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A SIMPLE PROOF OF THE CHAOTICITY OF SHIFT MAP UNDER A NEW DEFINITION OF CHAOS

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Abstract. Recently, Du has given a new strong definition of chaos by using the shift map. In this paper, we give a proof of the main theorem by constructing a dense uncountable invariant subset of the symbol space Σ_2 containing transitive points in a simpler way with the help of a different metric. We also provide two examples, which support this new definition.

Key words: *symbol space, shift map, δ-scrambled set, chaos, transitive points*

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1 Introduction

Nowadays the chaoticity of a dynamical system becomes a more demanding and challenging topic for both mathematicians and physicists. Li and Yorke[8] are the first people that connect the term 'chaos' with a map. There are various types of chaotic maps, namely, tent map [6, 10], quadratic map [2, 4, 6, 10] etc. If for a system the distance between the nearby points increases and the distance between the faraway points decreases with time, the system is said to be chaotic. According to Devaney[6], a map $f : V \to V$ is said to be chaotic if the following three properties hold:

(i) f has sensitive dependence on initial conditions;

(ii) f is topologically transitive; and

(iii) periodic points are dense in V.

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But in [3], it is shown that topological transitivity and dense periodic orbits together imply sensitive dependence on initial conditions. So the condition (i) of Devaney's definition is redundant. It is also known that in an interval (not necessarily finite) a continuous, topological transitive map is chaotic in the sense of Devaney[6,11]. Akin[1] introduced a linkage between the sensitivity and Li-Yorke[8] version of chaos.

Let $\Sigma_2 = \{\alpha : \alpha = (\alpha_0\alpha_1\cdots), \alpha_i = 0 \text{ or } 1\}$ be the symbol space containing two symbols 0 and 1. For any two points $s = (s_0s_1\cdots)$ and $t = (t_0t_1\cdots)$ of Σ_2, we define the distance between s and t by the metric

$$d(s,t) = \sum_{k=0}^{\infty} \frac{\delta(s_k,t_k)}{3^{k+1}},$$

where $\delta(s_k,t_k) = 0$, for $s_k = t_k$ and $\delta(s_k,t_k) = 1$, for $s_k \neq t_k$. It can be easily shown that Σ_2 is a metric space with the metric $d(s,t)$. We noted that the maximum distance between two points of Σ_2 with the metric is $\frac{1}{2}$, because

$$d(s,t) \leq \frac{1}{3} + \frac{1}{3^2} + \cdots = \frac{1}{2}.$$

The shift map [4, 6, 7, 9, 10] is often used to model the chaoticity of a dynamical system. Shift map has some remarkable and interesting properties, such as, it is topologically transitive, has dense periodic points and sensitive dependence on initial conditions. The present authors have extend the idea of the shift map to the generalized shift map in [5]. Recently. It is shown by Du[7] that the sensitive dependence on initial conditions tells only a part about the system to be chaotic and so he used another interesting property of the shift map called the extreme sensitive dependence on initial conditions to show a system to be more chaotic than the previous one.

In this paper, we use a metric different from the that used in [7] and with the help of this metric and a different but simple construction of the dense uncountable invariant subsets of Σ_2 containing transitive points, we are able to give a simpler proof of the chaoticity of the shift map. Further, we give two new examples, which illustrate the fact that (i) all topologically transitive maps or Li-Yorke sensitive maps are not chaotic and (ii) chaotic maps are not necessarily topologically transitive. We now state this new definition of chaos, which was given by Du.

2 A Strong Definition of Chaos[7]

Let (X,ρ) be an infinite compact metric space with the metric ρ and let f be a continuous map from X into itself. We say that f is chaotic if there exists a positive number δ such that for any point x and any non-empty open set V (not necessarily an open neighborhood of x) in X

there is a point y in V such that

$$\limsup \rho(f^n(x), f^n(y)) \geq \delta \text{ and } \liminf \rho(f^n(x), f^n(y)) = 0.$$

3 Mathematical Preliminaries

In this section, some definitions, lemmas and notations are given which will be used in the later section.

Definition 3.1(*Shift map*[6]). The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma(\alpha_0\alpha_1\cdots) = (\alpha_1\alpha_2\cdots)$, where $\alpha = (\alpha_0\alpha_1\cdots)$ is any point of Σ_2.

Definition 3.2 (δ-scrambled set[7,8]. Let $\sigma : S \to S$ be the shift map. Then the set S is called a δ-scrambled set for some $\delta > 0$ if

(i) for any $x \neq y$ in S, $\limsup d(\sigma^n(x), \sigma^n(y)) \geq \delta$ and $\liminf d(\sigma^n(x), \sigma^n(y)) = 0$,

(ii) for any periodic point p of σ and any y in S, $\limsup d(\sigma^n(p), \sigma^n(y)) \geq \frac{\delta}{1.5}$.

Note that here $S \subset \Sigma_2$ and the metric d is as defined above.

Definition 3.3 (Transitive point[6]). In the symbol space Σ_2 there are points whose orbits come arbitrarily close to any given sequence of Σ_2. That is the point with dense orbit. Those points are called transitive points. Obviously a point of Σ_2 which contains every finite sequence of 0's and 1's is a transitive point and vice versa.

We also require the following lemma.

Lemma 3.1. *If $s = (s_0s_1\cdots)$ and $t = (t_0t_1\cdots)$ are two elements of Σ_2 such that $s_i = t_i$, for $0 \leq i \leq n$, then $d(s,t) \leq \frac{1}{2}\frac{1}{3^{n+1}}$.*

Proof. The distance between $s = (s_0s_1\cdots)$ and $t = (t_0t_1\cdots)$ is

$$d(s,t) = \sum_{k=0}^{n} \frac{\delta(s_k,t_k)}{3^{k+1}} + \sum_{k=n+1}^{\infty} \frac{\delta(s_k,t_k)}{3^{k+1}} \leq \sum_{k=n+1}^{\infty} \frac{1}{3^{k+1}} = \frac{1}{2}\frac{1}{3^{n+1}}$$

Hence Lemma 3.1 follows

Some Notations.

1. If $K = k_0k_1\cdots k_l$ and $P = p_0p_1\cdots p_m$ are two finite sequences of 0's and 1's, then $KP = k_0k_1\cdots k_l p_0p_1\cdots p_m$. Further, if we suppose that $K_1, K_2, \cdots, K_n$ are n finite sequences of 0's and 1's, then $K_1K_2\cdots K_n$ can be defined in a similar manner as above.

2. For any binary numerals α_p, we define α'_p as the complement of α_p, that is, if $\alpha_p = 0$ or 1, then $\alpha'_p = 1$ or 0. Similarly, for any finite sequence $E = e_0e_1\cdots e_k$ of 0's and 1's, the complement of E can be written as $E' = e'_0e'_1\cdots e'_k$.

3. Let $\alpha = (\alpha_0\alpha_1\cdots)$ be a fixed transitive point of Σ_2. For any integer $k \geq 10$ and for any point $\beta = (\beta_0\beta_1\cdots)$ of Σ_2, we set

$$A(\beta,k) = \alpha_0\alpha_1\cdots\alpha_{k-1}(\beta_0)^{k-1}(\beta_1)^{k-1}\cdots(\beta_{k-1})^{k-1}(01)^{(k-2)!}(0011)^{(k-2)!}\cdots(0^{k-1}1^{k-1})^{(k-2)!}$$

where $(01)^2 = (0101), (0011)^2 = (00110011), (000111)^3 = (000111000111000111)$, and so on. From the above construction of $A(\beta,k)$, we find that the length of $A(\beta,k)$ is $k^2+k!$.

4. Let $X = \{x_i : x_i = x_{i,0}x_{i,1}\cdots, i \geq 1\}$ be any countable infinite set in Σ_2. Also we assume b_j $(0 \leq j \leq k-1)$ be any fixed k numbers composed of 0's and 1's. We now define $B(x_i,k) = (x_{i,t(k,i,j)}\ x_{i,t(k,i,j)+1}\cdots\ x_{i,t(k,i,j)+(k-1)})$ and

$$B'(x_i,k) = (x'_{i,t(k,i,j)+k}\ x'_{i,t(k,i,j)+(k+1)}\cdots\ x'_{i,t(k,i,j)+(2k-1)}),$$

where, $t(k,i,j) = k+k^2+k!+2k(i-1)-j$.

We denote the product $B(x_i,k)B'(x_i,k)$ as $\overline{B}(x_i,k)$.

5. Lastly we define,

$$\tau_\beta = b_0b_1\cdots b_{k-1}A(\beta,k)\overline{B}(x_1,k)\overline{B}(x_2,k)\cdots\overline{B}(x_k,k)A(\beta,k+1)\overline{B}(x_1,k+1)\overline{B}(x_2,k+1)$$
$$\cdots\overline{B}(x_{k+1},k+1)A(\beta,k+2)\cdots.$$

Here, we note the following facts:

a) The length of the sequence $b_0b_1\cdots b_{k-1}$ is k.

b) The length of the sequence $b_0b_1\cdots b_{k-1}A(\beta,k)\overline{B}(x_1,k)\overline{B}(x_2,k)\cdots\overline{B}(x_k,k)$ is $k!+3k^2+k$ and so on.

6. Let $S = \{\sigma^p(\tau_\beta) : p \geq 0, \beta \in \Sigma_2\}$. Then $\sigma(S) \subset S$ and S is uncountable. Also by our construction S consists of transitive points.

We shall now prove that S is a dense subset of Σ_2 and it is given in Lemma 3.2.

Lemma 3.2. *The set $S = \{\sigma^p(\tau_\beta) : p \geq 0, \beta \in \Sigma_2\}$ is a dense subset of Σ_2.*

Proof. To prove that S is a dense subset of Σ_2 we must produce a sequence of points from S such that an arbitrary point of Σ_2 is a limit point of this sequence. Let $\delta = (\delta_0\delta_1\cdots)$ be an arbitrary point of Σ_2. We consider the sequence $\sigma^p(\tau_\delta)$, since this sequence consists of transitive points we can get a $k \geq 0$ such that $\sigma^k(\tau_\delta)$ starts from $(\delta_0\delta_1\cdots)$. Let us assume that $\sigma^p(\tau_\delta)$ agrees with δ up to the n-th term. Then by Lemma 3.1 we know that $d(\sigma^k(\tau_\delta),\delta) \leq \frac{1}{2}\frac{1}{3^{n+1}}$ and by our construction of τ_β there are such infinitely many k's. Hence S is a dense subset of Σ_2. So S is a dense uncountable invariant subset of Σ_2.

4 The Main Theorem

Theorem 4.1. *For any given countable infinite subset X of Σ_2, there exists a dense uncountable $\frac{1}{2}$-scrambled set S of transitive points of Σ_2 such that, for every x in X and every y in S,* $\limsup d(\sigma^n(x),\sigma^n(y)) = \frac{1}{2}$ *and* $\liminf d(\sigma^n(x),\sigma^n(y)) = 0$.

Proof. Let us consider the set S defined in the notation **6** and $\beta = (\beta_0\beta_1\beta_2\cdots)$, $\gamma = (\gamma_0\gamma_1\gamma_2\cdots)$ be two different points of Σ_2 such that $\beta_s \neq \gamma_s$. Further, we take two integers i and j in such a way that $0 \leq i < j$ and $j \leq k-1$. After applying the shift map $2k + s.(k-1)$ times on the two points τ_β and τ_γ, we have

$$\sigma^{2k+s.(k-1)}(\tau_\beta) = (\beta_s)^{k-1}(\beta_{s+1})^{k-1}\cdots$$

and

$$\sigma^{2k+s.(k-1)}(\tau_\gamma) = (\gamma_s)^{k-1}(\gamma_{s+1})^{k-1}\cdots$$

So,

$$\begin{aligned} \limsup d(\sigma^n(\sigma^i(\tau_\beta)), \sigma^n(\sigma^i(\tau_\gamma))) &\geq \lim_{k\to\infty} d((\beta_s)^{k-1}\cdots, (\gamma_s)^{k-1}\cdots) \\ &\geq \lim_{k\to\infty}(\tfrac{1}{3} + \tfrac{1}{3^2} + \cdots + \tfrac{1}{3^{k-1}}) \\ &= \frac{1}{2}. \end{aligned} \tag{4.1}$$

Now, if we choose $t_k = k^2 + k + (j-i).(j-(i+1)).(k-2)!$ then,

$$\sigma^{t_k}(\tau_\beta) = (0^{j-i}1^{j-i})^{(k-2)!}\cdots = 0^{j-i}(1^{j-i}0^{j-i})^{(k-2)!-1}1^{j-i}$$

and

$$\sigma^{t_k}(\tau_\gamma) = (0^{j-i}1^{j-i})^{(k-2)!}\cdots = 0^{j-i}(1^{j-i}0^{j-i})^{(k-2)!-1}1^{j-i}.$$

So for different points τ_β and τ_γ of S, we observe that after a certain number of iterations they start with the same sequence of 0's and 1's. This happens because $A(\beta,k)$ or $A(\gamma,k)$ has finite sequence of the type $(0^{j-i}1^{j-i})^{(k-2)!}$ of 0's and 1's. Hence

$$\begin{aligned} \sigma^{t_k+(j-i)}(\tau_\gamma) &= (1^{j-i}0^{j-i})^{(k-2)!-1}1^{j-i}\cdots \\ &= 1^{j-i}(0^{j-i}1^{j-i})^{(k-2)!-1}\cdots. \end{aligned}$$

Consequently,

$$\begin{aligned} \limsup d(\sigma^n(\sigma^i(\tau_\beta)), \sigma^n(\sigma^j(\tau_\gamma))) &\geq \lim_{k\to\infty} d(\sigma^{t_k}(\tau_\beta), \sigma^{t_k+(j-i)}(\tau_\gamma)) \\ &\geq \lim_{k\to\infty}(\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{j-i+2(j-i)((k-2)!-1)}}) \\ &= \frac{1}{2}. \end{aligned} \tag{4.2}$$

Using equations (4.1) and (4.2) we can say that for any $x \neq y$ in S,

$$\limsup d(\sigma^n(x), \sigma^n(y)) = \frac{1}{2}. \tag{4.3}$$

We have chosen two points in such a way that the two points $\sigma^{n^m}(\tau_\beta)$ and $\sigma^{n^m}(\tau_\gamma)$ start with long sequence of zeros, for infinitely many integers n^m.

Hence,

$$\liminf d(\sigma^n(x),\sigma^n(y)) \le \lim_{p\to\infty}(\frac{0}{3}+\frac{0}{3^2}+\cdots+\frac{0}{3^p})=0,$$

for any $x \ne y$ in S and for some positive integer p.

So,

$$\liminf d(\sigma^n(x),\sigma^n(y)) = 0, \qquad \text{for any} \qquad x \ne y \quad \text{in} \quad S. \tag{4.4}$$

Moreover, if y is any periodic point of σ and for any x in S, we can always choose a positive integer m such that $\sigma^m(x)$ and $\sigma^m(y)$ are different in the first term of the sequence.

Hence,

$$\begin{aligned}\limsup d(\sigma^n(x),\sigma^n(y)) &\ge \lim_{m\to\infty} d(\sigma^m(x),\sigma^m(y)) \\ &\ge \frac{1}{3}=\frac{\frac{1}{2}}{1.5}.\end{aligned} \tag{4.5}$$

Now, from equations (4.3), (4.4) and (4.5), we get that S is a dense invariant uncountable $\frac{1}{2}$ -scrambled set for σ.

Lastly, to prove the final part of the main theorem, that is, for every x in X and every y in S, $\limsup d(\sigma^n(x),\sigma^n(y))=\frac{1}{2}$ and $\liminf d(\sigma^n(x),\sigma^n(y))=0$, let us take $m>k+i+j+10$ to be an integer.

Recall, $X=\{x_i : x_i = x_{i,0}x_{i,1}\cdots, i\ge 1\}$ and $\sigma^i(\tau_\gamma)$ is any point of S. Recall also from the notation **4**, $t(k,i,j)=k+k^2+k!+2k(i-1)-j$, we have

$$\sigma^{t(k,i,j)}(\sigma^j(\tau_\beta)) = x_{i,t(k,i,j)}x_{i,t(k,i,j)+1}\cdots$$

and

$$\sigma^{t(k,i,j)}(x_i) = x_{i,t(k,i,j)}x_{i,t(k,i,j)+1}\cdots$$

Similarly,

$$\sigma^{t(k,i,j)+k}(\sigma^j(\tau_\beta)) = x'_{i,t(k,i,j)+k}x'_{i,t(k,i,j)+(k+1)}\cdots$$

and

$$\sigma^{t(k,i,j)+k}(x_i) = x_{i,t(k,i,j)+k}x_{i,t(k,i,j)+(k+1)}\cdots$$

From the above arguments, we can obtain easily that

$$\limsup d(\sigma^n(x_i),\sigma^n(\sigma^j(\tau_\beta)))=\frac{1}{2} \text{ and } \liminf d(\sigma^n(x_i),\sigma^n(\sigma^j(\tau_\beta)))=0.$$

Hence the theorem is proved.

5 Two Supporting Examples of the Strong Definition of Chaos [7]

We first give an example which shows that every topologically transitive and Li-Yorke sensitive function is not chaotic.

Example 5.1. Let $f : [-1,1] \to [-1,1]$ be a map defined by

$$f(x) = \begin{cases} \frac{3}{2}x + \frac{3}{2}, & -1 \le x \le -\frac{1}{3}, \\ -3x, & -\frac{1}{3} \le x \le 0, \\ -x, & 0 \le x \le 1. \end{cases}$$

The function defined above is obviously a continuous map. Also it can be easily proved that the function is topologically transitive and Li-Yorke sensitive. But it is not chaotic since the period two point $-\frac{3}{5}$ and the closed interval $[0,1]$ are jumping alternatively and never get close to each other.

Next, we give an example, which shows that the chaotic functions are not necessarily topologically transitive.

Example 5.2. Let $F_4(x) = 4x(1-x)$ is the logistic map on the unit interval $[0,1]$. We consider the map $g(x)$ from $[-1,1]$ to itself defined by

$$g(x) = \begin{cases} -x, & -1 \le x \le 0, \\ F_4(x), & 0 \le x \le 1. \end{cases}$$

The map defined above is also a continuous map. Now $g(x)$ is chaotic but not topologically transitive, because if we choose $U = (0,1)$ and $V = (-1,0)$, then

$$f^k(U) \cap V = \phi$$

for all values of k.

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