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A THEORETICAL FRAMEWORK FOR THE CALCULATION OF HAUSDORFF MEASURE — SELF-SIMILAR SET SATISFYING OSC

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Abstract. A theoretical framework for the calculation of Hausdorff measure of self-similar sets satisfying OSC has been established.

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1 Introduction and Nan Results

It is well known that to calculate the Hausdorff measure of fractal sets is very difficult, even for simple sets, such as self-similar sets satisfying the open set condition(OSC,including SSC) and so there are few concrete results about computation of Hausdorff measure, unless the Hausdorff dimension is not larger than 1. Some authors have investigated the estimation and calculation and got some upper and lower bounds of the Hausdorff measure for self-similar sets satisfying OSC(see [6, 7, 8, 10, 11, 12, 14, 19]). A natural question is how to get the accurate value of them? In this paper we only discuss the case of self-similar sets satisfying OSC and our purpose is to establish a uniform theoretical framework for the calculation of Hausdorff measure for such fractal sets. Let $E \subset \mathbf{R}^n$ be a self-similar set satisfying OSC with $s = \dim_H E$. We have proved $H^s(E \cap U) \leq |U|^s$ (which will be called the measure-diameter's inequality) [12] for any $U \subset \mathbf{R}^n$. This inequality plays an important role for calculating the Hausdorff measure of the self-similar set satisfying OSC. The calcutation of the Hausdorff measure of E will be transformed to look for a solution U with $0 < |U|$ such that the equality holds in the above inequality, that is, $H^s(E \cap U) = |U|^s$. In this paper, we prove such a set U exists. Here, our proof is on existence behavior but not constructive and so in order to count genuinely the Hausdorff measure of E, we also must determine wholly U, including its diameter, geometric shape and location. We call this result the realization theorem with respect to the upper convex density 1. The upper convex density, introduced by Falconer [1], is an important notion giving rise to a series of new problems and opening a gate to understand deeply the structure of self-similar set and so far we work on it only a little. Our main results of this paper are as follows (for the definitions, terminologies and notations, see the next paragraph).

In this paper, we always let $E \subset \mathbf{R}^n (n > 0)$ be a self-similar set satisfying OSC and denote by $s = \dim_H E$ its Hausdorff dimension and $H^s(E)$ its $s-$dimensional Hausdorff measure.

Realization theorem. *There exists a set* $U \subset \mathbf{R}^n$ *with* $0 < |U|$ *such that*

$$\frac{H^s(E \cap U)}{|E \cap U|^s} = 1.$$

Corollary 1. $H^s(E) = |E \cap U|^s / \sum_{k>0} b_k$, *where* $|U| > 0$ *satisfies*

$$\frac{H^s(E \cap U)}{|E \cap U|^s} = 1$$

and each b_k *depends on* U.

Corollary 2. *There exists an almost everywhere best covering of* E, $\alpha = \{U_i : i > 0\}$, *such that*

$$H^s(E) = \sum_{i>0} |U_i|^s.$$

The proof of these results will be given later. Some discussions are given at the end of this paper.

2 Basic Concepts and Upper Convex Density

For some basic definitions and notations in Fractal Geometry, we refer to [1, 2, 3].

Denote by d the usual metric of $\mathbf{R}^n (n > 0)$. Let $D \subset \mathbf{R}^n$ be a bounded closed region and E a self-similar set yielded by $m (m > 0)$ (linear) similarities $S_j : D \to D$ with ratios $0 < c_j < 1, j = 1, 2, \ldots, m$, that is, $|S_j(x) - S_j(y)| = c_j |x - y|, \quad \forall\, x, y \in D,\ j = 1, 2, \cdots, m$ and E satisfies $E = \bigcup_j S_j(E)$. We say that E satisfies the open set condition (OSC) if there is a non-empty bounded open set $V \subset \mathbf{R}^n$ such that $\bigcup_j S_j(V) \subset V$ and

$$S_i(V) \cap S_j(V) = \emptyset,\ 1 \le i < j \le m.$$

Furthermore, we say that E also satisfies the strong separation condition (SSC),if

$$S_i(E)\cap S_j(E)=\emptyset, \qquad 1\le i<j\le m.$$

It is well known $E\subset\overline{V}$, where $\overline{V}$ is the closure of V, and $s=\dim_H(E)$ is determined by $\sum_{j=1}^{m}c_j^s=1$ and $0<H^s(E)<\infty$. So E is an $s-$set.

A covering $\alpha=\{U_i: i>0\}$ of E is called a best covering of E, if $H^s(E)=\sum_{i>0}|U_i|^s$;

A covering $\alpha=\{U_i: i>0\}$ is called an almost everywhere best covering of E if $H^(E-\bigcup_i U_i)=0$ and $H^s(E)=\sum_{i>0}|u_i|^s$.

Let $S=\{1,2,\dots,m\}$ be the state space with m symbols and

$$\Sigma_m=\{i=(i_1i_2\cdots):\ i_n\in S,\forall n>0\}$$

the one sided symbolic space on S [13]. Let $k>0$ and denote by J_k the set of all $k-$sequences on S.

Set

$$E_{i_1\cdots i_k}=S_{i_1}\cdots S_{i_k}(E),\ \forall i=(i_1\cdots i_k)\in J_k;$$

$$E_{i_1i_2\cdots}=\bigcap_{k=1}^{\infty}E_{i_1i_2\cdots i_k}=\bigcap_{k=1}^{\infty}S_{i_1}\cdots S_{i_k}(E)=\bigcap_{k=1}^{\infty}S_{i_1}\cdots S_{i_k}(\overline{V})=\{x_i\},\ \forall i=(i_1i_2\cdots)\in\Sigma_m.$$

Namely, the second term above is a singleton and $i=(i_1i_2\cdots)$ is called a representation of $x_i\in E$. Obviously, each point in E has a representation but unlikely unique. If, for each $k>0$, x_i is always an interior point of $S_{i_1}\cdots S_{i_k}(\overline{V})$, we will call x_i to be an interior point of E. It is easy to see that each interior point has unique representation. Consider the continuous mapping

$$\begin{cases}\zeta:\ \Sigma_m\ \to\ E\\ \zeta(i)=x_i,\ \forall i=(i_1i_2\cdots)\in\Sigma_m.\end{cases}$$

It is easy to see that if E satisfies SSC, the mapping is one-to-one and if E only satisfies OSC, it is many-to-one.

It is also easy to prove $H^s(\partial\overline{V})=0$,where $\partial\overline{V}$ denotes the boundary of $\overline{V}$. So we have

$$H^s\left(\bigcup_{k=1}^{\infty}\bigcup_{(i_1\cdots i_k)\in J_k}S_{i_1}\cdots S_{i_k}\left(\partial\overline{V}\right)\right)=0,$$

and the set consisting of all interior points is of H^s-full measure. We need the following simple

Claim 1. If x_i is an interior point of E, then there is at least $k>0$ such that $S_{i_1}\cdots S_{i_k}(\overline{V})\subset V$.

Let $A \subset \mathbf{R}^n$ and denote by $|A|$ the diameter A. Denote $U_x \subset \mathbf{R}^n$ a set containing x and δ a positive number.

Define

$$\overline{D_c^s}(E,x) = \lim_{\delta\to 0}\left\{\sup_{0<|U_x|\le\delta}\frac{H^s(E\cap U_x)}{|U_x|^s}\right\} \ge 0$$

and call it the upper convex density of E at x (see [1]). Evidently, here U_x may be taken to be convex. According to $H^s(E\cap U) \le |U|^s$, (see [12]), there holds always $\overline{D_c^s}(E,x) \le 1, \forall x \in E$. The computation and estimation of the upper convex density is more difficult than the case for the Hausdorff measure. Set $E_1 = \{x \in E:\ \overline{D_c^s}(E,x) = 1\}$.

Theorem A[1]**.** *E_1 is a measurable set and $H^s(E_1) = H^s(E)$.*

It is easy to see that there is an interior point x of E with $\overline{D_c^s}(E,x) = 1$.

3 Hausdorff Metric and Measure Convergence

Let $A \subset \mathbf{R}^n$ be non-empty and $\delta > 0$. Set $V(A,\delta) = \{x \in \mathbf{R}^n:\ d(A,x) < \delta\}$ and denote by $\mathcal{C}$ the set consisting of all compact subsets of $\mathbf{R}^n$. Suppose $A,B \in \mathcal{C}$ and define

$$\rho(A,B) = \inf\{\delta:\ B \subset V(A,\delta),\ A \subset V(B,\delta)\}.$$

It is not hard to prove that ρ is a complete metric on $\mathcal{C}$ and it is called the Hausdorff metric(see [1]).

Theorem B[1]**.** *Any uniformly bounded infinite set in $\mathcal{C}$ has convergent subsequence.*

Denote $F_i \xrightarrow{H} F$ if $\{F_i\}$ tends to F on $\mathcal{C}$ under Hausdorff metric and it is easy to see that $F_i \xrightarrow{H} F$ implies $|F_i| \to |F|$ and furthermore, if $x \in F_i, \forall i > 0$,then $x \in F$.

Suppose m is a finite measure on $\mathbf{R}^n$ with compact support and so is regular(see [9]).

Proposition 1. $F_i \xrightarrow{F} F \;\Rightarrow\; \limsup_{i\to\infty} m(F_i) \le m(F)$.

Proof. Given $r > 0$, according to the definition, for sufficient large $i > 0$, we have $F_i \subset V(F,r)$ and hence $m(F_i) \le m(V(F,r))$. By the regularity of the measure m, $\lim_{r\to 0} m(V(F,r)) = m(F)$. So it is easy to see that $\limsup_{i\to\infty} m(F_i) \le m(F)$. We are done.

Let $0 < \varepsilon < 1$ and $\mathcal{B} = \left\{U \subset \mathbf{R}^n:\ \dfrac{H^s(E\cap U)}{|E\cap U|^s} > 1-\varepsilon\right\}$. By the definition, it is easy to see that $\mathcal{B}$ is not empty and for given $\varepsilon > 0$, $d = \sup\{|U| : U \in \mathcal{B}\}$ exists.

Proposition 2. *There is a compact set $V \subset \mathbf{R}^n$ with $|E\cap V| = d > 0$ and $\frac{H^s(E\cap V)}{|E\cap V|^s} \ge 1-\varepsilon$.*

Proof. By the definition, for each integer $l > 0$, there is a compact $V_l \in \mathcal{B}$ such that $|E\cap V_l| > d - \dfrac{1}{l}$ and $\dfrac{H^s(E\cap V_l)}{|E\cap V_l|^s} > 1-\varepsilon$. Obviously, $\{E\cap V_l\}$ is uniformly bounded. Taking a subsequence if necessary, we may assume that $E\cap V_l \xrightarrow{H} V$ $(l\to\infty)$. Clearly, $\lim_{l\to\infty}|E\cap V_l| = |V| = d$ and by Proposition 1,

$$\limsup_{l\to\infty}\frac{H^s(E\cap V_l)}{|E\cap V_l|^s}\le\frac{H^s(E\cap V)}{|E\cap V|^s}$$

and hence

$$\frac{H^s(E\cap V)}{|E\cap V|^s}\ge\frac{H^s(E\cap V)}{|V|^s}\ge 1-\varepsilon.$$

We are done.

4 Proof of Realization Theorem

First of all, we introduce a class of similar enlargements as follows.

Let $x\in E$ be an interior point with $\overline{D_c^s}(E,x)=1$ and $i=(i_1\cdots i_l\cdots)$ its representation. An element $(j_1\cdots j_l)\in J_l$ $(l>0)$ is called an $l-$tuple of x, if there is some $m>0$ such that $j_1=i_{m+1},\cdots,j_l=i_{m+l}$ and denote it by $(j_1\cdots j_l)\prec x$.

$\forall l>0$, it is easy to see that

$$(S_{i_1}\cdots S_{i_l})^{-1}:\ S_{i_1}\cdots S_{i_l}(\overline{V})\to\overline{V}$$

is a similar enlargement from $S_{i_1}\cdots S_{i_l}(\overline{V})$ onto $\overline{V}$ with the similar ratio $\frac{1}{c_{i_1}\cdots c_{i_l}}$ and

$$(S_{i_1}\cdots S_{i_l})^{-1}(S_{i_1}\cdots S_{i_l}(E))=E.$$

Define a linear similar enlargement $T_l:\overline{V}\to\mathbf{R}^n$ such that the restriction of T_l on $S_{i_1}\cdots S_{i_l}(\overline{V})$ coinciding with $(S_{i_1}\cdots S_{i_l})^{-1}$, that is, $T_l|_{S_{i_1}\cdots S_{i_l}(\overline{V})}=(S_{i_1}\cdots S_{i_l})^{-1}$ or T_l is the linear extension of

$$(S_{i_1}\cdots S_{i_l})^{-1}:\ S_{i_1}\cdots S_{i_l}(\overline{V})\to\overline{V}$$

from $S_{i_1}\cdots S_{i_l}(\overline{V})$ to $\overline{V}$. Obviously, T_l is well defined and so we get a series of similar enlargements:

$$\begin{cases} T_l:\overline{V}\to\mathbf{R}^n\\ T_l(y)=(S_{i_1}\cdots S_{i_l})^{-1}(y),\quad \forall y\in S_{i_1}\cdots S_{i_l}(\overline{V})\\ T_l(S_{i_1}\cdots S_{i_l}(\overline{V}))=(S_{i_1}\cdots S_{i_l})^{-1}(S_{i_1}\cdots S_{i_l}(\overline{V}))=\overline{V}.\end{cases}$$

For convenience, sometimes write $T_l=(S_{i_1}\cdots S_{i_l})^{-1}$ and $T_l^{-1}=S_{i_1}\cdots S_{i_l}$. From the above definition, it is easy to see that there holds always $T_l(x)=(S_{i_1}\cdots S_{i_l})^{-1}(x)\in E\subset\overline{V}\subset T_l(\overline{V})$ and $T_l(x)$ is always an interior point of E. We call $T_l(\overline{V})$ an $l-$order enlargement of $\overline{V}$. For $(i_1\cdots i_l)\prec x$ and $(j_1\cdots j_l j_{l+1}\cdots j_k)\prec x$, the following Claim is simple.

Claim 2. $T_l(\overline{V}\cap T_{l+h}(\overline{V})=T_l(\overline{V}) \Leftrightarrow (i_1\cdots i_l)=(j_{1+h}\cdots j_{l+h}),\ \forall\ h>0,\ \forall l>0$, where $T_l=(S_{i_1}\cdots S_{i_l})^{-1}$ and $T_{l+h}=(S_{j_1}\cdots S_{j_{l+h}})^{-1}$.

We call $T_l(\overline{V})$ the $l-$order enlargement of $\overline{V}$ contained in $T_{l+h}(\overline{V})$. Especially, it is not hard to see

$$(S_{i_2}\cdots S_{i_l})^{-1}(\overline{V})\subset(S_{i_1}\cdots S_{i_l})^{-1}(\overline{V}),\ \forall i_1=1,2,\ldots,m.$$

In the following, we assume always that $x\in E$ is an interior point with $\overline{D_c^s}(E,x)=1$ and the unique representation $(i_1i_2\cdots)$ and define T_l by using $(i_1i_2\cdots)$ as the above. Noting that $T_l(E)$ is also a self-similar set and $H^s(T_l(E))=(c_{i_1}\cdots c_{i_l})^{-s}H^s(E)$. The following Claim is simple.

Claim 3. If $U\subset\overline{V}$, then $H^s(T_l(E)\cap U)=H^s(E\cap U)$. In general, if $U\subset T_l(\overline{V})$ and so $T_l^{-1}(U)\subset\overline{V}$, then

$$H^s(T_l(E)\cap U)=H^s(E\cap T_l^{-1}(U)).$$

According to the Scaling property [2, p.27], the following Claim is simple.

Claim 4. Let $U\subset\mathbf{R}^n$, then

$$\frac{H^s(E\cap U)}{|E\cap U|^s}=\frac{H^s(T_l(E\cap U))}{|T_l(E\cap U)|^s}=\frac{H^s(T_l^{-1}(E\cap U))}{|T_l^{-1}(E\cap U)|^s},\qquad \forall l\geq 0.$$

Proposition 3. *Let $F\subset\mathbf{R}^n$ be any compact subset, then there is the least integer $l>0$ such that $F\subset\ intT_l(\overline{V})$, where int $T_l(\overline{V})$ denotes the interior of $T_l(\overline{V})$.*

Proof. As $x\in E\subset\overline{V}$ is an interior point, so $d=d(x,\partial\overline{V})>0$. By the definition, it is easy to see that

$$d(T_l(x),T_l(\partial\overline{V})=d(x,\partial\overline{V})\,(c_{i_1}\cdots c_{i_l})^{-1}.$$

Because $T_l(x)\in E\subset\overline{V}$ and $0<c_i<1$ $(l>0, i=1,2,\ldots,m)$, obviously, the bounded set $F\subset$ *int* $T_l(\overline{V})$ if l is sufficient large. We are done.

Proof of the realization theorem. Next, using $U_x,U_{x,k}\subset\overline{V}\subset\mathbf{R}^n$ $(\forall k>0)$, denote the compact sets containing x and set

$$B_{x,k}=\left\{U_x:\ \frac{H^s(E\cap U_x)}{|E\cap U_x|^s}>1-\frac{1}{k}\right\},\qquad \forall k>0.$$

According to Proposition 2, we may set $0<r_{x,k}=\sup\{|E\cap U_x|:\ U_x\in B_{x,k}\}$. Take U_x^i in $B_{x,k}$ such that

$$|E\cap U_x^i|\to r_{x,k}\ (i\to\infty).$$

By Theorem B and Proposition 1, taking a subsequence if necessary, we may assume that

$$E\cap U_x^i\xrightarrow{H}E\cap U_{x,k}\subset E\ (i\to\infty).$$

It is easy to see that $|E\cap U_{x,k}|=r_{x,k}$ and

$$\frac{H^s(E\cap U_{x,k})}{|E\cap U_{x,k}|^s}\geq \limsup_{i\to\infty}\frac{H^s(E\cap U_x^i)}{|E\cap U_x^i|^s}\geq 1-\frac{1}{k}.$$

Suppose that $r_{x,k}\to 0$ $(k\to\infty)$ is not true. Taking a subsequence if necessary, we may assume $r_{x,k}\to r>0$ $(k\to\infty)$. Using Theorem B and Proposition 1, we may prove easily that there is $U\subset\overline{V}$ such that $|E\cap U|=|U|=r$ and $\frac{H^s(E\cap U)}{|E\cap U|^s}=1$.

Next, suppose $r_{x,k}\to 0$ $(k\to\infty)$.

For each $k\geq 0$, there is an integer $l(k)\geq 0$ depending on k such that (to stipulate: $S_{i_0}(\overline{V})=\overline{V}$)

$$E\cap U_{x,k}\subset S_{i_1}\cdots S_{i_{l(k)}}(\overline{V})$$

and $E\cap U_{x,k}$ not contained in $S_{i_1}\cdots S_{i_{l(k)+1}}(\overline{V})$, that is, $l(k)$ is the greatest integer such that

$$T_{l(k)}(E\cap U_{x,k})=\left(S_{i_1}\cdots S_{i_{l(k)}}\right)^{-1}(E\cap U_{x,k})\subset\overline{V}.$$

Setting $F_{l(k)}=T_{l(k)}(E\cap U_{x,k})$, then we have

$$\overline{r_k}=|F_{l(k)}|=r_{x,k}\left(c_{i_1}\cdots c_{i_{l(k)}}\right)^{-1},\ \forall k>0.$$

We have two cases as follows.

Firstly, suppose $\overline{r_k}\to 0$ $(k\to\infty)$ is not true. Taking a subsequence if necessary, we may assume that $\overline{r_k}\to r>0$ $(k\to\infty)$. By Theorem B and Proposition 1, we may prove that there is a compact subset $U\subset\overline{V}$ such that $|U|=r$ and $\frac{H^s(E\cap U)}{|E\cap U|^s}=1$.

Next, suppose $\overline{r_k}\to 0$ $(k\to\infty)$. Set $c=\min\{c_1,c_2,\ldots,c_m\}>0$ and arbitrarily given $d>0$. For each $k>0$, there is a unique integer $h(k)>0$ depending on k such that

$$r_{x,k}\left(c_{i_1}\cdots c_{i_{l(k)+h(k)}}\right)^{-1}<d\leq r_{x,k}\left(c_{i_1}\cdots c_{i_{l(k)+h(k)+1}}\right)^{-1}<dc^{-1}_{i_{l(k)+h(k)+1}}\leq\frac{d}{c}$$

or

$$\overline{r_k}\left(c_{i_{l(k)+1}}\cdots c_{i_{l(k)+h(k)}}\right)^{-1}<d<\overline{r_k}\left(c_{i_{l(k)+1}}\cdots c_{i_{l(k)+h(k)+1}}\right)^{-1}<dc^{-1}_{i_{l(k)+h(k)+1}}\leq\frac{d}{c}$$

As a clear consequence, noting that $(T_{l(k)+h(k)+1}(E\cap U_{x,k})\cap\overline{V}\neq\emptyset$, we have

Proposition 4. *$T_{l(k)+h(k)+1}(E\cap U_{x,k})$ is uniformly bounded for all $k>0$.*

Thus, taking a subsequence if necessary, without loss of generality, we may assume, by Theorem B, that $T_{l(k)+h(k)+1}(E\cap U_{x,k})$ is convergent under the Hausdorff metric, say $T_{l(k)+h(k)+1}(E\cap U_{x,k})\xrightarrow{H}F$ $(k\to\infty)$, where F is compact and $T_{l(k)+h(k)+1}(x)$ is also convergent, say $T_{l(k)+h(k)+1}(x)\to x_0\in E$ $(k\to\infty)$. Note that $T_{l(k)+h(k)+1}(x)$ is an interior point for all $k>0$ and x_0 is unlikely.

By Proposition 3,there is at least $l > 0$ such that $F \subset (\text{int } T_l)(\overline{V})$ and hence $T_l^{-1}(F) \subset int\overline{V}$. Though x_0 is unlikely an interior point of E, but we have the following simple

Claim 5. There is some i with $0 < i \leq m$ such that $x_0 \in S_i(E)$ and $T_{l(k)+h(k)}(x) \in S_i(V)$ for infinitely many k and so $S_{l(k)+h(k)+1}$ is the same for infinitely many k. Furthermore, by induction, there is a constant $l-$sequence

$$l(k)+h(k)+1-l, l(k)+h(k)+1-l+1, \ldots, l(k)+h(k)+1 \in J_l$$

for infinitely many $k > 0$ and so $S_{l(k)+h(k)+1}$ is the same for infinitely many k.

Obviously, $T_{l(k)+h(k)+1}(E \cap U_{x,k})$ is in the same $l-$order enlargement of $\overline{V}$ for infinitely many k and so

$$\frac{H^s(T_{l(k)+h(k)+1}(E \cap U_{x,k}))}{|T_{l(k)+h(k)+1}(E \cap U_{x,k}|^s}$$

is defined well for infinitely many k. Taking a subsequence if necessary, without lose of generality, we may assume that it holds for all k. It is easy to see that

$$T_l^{-1} = S_{l(k)+h(k)+1-l} S_{l(k)+h(k)+1-l+1} \cdots S_{l(k)+h(k)+1}$$

for all $k > 0$. Obviously, we have $T_{l(k)+h(k)+1}(E \cap U_{x,k}) \subset T_l(V)$ for sufficient large $k > 0$. According to Claims 3 and 4,

$$\frac{H^s(T_{l(k)+h(k)+1}(E \cap U_{x,k}))}{|T_{l(k)+h(k)+1}(E \cap U_{x,k})|^s} = \frac{H^s(E \cap U_{x,k})}{|E \cap U_{x,k}|^s} \geq 1 - \frac{1}{k}$$

for sufficient large $k > 0$. According to Proposition 1, it is easy to see that

$$1 = \limsup_{k \to \infty} \frac{H^s(T_{l(k)+h(k)+1}(E \cap U_{x,k})}{|T_{l(k)+h(k)+1}(E \cap U_{x,k})|^s} \leq \frac{H^s(T_{l(k)+h(k)+1}(E) \cap F)}{|E \cap T^{-1}_{l(k)+h(k)+1}(F)|^s} = 1$$

and obviously,

$$\frac{H^s(T_{l(k)+h(k)+1}(E \cap F))}{|T_{l(k)+h(k)+1}(E \cap F)|^s} = \frac{H^s(E \cap T^{-1}_{l(k)+h(k)+1}(F))}{|E \cap T^{-1}_{l(k)+h(k)+1}(F)|^s} = 1$$

and $0 < |E \cap T^{-1}_{l(k)+h(k)+1}(F)|$. We are done.

Remark. It is easy to see that we proved that the case $\overline{r_k} \to 0$ is impossible and hence $r_{x,k} \to 0$ $(k \to \infty)$ also is impossible in the above and hence we have proved that there is a set U_x with $|U_x| > 0$ such that

$$\overline{D^s_c}(E,x) = \frac{H^s(E \cap U_x)}{|U_x|^s}$$

for all interior point $x \in E$ with $\overline{D^s_c}(E,x) = 1$. In fact, by the same argument, we may prove that this conclusion holds for all $x \in E$ with $\overline{D^s_c}(E,x) = 1$. Such a set U_x is called a best shape at x in

[14]. But, to look for a best shape is very difficult, even for some special point if the Hausdorff dimension is larger than 1.For example, so far we cannot determine a best shape at a vertex of $\mathbf{C}\times\mathbf{C}$, where $\mathbf{C}$ is the middle third Cantor set, even we cannot prove that it is symmetric with respect to the diagonal of the square, passing the vertex, therein $\mathbf{C}\times\mathbf{C}$ is yielded (see [14]),but we conjecture so.

5 Proof of Corollary 1

For convenience, we only prove a special case with $c_j = c = \text{const.}, j = 1,2,\cdots,m$. For the general case, there is no any essential difference except more complecated.

Let the convex set $U \subset \overline{V}$ such that $|U| > 0$ and $\frac{H^s(E\cap U)}{|E\cap U|^s} = 1$ as the above. Let $k > 0$ and $\forall\ (i_1\cdots i_k) \in J_k$, we call $S_{i_1}\cdots S_{i_k}(\overline{V})$ a $k-$th order copy of $\overline{V}$. We have

$$H^s(S_{i_1}\cdots S_{i_k}(\overline{V}(E)) = c^{-ks}H^s(E), \qquad \forall\ (i_1\cdots i_k) \in J_k,\ \forall k > 0.$$

Denote a_1 the number of all first order copies of $\overline{V}$ contained in U and a_2 the number of all second order copies of $\overline{V}$ contained in U except those contained in some first order copy contained in U. Inductively, denote by a_k the number of all a_k-th order copies of $\overline{V}$ contained in U, except those contained in some $(k-1)-$th order copy contained in U, for all $k > 0$.

Denote by M the union of all copies of $\overline{V}$ contained in U and M_k the union of all copies of $\overline{V}$ contained in U whose order is not larger than $k > 0$. As U is convex, we may prove that the Hausdorff dimension of $E\cap\partial U$ is strictly smaller than s and so $H^s(E\cap\partial U) = 0$ (we refer to [5]). It is easy to see that

$$E\cap(\text{int}\,U - \overline{M}) \subset E\cap\left(\bigcup_{k=1}^{\infty}\ \bigcup_{(i_1\cdots i_k)\in J_k} S_{i_1}\cdots S_{i_k}(\partial\overline{V})\right),$$

that is, there is no any interior point of E in $E\cap(\text{int}\,U-\overline{M})$, because if there is an interior point in int $U-\overline{M}$, then, noting that int $U-\overline{M}$ is an open set, it is easy to see that there is a copy $S_{i_1}S_{i_2}\cdots S_{i_k}(\overline{V})$ of $\overline{V}$ in int $U-\overline{M}$ for some $k > 0$ and it contradicts the above construction. It is easy to see

$$H^s(E\cap(U-\overline{M})) = H^s(E\cap(\text{int}\,U-\overline{M})) = 0$$

and

$$H^s(E\cap U) = H^s(E\cap\overline{M}).$$

But, obviously, $M_k \xrightarrow{H} \overline{M}$ $(k \to \infty)$ and $H^s(E \cap M_k) \to H^s(E \cap \overline{M}) = H^s(E \cap U) = \sum_{k>0} a_k c^{ks} H^s(E)$. Hence

$$\frac{H^s(E \cap U)}{|E \cap U|^s} = \sum_{k>0} a_k c^{ks} H^s(E)|E \cap U|^{-s} = 1, \text{ or } H^s(E) = |E \cap U|^s \left(\sum_{k>0} a_k c^{ks} \right)^{-1}.$$

Take $b_k = a_k c^{-ks}$ and we are done.

Remark. If all of the ratios are not the same, then the different $k-$th copy has the different Hausdorff measure and so the formula of b_k is more complex, but the difficulty is not essential.

6 Proof of Corollary 2

Let U and $\frac{H^s(E \cap U)}{|E \cap U|^s} = 1$ as the above. Obviously,

$$\frac{H^s(E \cap U)}{|E \cap U|^s} = \frac{H^s(E \cap S_{i_1} \cdots S_{i_k}(U))}{|E \cap S_{i_1} \cdots S_{i_k}(U)|^s} = 1, \ \forall k > 0, \forall (i_1 \cdots i_k) \in J_k.$$

For convenience, we first introduce the following notations. Let $A \subset \mathbf{R}^n$ and ζ, ξ be two sets consisting of some subsets in $\mathbf{R}^n$, respectively. Set

$$\zeta \cup \xi = \{C : \ C \in \zeta \text{ or } C \in \xi\},$$

$$A \cap \zeta = \emptyset \ \Leftrightarrow \ H^s(E \cap A \cap B) = 0, \ \forall B \in \zeta.$$

Set

$$\begin{aligned} \alpha_0 &= \{U\}, \\ \alpha_1 &= \{S_i(U) : \ S_i(U) \cap \alpha_0 = \emptyset, \ 0 < i \le m\}, \\ \alpha_2 &= \{S_{i_1} S_{i_2}(U) : \ S_{i_1} S_{i_2}(U) \cap (\alpha_0 \cup \alpha_1) = \emptyset, \ (i_1 i_2) \in J_2\}, \end{aligned}$$

Inductively, let α_{k-1} be defined well for $k > 2$,set

$$\alpha_k = \{S_{i_1} \cdots S_{i_k}(U) : \ S_{i_1} \cdots S_{i_k}(U) \cap (\alpha_0 \cup \cdots \cup \alpha_{k-1}) = \emptyset, \ (i_1 \cdots i_k) \in J_k\}.$$

Thus, we get $\{\alpha_i\}_{i=0}^{\infty}$ and set $\alpha = \bigcup_{i \ge 0} \alpha_i$. From the above form, it is easy to see that for any $M \in \alpha$, we have

$$\frac{H^s(E \cap M)}{|E \cap M|^s} = 1$$

and the intersection of any different two elements in α has H^s-zero measure. By the same argument as in § 4, it is not hard to prove

$$H^s(E) = \sum_{M \in \alpha} H^s(E \cap M) = \sum_{M \in \alpha} |E \cap M|^s.$$

It is easy to see that α is an almost everywhere best covering of E. We are done.

7 Discussions

Let $E \subset \mathbf{R}^n$ and $\dim_H(E) = s$ be the same as the above. In order to calculate the Hausdorff measure of E, it suffices to look for $U \subset \mathbf{R}^n$ such that $|U| > 0$ and $\frac{H^s(E\cap U)}{|E\cap U|^s} = 1$,because if U (including its diameter, position and shape) is determined well, then the series in Corollary 1 is also determined and the remainder is only some calculations. But, in general, it is very difficult to look for such a set U. In fact, using our model, Refs [10, 11, 12] and [8] investigate the computation of the Hausdorff measure for self-similar sets but there U satisfies only $\frac{H^s(E\cap U)}{|E\cap U|^s} < 1$ and so the results obtained are only some upper limits of the corresponding Hausdorff measure but not the exact values. Marion [4] has posed two conjectures about the exact values of the Hausdorff measure of the Sierpinski gasket and Koch curve respectively and we have negated them [11, 12]. In essence, the set U obtained by him does not satisfy the above equality and so his results are only the upper limit and not the exact value of the corresponding Hausdorff measure. Clearly, the smaller $1 - \frac{H^s(E\cap U)}{|E\cap U|^s}$ is, the better the corresponding upper limit is. When the Hausdorff dimension is not larger than 1, we have some examples for that the corresponding U have been obtained (see Ref. [17, 18]), but for the case of that the Hausdorff dimension is larger than 1(non-integer), so far no any such examples have be found. It is why we can not get any example for that the exact values of the Hausdorff measure is obtained when its Hausdorff dimension is larger than 1(non-integer).Finally,we pose

A problem. To form a self-similar set with OSC and $dim_H > 1$,whose a best shape (including its diameter,geometric shape and location)is determined wholly.

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