdispahic and M. Pepic [2] obtained its analog in a more general case.

The following Lemma is due to A. V. Efimov (see [8, Section 10.5]).

Lemma 2.4. *Let $f\in L^p[0,1)$, $1\le p<\infty$, or $f\in C^*[0,1)$. Then*

$$2^{-1}\omega^*(f,1/m_n)_p \le E_{m_n}(f)_p \le \|f-S_{m_n}(f)\|_p \le \omega^*(f,1/m_n)_p, \quad n\in\mathbb{N}.$$

Lemma 2.5. *If $\omega(t)\in\Omega$ satisfies the Δ_2-condition, then from $f\in H_p^{\omega}$ it follows that $E_n(f)_p\le C\omega(1/n)$, $n\in\mathbb{N}$.*

Proof. Let $\|f\|_{p,\omega}=C_1$, $\omega(t)\le C_2\omega(t/2)$, $t\in[0,1)$, and $n\in[m_k,m_{k+1})$, $k\in\mathbb{Z}_+$. Then by Lemma 2.4

$$E_n(f)_p \le E_{m_k}(f)_p \le \omega^*(f,1/m_k)_p \le C_1\omega(1/m_k) \le C_1C_2^{[\log_2 N]+1}\omega(1/m_{k+1}) \le C_3\omega(1/n).$$

Thus, Lemma 2.5 is proved. □

Lemma 2.6. *(i) Let a matrix A satisfies conditions* (1.3) *and* (1.6)*. Then $a_{n,i} \le (K+1)a_{n,m}$ for $m \le i \le 2m \le n$, where K is the constant from* (1.6)*.*

(ii) Let a matrix A satisfies conditions (1.3) *and* (1.7)*. Then $a_{n,i} \le (K+1)a_{n,m}$ for $[m/2] \le i \le m$, where K is the constant from* (1.7)*.*

Proof. Part (i) may be found in [18]. In order to establish (ii), we find for $[m/2] \le i < m$ that

$$Ka_{n,m} \ge \sum_{k=[m/2]}^{m-1} |a_{n,k} - a_{n,k+1}| \ge \Big| \sum_{k=i}^{m-1} (a_{n,k} - a_{n,k+1}) \Big| \ge a_{n,i} - a_{n,m},$$

whence $(K+1)a_{n,m} \ge a_{n,i}$. In the case $i=m$, the statement (ii) is evident. Thus, Lemma 2.6 is proved. □

The trigonometric counterpart of Lemma 2.7 is due to L. Leindler [11].

Lemma 2.7. *Let $f \in C^*[0,1)$, $1 \le r < \infty$. Then*

$$\|\sigma_n(f,r)\|_\infty := \Big\| \Big(n^{-1} \sum_{k=1}^n |S_k(f)(\cdot)|^r \Big)^{1/r} \Big\|_\infty \le M \|f\|_\infty, \quad n \in \mathbb{N}, \tag{2.2}$$

where M does not depend on $n \in \mathbb{N}$ and f.

Proof. Let us consider $i \in \mathbb{N}$ such that $n \in [m_{i-1}, m_i)$. Then

$$n^{-1} \sum_{k=1}^n |S_k(f)(x)|^r \le N m_i^{-1} \sum_{k=1}^n |S_k(f)(x)|^r = N \|h\|_r,$$

where $h(t)$ equals to $S_k(f)(x)$ on $I_k^i = [k/m_i, (k+1)/m_i)$, $1 \le k \le n$, and $h(t) = 0$ on other I_k^i. It is clear that $\|h\|_r = \sup \int_0^1 h(t)g(t)dt$, where sup is taken over constant on I_k^i functions $g(t)$ with the property $\|g\|_{r'} \le 1$, $1/r + 1/r' = 1$. In other words, if $g(t) = a_k$ for $t \in I_k^i$, $1 \le k \le n$, then

$$\Big(\sum_{k=1}^n |a_k|^{r'} \Big)^{1/r'} \le m_i^{1/r'} \quad \Big(\sup_{1 \le k \le n} |a_k| \le 1 \text{ for } r=1 \Big). \tag{2.3}$$

We have

$$\begin{aligned}\int_0^1 h(t)g(t)dt &= m_i^{-1} \sum_{k=1}^n a_k S_k(f)(x) = m_i^{-1} \int_0^1 \sum_{k=1}^n a_k D_k(t) f(x \ominus t) dt \\ &\le m_i^{-1} \|f\|_\infty \Big\| \sum_{k=1}^n a_k D_k \Big\|_1.\end{aligned}$$

Using (2.1) and (2.3), we find that

$$\|\sigma_n(f,r)\|_\infty \le C_1 m_i^{-1} \|f\|_\infty m_i^{1/r} \Big(\sum_{k=1}^n |a_k|^{r'} \Big)^{1/r'} = C_1 \|f\|_\infty.$$

So, Lemma 2.7 is proved. □

The inequality (2.5) of Lemma 2.8 in the case $m=[n/2]$ is stated without proof by S. Fridli and F. Schipp [6] for some general systems. In [6] also one can find the idea of application of (2.1) to problems of strong approximation (see also [7]).

Lemma 2.8. *Let* $f\in C^*[0,1)$, $1\le r<\infty$, $\nu n\le m<n$, *where* $\nu\in(0,1)$. *Then*

$$\|U_{n,m}(f,r)\|_\infty:=\Big\|\Big((m+1)^{-1}\sum_{k=n-m}^{n}|S_k(f)(\cdot)|^r\Big)^{1/r}\Big\|_\infty\le M(\nu)\|f\|_\infty \tag{2.4}$$

and

$$\|V_{n,m}(f,r)\|_\infty:=\Big\|\Big((m+1)^{-1}\sum_{k=n-m}^{n}|S_k(f)(\cdot)-f(\cdot)|^r\Big)^{1/r}\Big\|_\infty\le(M(\nu)+1)E_{n-m}(f)_\infty, \tag{2.5}$$

where $M(\nu)$ *does not depend on* $n,m\in\mathbb{N}$ *and* f.

Proof. By (2.2) we have

$$\begin{aligned}(m+1)^{1/r}\|U_{n,m}(f,r)\|_\infty&=\Big\|\Big(\sum_{k=n-m}^{n}|S_k(f)(\cdot)|^r\Big)^{1/r}\Big\|_\infty\\&\le\Big\|\Big(\sum_{k=1}^{n}|S_k(f)(\cdot)|^r\Big)^{1/r}\Big\|_\infty+\Big\|\Big(\sum_{k=1}^{n-m-1}|S_k(f)(\cdot)|^r\Big)^{1/r}\Big\|_\infty\\&\le C_1(n^{1/r}+(n-m-1)^{1/r})\|f\|_\infty,\end{aligned}$$

whence (2.4) follows in virtue of inequality $\nu n\le m$.

The inequality (2.5) is derived from (2.4) by substitution $f-t_{n-m}$ instead of f, where $t_{n-m}\in\mathcal{P}_{n-m}$ and $\|f-t_{n-m}\|_\infty=E_{n-m}(f)_\infty$. Here we use the equality $S_k(t_{n-m})=t_{n-m}$ for $k\ge n-m$ and Minkowski inequality in l^r as follows:

$$\begin{aligned}\|V_{n,m}(f,r)\|_\infty\le&\Big\|\Big((m+1)^{-1}\sum_{k=n-m}^{n}|S_k(f-t_{n-m})(\cdot)|^r\Big)^{1/r}\Big\|_\infty\\&+\Big\|\Big((m+1)^{-1}\sum_{k=n-m}^{n}|(f-t_{n-m})(\cdot)|^r\Big)^{1/r}\Big\|_\infty\\=&\|U_{n,m}(f-t_{n-m},r)\|_\infty+E_{n-m}(f)_\infty\le C_2E_{n-m}(f)_\infty.\end{aligned} \tag{2.6}$$

So, Lemma 2.8 is proved. ☐

Remark 2.1. The counterparts of (2.4) and (2.5) for $\|\cdot\|_p$ and $p\ge r$ are easily follows from Lemma 2.1 and Lemma 2.2 (see the proof of Theorem 3.2).

The following lemma is an analog of Leindler-Meir-Totik theorem [12].

Lemma 2.9. *Let $\omega,\mu\in\Omega$ be such that $\lambda(t)=\omega(t)/\mu(t)$ is increasing on $(0,1)$. Then for an operator $A_n(f)=K_n*f$, $K_n\in L^1[0,1)$, and $f\in H_p^\omega$ the inequality*

$$\|A_n(f)-f\|_{p,\mu}\le C(\|A_n(f)-f\|_p/\mu(n^{-1})+\lambda(n^{-1})(1+\|A_n\|_{L^p\to L^p}))$$

holds.

The proof of Lemma 2.9 is similar to one of Theorem 8 in [9].

Lemma 2.10. *Let $\omega,\mu\in\Omega$ be such that $\lambda(t)=\omega(t)/\mu(t)$ is increasing on $(0,1)$. If ω satisfies Δ_2-condition and $f\in H_p^\omega$, then $E_n(f)_{p,\mu}\le C\lambda(1/n)$, $n\in\mathbb{N}$.*

Proof. Let $K_n=\sum_{k=n}^{2n-1}D_k/n$ and $A_n(f)=K_n*f$. Then for any $t_n\in\mathcal{P}_n$, we have $K_n*t_n=t_n$. In virtue of Lemma 2.5 and by the standard procedure, we deduce $\|A_n(f)-f\|_p\le C_1E_n(f)_p\le C_2\omega(1/n)$. In addition, $\|A_n(f)\|_{L^p\to L^p}\le\|K_n\|_1\le C_3$ (see, for example, [9]). By Lemma 2.9, we obtain

$$\|A_n(f)-f\|_{p,\mu}\le C_4(\omega(n^{-1})/\mu(n^{-1})+\lambda(n^{-1}))=2C_4\lambda(n^{-1}).$$

Thus, $E_{2n}(f)_{p,\mu}\le 2C_4\lambda(n^{-1})$. Using monotonicity of best approximations and Δ_2-condition, we get the inequality of Lemma. □

Remark 2.2. The condition of increasing of $\omega(t)/\mu(t)$ introduced by J. Prestin and S. Prössdorf [13] is suitable for some applications, for example, the theory of multiplicators of Lipschitz classes (see [1]).

3 Main results

Theorem 3.1. *Let a matrix A satisfies conditions* (1.3) *and* (1.7)*, $f\in C^*[0,1)$, $r\ge 1$. Then*

$$\|R_n(f,r)\|_\infty=\mathcal{O}\Big(\sum_{k=0}^{[\log_2 n]-1}2^kE_{2^k}^r(f)_\infty a_{n,2^{k+1}}+na_{nn}E_{[(n+1)/2]}^r(f)_\infty\Big)^{1/r}.$$

Proof. Let $n\in\mathbb{N}$ and $j=j(n)\in\mathbb{Z}_+$ be defined by inequality $2^j\le n<2^{j+1}$, i.e., $j=[\log_2 n]$. Then we have

$$|R_n(f,r)(x)|^r==\sum_{k=1}^{j}\sum_{i=2^{k-1}}^{2^k-1}a_{n,i}|S_i(f)(x)-f(x)|^r+\sum_{i=2^j}^{n}a_{n,i}|S_i(f)(x)-f(x)|^r=:I_1+I_2.$$

Using Abel's transform (summation by parts), (1.7) and Lemma 2.6, we obtain

$$\begin{aligned}I_1\le&\sum_{k=1}^{j}\Big(\sum_{i=2^{k-1}}^{2^k-2}|a_{n,i}-a_{n,i+1}|\sum_{l=2^{k-1}}^{i}|S_l(f)(x)-f(x)|^r++a_{n,2^k-1}\sum_{i=2^{k-1}}^{2^k-1}|S_i(f)(x)-f(x)|^r\Big)\\ \le&C_1\sum_{k=1}^{j}a_{n,2^k}\sum_{i=2^{k-1}}^{2^k-1}|S_i(f)(x)-f(x)|^r.\end{aligned}$$

According to (2.5),

$$I_1 \le C_2 \sum_{k=1}^{j} a_{n,2^k} 2^{k-1} E^r_{2^{k-1}}(f)_\infty = C_2 \sum_{k=0}^{j-1} a_{n,2^{k+1}} 2^k E^r_{2^k}(f)_\infty. \tag{3.1}$$

It is clear that (1.7) implies $\sum_{i=[(n+1)/2]}^{n-1} |a_{n,i} - a_{n,i+1}| \le C_3 a_{n,n}$. Since $[(n+1)/2] \le (n+1)/2 \le 2^j$, using of Abel's transform and (2.5) gives

$$\begin{aligned} I_2 \le & \sum_{k=[(n+1)/2]}^{n-1} |a_{n,k} - a_{n,k+1}| \sum_{i=[(n+1)/2]}^{k} |S_i(f)(x) - f(x)|^r + a_{n,n} \sum_{i=[(n+1)/2]}^{n} |S_i(f)(x) - f(x)|^r \\ \le & C_5 n a_{n,n} E^r_{[(n+1)/2]}(f)_\infty. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), the statement of theorem follows. □

Theorem 3.2. *Let a matrix A satisfies conditions* (1.3) *and* (1.7), $f \in L^p[0,1)$, $1 < p < \infty$, $p \ge r \ge 1$. *Then*

$$\|R_n(f,r)\|_p = \mathcal{O}\Big(\sum_{k=1}^{n} a_{n,k} E^r_k(f)_p \Big)^{1/r}. \tag{3.3}$$

Proof. Applying Lemma 2.2, we have

$$\|R_n(f,r)\|_p = \Big\| \Big(\sum_{k=1}^{n} a_{n,k} |S_k(f)(\cdot) - f(\cdot)|^r \Big)^{1/r} \Big\|_p \le \Big(\sum_{k=1}^{n} a_{n,k} \|S_k(f)(\cdot) - f(\cdot)\|^r_p \Big)^{1/r}.$$

Therefore by Lemma 2.1, $\|R_n(f,r)\|^r_p \le C \sum_{k=1}^{n} a_{n,k} E^r_k(f)_p$, whence the inequality of theorem follows. □

Theorem 3.3. *Let a matrix A satisfies conditions* (1.3) *and* (1.6), $f \in C^*[0,1)$, $r \ge 1$. *Then*

$$\|R_n(f,r)\|_\infty = \mathcal{O}\Big(\sum_{k=1}^{n} a_{n,k} E^r_k(f)_\infty \Big)^{1/r}.$$

Proof. We shall use again $j = j(n)$ with property $2^j \le n < 2^{j+1}$, i.e., $j = [\log_2 n]$. Applying Abel's transform, we obtain

$$\begin{aligned} & (R_n(f,r)(x))^r \\ \le & \sum_{k=1}^{j} \Big(\sum_{i=2^{k-1}}^{2^k-2} |a_{n,i} - a_{n,i+1}| \sum_{l=2^{k-1}}^{i} |S_l(f)(x) - f(x)|^r + a_{n,2^k-1} \sum_{i=2^{k-1}}^{2^k-1} |S_i(f)(x) - f(x)|^r \Big) \\ & + \sum_{k=2^j}^{n-1} |a_{n,k} - a_{n,k+1}| \sum_{l=2^j}^{k} |S_l(f)(x) - f(x)|^r + a_{n,n} \sum_{k=2^j}^{n} |S_k(f)(x) - f(x)|^r. \end{aligned}$$

By (1.6), Lemma 2.6 and (2.5), we have

$$
\begin{aligned}
(R_n(f,r)(x))^r \le & C_1\Big(\sum_{k=1}^{j}\Big(2^{k-1}a_{n,2^{k-1}}E^r_{2^{k-1}}(f)_\infty+2^{k-1}a_{n,2^{k-1}}E^r_{2^{k-1}}(f)_\infty\Big)\Big)\\
&+C_2a_{n,2^j}2^jE^r_{2^j}(f)_\infty \le C_3\sum_{k=0}^{j}2^ka_{n,2^k}E^r_{2^k}(f)_\infty.
\end{aligned}
$$

Since

$$2^{k-1}a_{n,2^k}\le C_5\sum_{i=2^{k-1}}^{2^k-1}a_{n,i}$$

by Lemma 2.6, we find that

$$(R_n(f,r)(x))^r\le C_6\Big(a_{n,1}E^r_1(f)_\infty+\sum_{k=1}^{j}\sum_{i=2^{k-1}}^{2^k-1}a_{ni}E^r_i(f)_\infty\Big),$$

whence the inequality of theorem follows. □

Similarly to Theorem 3.2, one can prove

Theorem 3.4. *If a matrix A satisfies conditions* (1.3) *and* (1.6), $f\in L^p[0,1)$, $1<p<\infty$, $p\ge r\ge 1$, *then* (3.3) *holds.*

Theorems 3.3 and 3.4 imply

Corollary 3.1. Let $f\in L^p[0,1)$, $1<p<\infty$, $1\le r\le p$, or $f\in C^*[0,1)$ $(p=\infty)$, $1\le r<\infty$. Then

$$\Big\|\Big(n^{-1}\sum_{k=1}^{n}|S_k(f)-f|^r\Big)^{1/r}\Big\|_p=\mathcal{O}\Big(\sum_{k=1}^{n}E^r_k(f)_p/n\Big)^{1/r},\quad n\in\mathbb{N}.$$

In particular, for $r=1$ and $f\in Lip^*(\alpha,p)$ (i.e., $\omega^*(f,h)_p=\mathcal{O}(h^\alpha)$) we obtain

$$\Big\|n^{-1}\sum_{k=1}^{n}|S_k(f)-f|\Big\|_p=\begin{cases}\mathcal{O}(n^{-\alpha}), & 0<\alpha<1,\\ \mathcal{O}(\ln(n+1)/(n+1)), & \alpha=1,\\ \mathcal{O}(n^{-1}), & \alpha>1.\end{cases}$$

Remark 3.1. It is well known that for $f\in h^\omega_p$ and $\sigma_n(f)=\sum_{k=1}^{n}S_k(f)/n$, the equality $\lim_{n\to\infty}\|f-\sigma_n(f)\|_{p,\omega}=0$ holds (see [9] for $\omega(h)=h^\alpha$). In particular, for $f\in h^\omega_p$ we have $\lim_{n\to\infty}E_n(f)_{p,\omega}=0$.

Theorem 3.5 gives an analog of the estimate (2.5) for Hölder metric.

Theorem 3.5. *Let $f\in h^\omega_p$, $1<p\le\infty$, $p\ge r\ge 1$ for $p<\infty$ and $1\le r<\infty$ for $p=\infty$. If $\nu n\le m<n$, $\nu\in(0,1)$, then we have*

$$\|V_{n,m}(f,r)\|_{p,\omega}\le C(\nu)E_{n-m}(f)_{p,\omega}.$$

Proof. By Minkowski inequality and commutativity of translation and convolution, we have

$$\|U_{n,m}(f,r)(\cdot\ominus h)-U_{n,m}(f,r)(\cdot)\|_p\le\|U_{n,m}(f(\cdot\ominus h)-f(\cdot),r)\|_p. \tag{3.4}$$

Hence, in virtue of (2.4) and Remark 2.1, it follows that

$$\begin{aligned}\|U_{n,m}(f,r)\|_{p,\omega}\le&\|U_{n,m}(f,r)\|_p+\sup_{0<h<1}\frac{\|U_{n,m}(f(\cdot\ominus h)-f(\cdot),r)\|_p}{\omega(h)}\\ \le&C_1\Big(\|f\|_p+\sup_{0<h<1}\frac{\|(f(\cdot\ominus h)-f(\cdot)\|_p}{\omega(h)}\Big)=C_1\|f\|_{p,\omega},\end{aligned}$$

where in the case $p=\infty$, the constant C_1 is equal to $M(\nu)$ from Lemma 2.8. Let $t_{n-m}\in\mathcal{P}_{n-m}$ be such that $\|f-t_{n-m}\|_{p,\omega}=E_{n-m}(f)_{p,\omega}$. Using equality $S_k(t_{n-m})=t_{n-m}$ for $k\ge n-m$, we obtain similarly to (2.6)

$$\begin{aligned}\|V_{n,m}(f,r)\|_p\le&\|U_{n,m}(f-t_{n-m},r)\|_p+\|f-t_{n-m}\|_p\\ \le&C_1\|f-t_{n-m}\|_p+\|f-t_{n-m}\|_p\le(C_1+1)E_{n-m}(f)_{p,\omega}.\end{aligned} \tag{3.5}$$

On the other hand, by (3.4) and (3.5) (we use notation $\Delta_h f=f(\cdot\ominus h)-f(\cdot)$)

$$\begin{aligned}&\sup_{0<h<1}\|\Delta_h V_{n,m}(f,r)\|_p/\omega(h)\\ \le&\sup_{0<h<1}\|V_{n,m}(\Delta_h f,r)\|_p/\omega(h)\\ \le&\sup_{0<h<1}\big(\|U_{n,m}(\Delta_h(f-t_{n-m},r)\|_p+\|\Delta_h(f-t_{n-m})\|_p\big)/\omega(h)\\ \le&(C_1+1)\|f-t_{n-m}\|_{p,\omega}=(C_1+1)E_{n-m}(f)_{p,\omega}.\end{aligned} \tag{3.6}$$

Combining estimates (3.5) and (3.6), we finish the proof of theorem. □

Corollary 3.2. Let $1<p\le\infty$, $\omega,\mu\in\Omega$, where $\omega(t)$ satisfies Δ_2-condition, while $\lambda(t)=\omega(t)/\mu(t)$ is increasing on $(0,1)$ and $\lim_{t\to0}\lambda(t)=0$. If $f\in H_p^\omega$, $p\ge r\ge1$, and numbers $n,m\in\mathbb{N}$ are such that $\nu n\le m\le n$, $\nu\in(0,1)$, then $\|V_{n,m}(f,r)\|_{p,\mu}\le C\lambda((n-m)^{-1})$, $(n-m)\in\mathbb{N}$.

Proof. In virtue of Theorem 3.5, $\|V_{n,m}(f,r)\|_{p,\mu}\le C_1(\nu)E_{n-m}(f)_{p,\mu}$, while by Lemma 2.10, we have $E_{n-m}(f)_{p,\mu}\le C_2\lambda(1/(n-m))$. Substituting the second inequality into first one, we prove the theorem.

Following the idea of Szal [16], we assume in two last theorems that there exists $\alpha\in(0,1)$, such that $\omega^\alpha(t)/\mu(t)$ is increasing on $(0,1)$. We also require that $\omega,\mu\in\Omega$ and ω satisfies Δ_2-condition. □

Theorem 3.6. *Let a matrix A satisfies conditions* (1.3) *and* (1.7)*,* $f\in H_\infty^\omega[0,1)$*,* $r\ge1$*. Then*

$$\|R_n(f,r)\|_{\infty,\mu}\le C(1+na_{n,n})^{\alpha/r}\Big(\sum_{k=0}^{[\log_2 n]-1}2^k a_{n,2^{k+1}}\omega^r(2^{-k})+na_{n,n}\omega^r(n^{-1})\Big)^{(1-\alpha)/r}.$$

Proof. In virtue of Theorem 3.1 and Minkowski inequality

$$\begin{aligned}&\sup_{0<h<1}\|\Delta_h R_n(f,r)\|_\infty/\mu(h)\le \sup_{0<h<1}\|R_n(\Delta_h f,r)\|_\infty/\mu(h)\\ \le&\sup_{0<h<1}C_1\Big(\sum_{k=0}^{[\log_2 n]-1}2^kE^r_{2^k}(\Delta_h f)_\infty a_{n,2^{k+1}}+na_{n,n}E^r_{[(n+1)/2]}(\Delta_h f)_\infty\Big)^{1/r}\Big/\mu(h)\\ =:&\sup_{0<h<1}C_1A_n^{1/r}(h)/\mu(h).\end{aligned}$$

By Lemmas 2.4 and 2.5, the estimates

$$E_{2^k}(\Delta_h f)_\infty\le 2E_{2^k}(f)_\infty\le C_2\omega(2^{-k}),\quad E_{[(n+1)/2]}(\Delta_h f)_\infty\le C_3\omega((n+1)^{-1}),\tag{3.7}$$

hold. On the other hand, $E_k(\Delta_h f)_\infty\le\|\Delta_h f\|_\infty\le\omega(h)$, $k\in\mathbb{N}$, and as Corollary,

$$A_n(h)\le\omega^r(h)\Big(\sum_{k=0}^{[\log_2 n]-1}2^ka_{n,2^{k+1}}+na_{n,n}\Big)\le C_4\omega^r(h)(1+na_{n,n}),\tag{3.8}$$

since $2^ka_{n,2^{k+1}}\le C_5\sum_{i=2^{k+1}}^{2^{k+2}-1}a_i$ by Lemma 2.6 and $\sum_{k=1}^n a_{nk}=1$ by (1.3). Writing $A_n(h)$ as $A_n(h)=A_n(h)^\alpha A_n(h)^{1-\alpha}$ and applying (3.8) to the first factor and (3.7) to the second one, we obtain the required estimate for $\sup_{0<h<1}\|\Delta_h R_n(f,r)\|_\infty/\mu(h)$. For $\|R_n(f,r)\|_\infty$ similar result follows from Theorem 3.1 and second inequality (3.8). The theorem is proved. $\square$

Theorem 3.7. *Let a matrix A satisfies conditions* (1.3) *and* (1.6), $f\in L^p[0,1)$, $1<p<\infty$, *or* $f\in C^*[0,1)$ *(for $p=\infty$), $p\ge r\ge 1$. If $f\in H_p^\omega$, then*

$$\|R_n(f,r)\|_{\infty,\mu}\le C\Big(\sum_{k=0}^n a_{n,k}\omega^r(k^{-1})\Big)^{(1-\alpha)/r}.$$

The proof of Theorem 3.7 is similar to the one of Theorem 3.6, and uses Theorems 3.3 and 3.4 instead of Theorem 3.1.

Remark 3.2. The conterparts of Theorems 3.1 and 3.7, proved in [16], contain the term $\ln 2na_{n,n}$ instead of $na_{n,n}$ in the present paper (by authors opinion, it is more correctly to write $1+\ln^+ na_{n,n}$). Such estimates may have a better order of decreasing (for example, if $a_{n,n}=1$, $a_{n,k}=0$, $1\le k<n$). It will be interesting to refine Theorems 3.1 and 3.7 in a similar manner and to study $\|R_n(f,r)\|_p$ in the case of $p=1$.

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