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ON CONVERGENCE OF GENERAL GAMMA TYPE OPERATORS

Harun Karsli

(*Abant Izzet Baysal University, Turkey)*

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Abstract. The present paper deals with the new type of Gamma operators, here we estimate the rate of pointwise convergence of these new Gamma type operators $M_{n,k}$ for functions of bounded variation, by using some techniques of probability theory.

Key words: *rate of convergence, gamma operator, bounded variation, total variation*

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1 Introduction

Let $BV_{\gamma}[0,\infty)$, $(\gamma \geq 0)$ be the class of all functions defined on $[0,\infty)$, being bounded variation on every finite subinterval of $[0,\infty)$ and satisfying the growth condition $|f(t)| \leq Mt^{\gamma}$ for every $t > 0$ and some constant $M > 0$.

For a measurable complex valued locally bounded function f defined on $[0,\infty)$, Lupas and Müller[1] introduced and investigated some approximation properties of the sequence of linear positive operators $\{G_n\}$ defined by

$$G_n(f;x) = \int_0^{\infty} g_n(x,u) f\left(\frac{n}{u}\right) \mathrm{d}u,$$

which is called Gamma operator, where $g_n(x,u) = \dfrac{x^{n+1}}{n!} e^{-xu} u^n$, $x > 0$. Some approximation properties of these operators were studied by Chen and Guo[2] for functions in $BV[0,\infty)$ and recently for functions in $BV[0,\infty)$ and $DBV[0,\infty)$ by Zeng[3].

In [4], Mazhar defined and studied some approximation properties of the following sequence of linear positive operators

$$\begin{aligned} F_n(f;x) &= \int_0^\infty g_n(x,u)\mathrm{d}u \int_0^\infty g_{n-1}(u,t)f(t)\mathrm{d}t \\ &= \frac{(2n)!x^{n+1}}{n!(n-1)!}\int_0^\infty \frac{t^{n-1}}{(x+t)^{2n+1}}f(t)\mathrm{d}t, \quad n>1, \quad x>0, \end{aligned}$$

where $g_n(x,u)$ is the same function, which has been used by Lupas and Müller in paper [1].

Recently, by using the techniques due to Mazhar, Izgi and Buyukyazici[5] and independently Karsli[6] considered the following Gamma type linear and positive operators

$$\begin{aligned} L_n(f;x) &= \int_0^\infty g_{n+2}(x,u)\mathrm{d}u \int_0^\infty g_n(u,t)f(t)\mathrm{d}t \\ &= \frac{(2n+3)!x^{n+3}}{n!(n+2)!}\int_0^\infty \frac{t^n}{(x+t)^{2n+4}}f(t)\mathrm{d}t, \qquad x>0. \end{aligned}$$

For a very recent results on the local and global approximation results on $L_n(f;x)$ see Karsli and Ozarslan[15].

In 2007 Mao[14] defined the following gamma type operators

$$\begin{aligned} (M_{n,k}f)(x) &= \int_0^\infty g_n(x,u)\mathrm{d}u \int_0^\infty g_{n-k}(u,t)f(t)\mathrm{d}t \\ &= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}}f(t)\mathrm{d}t, \qquad x>0, \end{aligned}$$

whose special cases are:

If $k=1$, then $(M_{n,1}f)(x) = F_n(f;x)$,

If $k=2$, then $(M_{n,2}f)(x) = L_{n-2}(f;x)$.

In addition, if f is right-side continuous at $x=0$, we define

$$(M_{n,k}f)(0) := f(0), \qquad n,k \in \mathbf{N}.$$

For the convenience we can rewrite the operators $(M_{n,k}f)(x)$ as

$$(M_{n,k}f)(x) = \int_0^\infty K_{n,k}(x,t)f(t)\mathrm{d}t, \tag{1}$$

where

$$K_{n,k}(x,t) := \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\frac{t^{n-k}}{(x+t)^{2n-k+2}}, \qquad x,t \in (0,\infty).$$

The rate of approximation for functions of bounded variation is an interesting topic in approximation theory, several researchers have studied on these subjects for three decades. We mention the work of Bojanic-Vuilleumier and Cheng (see [9,10]) who estimated the rate of convergence of bounded variation for Fourier- Legendre series and Bernstein polynomials by using different methods. We also mention some further articles devoted to this subject were written for different operators, some of the important papers on this topic are due to Pych-Taberska[11], Gupta[12] and Gupta et al.[13].

In this paper, we shall estimate the rate of convergence of operators $M_{n,k}$ for functions of bounded variation defined on $[0,\infty)$ at points x where $f(x+)$ and $f(x-)$ exist, we shall prove that the operators (1) converge to the limit $\dfrac{f(x+)+f(x-)}{2}$.

2 Auxiliary Results

In this section we give some results, which are necessary to prove our main theorem.

Lemma 1[14]**.** *For any $p\in\mathbf{N}$, $p\le n-k$*

$$\left(M_{n,k}t^p\right)(x)=\frac{[n-k+p]_p}{[n]_p}x^p. \tag{2}$$

where $n,k\in\mathbf{N}$ and $[x]_p:=x(x-1)\cdots(x-p+1)$, $[x]_0:=1$, $x\in\mathbf{R}$, is the falling difference polynomial.

Proof. By (1) we can write

$$\left(M_{n,k}t^p\right)(x)=\int_0^\infty K_{n,k}(x,t)t^p\,\mathrm{d}t.$$

If we set $u=\dfrac{t}{x+t}$, then we get

$$\begin{aligned}
\left(M_{n,k}t^p\right)(x) &= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\int_0^\infty \frac{t^{n-k+p}}{(x+t)^{2n-k+2}}\mathrm{d}t\\
&= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\int_0^1 \frac{u^{n-k+p}(1-u)^{n-p}}{x^{n+1-p}}\mathrm{d}u\\
&= \frac{(2n-k+1)!}{n!(n-k)!}x^p\int_0^1 u^{n-k+p}(1-u)^{n-p}\mathrm{d}u\\
&= \frac{(2n-k+1)!}{n!(n-k)!}\frac{1}{\Gamma(2n-k+2)}\Gamma(n-p+1)\Gamma(n-k+1+p)x^p\\
&= \frac{\Gamma(n-p+1)\Gamma(n-k+1+p)}{\Gamma(n+1)\Gamma(n-k+1)}x^p\\
&= \frac{(n-k+p)(n-k+p-1)\cdots(n-k+p-p+1)}{(n-p+1)\cdots n}x^p\\
&= \frac{[n-k+p]_p}{[n]_p}x^p.
\end{aligned}$$

Specially if we take $p = 0, 1$, and 1 in (2) we get

$$\begin{aligned}
(M_{n,k}1)(x) &= 1\\
(M_{n,k}t)(x) &= \frac{\Gamma(n)\Gamma(n-k+2)}{\Gamma(n+1)\Gamma(n-k+1)}x = \frac{n-k+1}{n}x\\
\left(M_{n,k}t^2\right)(x) &= \frac{\Gamma(n-1)\Gamma(n-k+3)}{\Gamma(n+1)\Gamma(n-k+1)}x^2 = \frac{(n-k+2)(n-k+1)}{n(n-1)}x^2.
\end{aligned}$$

From (2) we easily find the following equalities:

$$\begin{aligned}
(M_{n,k}(t-x))(x) &= \frac{1-k}{n}x\\
\left(M_{n,k}(t-x)^2\right)(x) &= \frac{1}{n(n-1)}\left(k^2-5k+2n+4\right)x^2\\
&\le \frac{3}{n-1}x^2 \quad (n \ge (k-1)(k-4)). \qquad (3)
\end{aligned}$$

Note that in the paper [14] the inequality (3) is given for sufficiently large n as;

$$\left(M_{n,k}(t-x)^2\right)(x) \le \frac{2}{n-1}x^2. \qquad (4)$$

But the inequality (4) of Mao is not correct for sufficiently large n. Indeed, for $p = 2$, $k = 5$, then the last inequality will not correct for $n \in \mathbf{N}$.

Lemma 2. *For all $x \in (0,\infty)$ and $n \ge (k-1)(k-4)$, we have*

$$\begin{aligned}
\lambda_{n,k}(x,t) \ &: = \int_0^t K_{n,k}(x,u)\mathrm{d}u \le \frac{1}{(x-t)^2}\frac{3}{n-1}x^2, \quad 0 \le t < x, \qquad (5)\\
1-\lambda_{n,k}(x,z) \ &: = \int_z^\infty K_{n,k}(x,u)\mathrm{d}u \le \frac{1}{(z-x)^2}\frac{3}{n-1}x^2, \quad x < z < \infty.
\end{aligned}$$

Lemma 3[12]**.** *For $x \in (0,\infty)$ and $l \in N$, we have*

$$\sum_{l=0}^{\infty} b_{n,l}(x) = 1 \quad \text{and} \quad b_{n,l}(x) \le \frac{\sqrt{1+x}}{\sqrt{2enx}}$$

where $b_{n,l}(x)$ is the Baskakov basis defined by

$$b_{n,l}(x) = \left(\begin{array}{c} n+l-1 \\ l \end{array}\right) \frac{x^l}{(1+x)^{n+l}}.$$

Lemma 4[12]**.** *For $x \in (0,\infty)$ and $\alpha \le 1$, we have*

$$\left|\left(\sum_{j>nx} b_{n,j}(x)\right)^{\alpha} - \frac{1}{2^{\alpha}}\right| \le \frac{3\sqrt{1+x}}{\sqrt{nx}}.$$

By choosing $\alpha = 1$ *in Lemma* 4, *we obtain;*

Lemma 5. *One has*

$$\int_0^x K_{n,k}(x,t)\mathrm{d}t = \sum_{l=n+1}^{\infty} b_{n-k+1,l}(1),$$

and

$$|M_{n,k}(sgn(t-x);x)| \leq \frac{6\sqrt{2}}{\sqrt{n-k+1}}. \tag{6}$$

Proof. By direct calculation, we get

$$(M_{n,k}sgn(t-x))(x) = \int_0^{\infty} K_{n,k}(x,t)\mathrm{d}t - 2\int_0^x K_{n,k}(x,t)\mathrm{d}t = 1 - 2\int_0^x K_{n,k}(x,t)\mathrm{d}t.$$

Integration by parts gives

$$\int_0^x K_{n,k}(x,t)\mathrm{d}t = \frac{(2n-k+1)!}{n!(n-k)!}x^{n+1}\int_0^x \frac{t^{n-k}}{(x+t)^{2n-k+2}}\mathrm{d}t$$

$$\begin{aligned}
&= \frac{(2n-k+1)!}{n!(n-k)!}x^{n+1}\left[-\frac{t^{n-k}}{(2n-k+1)(x+t)^{2n-k+1}}\Big|_0^x + \frac{n-k}{2n-k+1}\int_0^x \frac{t^{n-k-1}}{(x+t)^{2n-k+1}}\mathrm{d}t\right] \\
&= -\frac{(2n-k)!}{n!(n-k)!}\frac{x^{2n-k+1}}{(2x)^{2n-k+1}} + \frac{(2n-k)!}{(n-k-1)!n!}x^{n+1}\int_0^x \frac{t^{n-k-1}}{(x+t)^{2n-k+1}}\mathrm{d}t \\
&= -\frac{1}{2^{2n-k+1}}\frac{(2n-k)!}{n!(n-k)!} + \frac{(2n-k)!}{(n-k-1)!n!}x^{n+1}\int_0^x \frac{t^{n-k-1}}{(x+t)^{2n-k+1}}\mathrm{d}t.
\end{aligned}$$

By the same way, if we apply $(n-1)$ times partial integration, one has

$$\begin{aligned}
\int_0^x K_{n,k}(x,t)\mathrm{d}t &= 1 - \left[\frac{1}{2^{2n-k+1}}\frac{(2n-k)!}{n!(n-k)!} + \cdots + \frac{1}{2^{n-k+2}}\frac{(n-k+1)!}{(n-k)!} + \frac{1}{2^{n-k+1}}\right] \\
&= 1 - [b_{n-k+1,n}(1) + b_{n-k+1,n-1}(1) + \cdots + b_{n-k+1,1}(1) + b_{n-k+1,0}(1)] \\
&= 1 - \sum_{l=0}^{n} b_{n-k+1,l}(1) \\
&= \sum_{l=n+1}^{\infty} b_{n-k+1,l}(1).
\end{aligned}$$

Since

$$\begin{aligned}
|(M_{n,k}\mathrm{sgn}(t-x))(x)| &= \left|1 - 2\int_0^x K_{n,k}(x,t)\mathrm{d}t\right| = 2\left|\sum_{l=n+1}^{\infty} b_{n-k+1,l}(1) - \frac{1}{2}\right| \\
&\leq 2\left|\sum_{l>n-k+1}^{\infty} b_{n-k+1,l}(1) - \frac{1}{2}\right|
\end{aligned}$$

from Lemma 3 and Lemma 4, we reach the desired result.

Lemma 6. *Let* $T_{n,k,m}(x) := (M_{n,k}(t-x)^m)(x)$, $m \in N_0$ *(the set of non-negative integers). The central moments of order* $m \in N_0$, *any fixed* $x \in [0,\infty)$, *then*

$$T_{n,k,m}(x) = \left(\sum_{l=0}^{m} (-1)^l \begin{pmatrix} m \\ l \end{pmatrix} \frac{\Gamma(n-m+l+1)\Gamma(n-k+1+m-l)}{\Gamma(n+1)\Gamma(n-k+1)} \right) x^m.$$

Proof.

$$\begin{aligned} T_{n,k,m}(x) \quad : &= (M_{n,k}(t-x)^m)(x) = \int_0^\infty K_{n,k}(x,t)(t-x)^m \mathrm{d}t \\ &= \int_0^\infty K_{n,k}(x,t) \sum_{l=0}^{m} (-1)^l \begin{pmatrix} m \\ l \end{pmatrix} x^l t^{m-l} \mathrm{d}t \\ &= \sum_{l=0}^{m} (-1)^l \begin{pmatrix} m \\ l \end{pmatrix} x^l \left(M_{n,k} t^{m-l} \right)(x). \end{aligned}$$

By (2), we reach the result.

Furthermore, the following recurrence relation holds:

$$(m+2-n-2)T_{n,k,m+1} = -x(5+3m+k)T_{n,k,m} - 2x^2 m T_{n,k,m-1}, \quad m \geq 1.$$

Proof of the Recurrence Relation. Alternatively we can rewrite the operators $(M_{n,k}f)(x)$ as

$$\begin{aligned} (M_{n,k}f)(x) &= \int_0^\infty K_{n,k}(x,t) f(t) \mathrm{d}t \\ &= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!} \int_0^\infty \frac{t^{n-k}}{(x+t)^{2n-k+2}} f(t) \mathrm{d}t, \qquad x > 0, \end{aligned}$$

where

$$K_{n,k}(x,t) := \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!} \frac{t^{n-k}}{(x+t)^{2n-k+2}}, \qquad x,t \in (0,\infty).$$

Here we differentiate the kernel function with respect to t, we get

$$\begin{aligned}
\frac{\partial}{\partial t}K_{n,k}(x,t) &= \frac{\partial}{\partial t}\left[\frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\frac{t^{n-k}}{(x+t)^{2n-k+2}}\right] \\
&= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\left[\frac{(n-k)t^{n-k-1}}{(x+t)^{2n-k+2}}-\frac{(2n-k+2)t^{n-k}}{(x+t)^{2n-k+3}}\right] \\
&= \frac{(2n-k+1)!x^{n+1}}{n!(n-k)!}\left[\frac{n-k}{t}\frac{t^{n-k}}{(x+t)^{2n-k+2}}-\frac{2n-k+2}{x+t}\frac{t^{n-k}}{(x+t)^{2n-k+2}}\right] \\
&= \frac{n-k}{t}K_{n,k}(x,t)-\frac{2n-k+2}{x+t}K_{n,k}(x,t) \\
&= K_{n,k}(x,t)\left[\frac{n-k}{t}-\frac{2n-k+2}{x+t}\right] \\
&= K_{n,k}(x,t)\left[\frac{nx-kx-nt-2t}{t\,(x+t)}\right].
\end{aligned}$$

Hence we have

$$t(x+t)\frac{\partial}{\partial t}K_{n,k}(x,t)=K_{n,k}(x,t)\,[nx-kx-nt-2t]\,.$$

Since

$$nx-kx-nt-2t=-\,(2x+kx)-(t-x)\,(n+2)$$

and

$$t(x+t)=(t-x+x)(t-x+2x)=(t-x)^2+3x(t-x)+2x^2,$$

we obtain

$$\begin{aligned}
&\int_0^\infty t(x+t)\left[\frac{\partial}{\partial t}K_{n,k}(x,t)\right](t-x)^m\mathrm{d}t \\
=\ &\int_0^\infty \left[(t-x)^2+3x(t-x)+2x^2\right]\left[\frac{\partial}{\partial t}K_{n,k}(x,t)\right](t-x)^m\mathrm{d}t \\
=\ &\int_0^\infty \left[\frac{\partial}{\partial t}K_{n,k}(x,t)\right](t-x)^{m+2}\mathrm{d}t+ \\
&+3x\int_0^\infty \left[\frac{\partial}{\partial t}K_{n,k}(x,t)\right](t-x)^{m+1}\mathrm{d}t+2x^2\int_0^\infty \left[\frac{\partial}{\partial t}K_{n,k}(x,t)\right](t-x)^m\mathrm{d}t.
\end{aligned}$$

Using partial integration to the right handside of the last equality, we have

$$\begin{aligned}
\int_0^\infty t(x+t)\left[\frac{\partial}{\partial t}K_{n,k}(x,t)\right](t-x)^m\mathrm{d}t &= -(m+2)\int_0^\infty K_{n,k}(x,t)(t-x)^{m+1}\mathrm{d}t \\
&\quad -3x(m+1)\int_0^\infty K_{n,k}(x,t)(t-x)^m\mathrm{d}t \\
&\quad -2x^2m\int_0^\infty K_{n,k}(x,t)(t-x)^{m-1}\mathrm{d}t
\end{aligned}$$

$$=-(m+2)T_{n,k,m+1}-3x(m+1)T_{n,k,m}-2x^2mT_{n,k,m-1}.$$

On the other hand

$$\begin{aligned}&\int_0^\infty t(x+t)\left[\frac{\partial}{\partial t}K_{n,k}(x,t)\right](t-x)^m\mathrm{d}t\\&\quad=\int_0^\infty K_{n,k}(x,t)\left[-(2x+kx)-(t-x)(n+2)\right](t-x)^m\mathrm{d}t.\end{aligned}$$

Thus we have

$$\begin{aligned}&= -(n+2)\int_0^\infty K_{n,k}(x,t)(t-x)^{m+1}\mathrm{d}t-(2x+kx)\int_0^\infty K_{n,k}(x,t)(t-x)^m\mathrm{d}t\\&= -(n+2)T_{n,k,m+1}-(2x+kx)T_{n,k,m}.\end{aligned}$$

Consequently we obtain the following recurrence relation;

$$\begin{aligned}&-(n+2)T_{n,k,m+1}-(2x+kx)T_{n,k,m}\\&\quad=-(m+2)T_{n,k,m+1}-3x(m+1)T_{n,k,m}-2x^2mT_{n,k,m-1}\end{aligned}$$

or

$$\begin{aligned}(m+2-n-2)T_{n,k,m+1} &= [-3x(m+1)-2x-kx]T_{n,k,m}-2x^2mT_{n,k,m-1}\\&= -x(5+3m+k)T_{n,k,m}-2x^2mT_{n,k,m-1}.\end{aligned}$$

This completes the proof of the Lemma.

It follows from the recurrence relation by using the principle of mathematical induction that

$$T_{n,k,m}(x)\le A(m,k)\frac{x^m}{n^{[(m+1)/2]}},\qquad n\to\infty,\ x\in[0,\infty),\ m\in\mathbf{N}_0,$$

where $A(m,k)$ is a constant depending only on m and k and $[a]$ denotes the integral part of a.

3 Main Result

The main theorem of this paper is stated as:

Theorem. *Let $f\in BV_\gamma[0,\infty)$, $\gamma\ge 0$. Then for every $x\in(0,\infty)$, for $r\in N$ $(2r\ge\gamma)$ and for n sufficiently large, we have*

$$\begin{aligned}&\left|(M_{n,k}f)(x)-\frac{f(x+)+f(x-)}{2}\right|\\&\quad\le\frac{6}{n-1}\sum_{l=1}^{n}\bigvee_{x-\frac{x}{\sqrt{l}}}^{x+\frac{x}{\sqrt{l}}}(f_x)+\frac{6\sqrt{2}}{\sqrt{n-k+1}}\left|\frac{f(x+)-f(x-)}{2}\right|\\&\qquad+M2^{2r}A(2r,k)\frac{x^{2r}}{n^r},\end{aligned}\tag{7}$$

where

$$f_x(t)=\begin{cases} f(t)-f(x+) & , \; x<t<\infty \\ 0 & , \; t=x \\ f(t)-f(x-) & , \; 0\le t<x \end{cases}, \tag{8}$$

$\bigvee_a^b (f_x)$ *is the total variation of* f_x *on* $[a,b]$ *and* $A(2r,k)$ *is a constant depending on* r *and* k.

Proof. For $t\in[0,\infty)$, it is known that from (8)

$$\begin{aligned} f(t) &= \frac{f(x+)+f(x-)}{2}+f_x(t)+\frac{f(x+)-f(x-)}{2}\operatorname{sgn}(t-x) \\ &\quad +\delta_x(t)\left[f(x)-\frac{f(x+)+f(x-)}{2}\right], \end{aligned} \tag{9}$$

where

$$\delta_x(t)=\begin{cases} 1 \, , \; x=t \\ 0 \, , \; x\neq t. \end{cases}$$

If we use (9) in (1), we have the following expressions.

$$\left(M_{n,k}f\right)(x)=\frac{f(x+)+f(x-)}{2}\left(M_{n,k}1\right)(x)+\left(M_{n,k}f_x\right)(x)$$

$$+\frac{f(x+)-f(x-)}{2}\left(M_{n,k}\operatorname{sgn}(t-x)\right)(x)+\left[f(x)-\frac{f(x+)+f(x-)}{2}\right]\left(M_{n,k}\delta_x\right)(x)$$

For operators $\left(M_{n,k}f\right)(x)$, it is obvious that $\left(M_{n,k}\delta_x\right)(x)=0$ and $\left(M_{n,k}1\right)(x)$. Hence we have

$$\left|\left(M_{n,k}f\right)(x)-\frac{f(x+)+f(x-)}{2}\right|\le\left|\left(M_{n,k}f_x\right)(x)\right|+\left|\frac{f(x+)-f(x-)}{2}\right|\left|\left(M_{n,k}\operatorname{sgn}(t-x)\right)(x)\right|. \tag{10}$$

In order to prove the Theorem 1, we need the estimates for $\left(M_{n,k}f_x\right)(x)$ and $\left(M_{n,k}\operatorname{sgn}(t-x)\right)(x)$ in (10).

We first estimate $\left(M_{n,k}f_x\right)(x)$ as follows:

$$\left|\left(M_{n,k}f_x\right)(x)\right|\le\left|I_1(n,x)\right|+\left|I_2(n,x)\right|+\left|I_3(n,x)\right|, \tag{11}$$

where

$$\left|I_1(n,x)\right|=\left|\int_0^{x-\frac{x}{\sqrt{n}}} f_x(t)\,\mathrm{d}_t\left(\lambda_{n,k}(x,t)\right)\right|, \; \left|I_2(n,x)\right|=\left|\int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} f_x(t)\,\mathrm{d}_t\left(\lambda_{n,k}(x,t)\right)\right|,$$

$$\left|I_3(n,x)\right|=\left|\int_{x+\frac{x}{\sqrt{n}}}^{\infty} f_x(t)\,\mathrm{d}_t\left(\lambda_{n,k}(x,t)\right)\right|$$

and

$$\lambda_{n,k}(x,t) := \int_0^t K_{n,k}(x,u)\mathrm{d}u.$$

First we estimate $I_2(n,x)$. For $t \in \left[x - \frac{x}{\sqrt{n}}, x + \frac{x}{\sqrt{n}}\right]$, we have

$$|I_2(n,x)| \le \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} |f_x(t) - f_x(x)|\,|d_t\left(\lambda_{n,k}(x,t)\right)| \le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}(f_x) \le \frac{1}{n}\sum_{l=1}^{n}\bigvee_{x-\frac{x}{\sqrt{l}}}^{x+\frac{x}{\sqrt{l}}}(f_x). \tag{12}$$

Next, we estimate $I_1(n,x)$. Using partial Lebesgue-Stieltjes integration, we have

$$I_1(n,x) = f_x\left(x - \frac{x}{\sqrt{n}}\right)\lambda_{n,k}\left(x, x - \frac{x}{\sqrt{n}}\right) - f_x(0)\lambda_{n,k}(x,0) - \int_0^{x-\frac{x}{\sqrt{n}}} \lambda_{n,k}(x,t)\mathrm{d}_t\left(f_x(t)\right)$$

Since

$$\left|f_x\left(x - \frac{x}{\sqrt{n}}\right)\right| = \left|f_x\left(x - \frac{x}{\sqrt{n}}\right) - f_x(x)\right| \le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_x),$$

it follows that

$$|I_1(n,x)| \le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_x)\left|\lambda_{n,k}\left(x, x - \frac{x}{\sqrt{n}}\right)\right| + \int_0^{x-\frac{x}{\sqrt{n}}} \lambda_{n,k}(x,t)\mathrm{d}_t\left(-\bigvee_t^x(f_x)\right).$$

By (5), it is clear that

$$\lambda_{n,k}\left(x, x - \frac{x}{\sqrt{n}}\right) \le \frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2}\frac{3}{n-1}x^2.$$

For simplicity we define $\frac{3}{n-1}x^2 := B_n(x)$. It follows that

$$\begin{aligned} |I_1(n,x)| &\le \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_x)\frac{B_n(x)}{\left(\frac{x}{\sqrt{n}}\right)^2} + \int_0^{x-\frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2}B_n(x)\mathrm{d}_t\left(-\bigvee_t^x(f_x)\right) \\ &= \bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_x)\frac{B_n(x)}{\left(\frac{x}{\sqrt{n}}\right)^2} + B_n(x)\int_0^{x-\frac{x}{\sqrt{n}}} \frac{1}{(x-t)^2}\mathrm{d}_t\left(-\bigvee_t^x(f_x)\right). \end{aligned}$$

Integration by parts gives

$$\begin{aligned} \int_0^{x-\frac{x}{\sqrt{n}}} \frac{\mathrm{d}_t\left(-\bigvee_t^x(f_x)\right)}{(x-t)^2} &= -\frac{\bigvee_t^x(f_x)}{(x-t)^2}\Bigg|_0^{x-\frac{x}{\sqrt{n}}} + \int_0^{x-\frac{x}{\sqrt{n}}} \frac{2\bigvee_t^x(f_x)}{(x-t)^3}\mathrm{d}t \\ &= -\frac{1}{\left(\frac{x}{\sqrt{n}}\right)^2}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}(f_x) + \frac{1}{x^2}\bigvee_0^x(f_x) + \int_0^{x-\frac{x}{\sqrt{n}}} \frac{2\bigvee_t^x(f_x)}{(x-t)^3}\mathrm{d}t. \end{aligned}$$

Putting $t = x - \dfrac{x}{\sqrt{u}}$ in the last integral, we get

$$\int_0^{x-\frac{x}{\sqrt{n}}} \frac{2\bigvee_t^x(f_x)}{(x-t)^3}\mathrm{d}t = \frac{1}{x^2}\int_1^n \bigvee_{x-\frac{x}{\sqrt{u}}}^{x}(f_x)\mathrm{d}u = \frac{1}{x^2}\sum_{l=1}^{n}\bigvee_{x-\frac{x}{\sqrt{l}}}^{x}(f_x).$$

Consequently we have

$$|I_1(n,x)| \le B_n(x)\left(\frac{1}{x^2}\bigvee_0^x(f_x) + \frac{1}{x^2}\sum_{l=1}^{n}\bigvee_{x-\frac{x}{\sqrt{l}}}^{x}(f_x)\right) \le \frac{6}{n-1}\sum_{l=1}^{n}\bigvee_{x-\frac{x}{\sqrt{l}}}^{x}(f_x). \tag{12}$$

Finally, we estimate $|I_3(n,x)|$ by setting

$$\hat{g}_x(t) := \begin{cases} f(t), & 0 \le t \le 2x \\ f(2x), & 2x < t < \infty \end{cases}$$

we rewrite $|I_3(n,x)|$ as

$$\begin{aligned} |I_3(n,x)| &= \int_{x+\frac{x}{\sqrt{n}}}^{\infty} \hat{g}_x(t)\mathrm{d}_t\left(\lambda_{n,k}(x,t)\right) + \int_{2x}^{\infty}[f(t)-f(2x)]\,\mathrm{d}_t\left(\lambda_{n,k}(x,t)\right) \\ &= : I_{3,1}(n,x) + I_{3,2}(n,x). \end{aligned}$$

$I_{3,1}(n,x)$ can be evaluated as follows:

$$\begin{aligned} I_{3,1}(n,x) &= \lim_{a\to\infty}\left\{ f\left(x+\frac{x}{\sqrt{n}}\right)\left[1-\lambda_{n,k}\left(x, x+\frac{x}{\sqrt{n}}\right)\right] + \hat{g}_x(a)\left[\lambda_{n,k}(x,a)-1\right] \right. \\ &\quad \left. + \int_{2x}^{a} f(t)\mathrm{d}_t\left(\lambda_{n,k}(x,t)\right)\right\}. \end{aligned}$$

Using (5), we obtain

$$\begin{aligned} I_{3,1}(n,x) &= \frac{3x^2}{n-1}\lim_{a\to\infty}\left\{\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f_x)\frac{n}{x^2} + \frac{\hat{g}_x(a)}{(a-x)^2} + \int_0^x \frac{1}{(t-x)^2}\mathrm{d}_t\left(\bigvee_x^t(\hat{g}_x)\right)\right\} \\ &= \frac{3x^2}{n-1}\left\{\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}(f_x)\frac{n}{x^2} + \int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}\mathrm{d}_t\left(\bigvee_x^t(f)\right)\right\}. \end{aligned}$$

So

$$\begin{aligned}\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{1}{(t-x)^2}\mathrm{d}_t\left(\bigvee_x^t(f_x)\right) &= \frac{\bigvee_x^t(f_x)}{(t-x)^2}\Big|_{x+\frac{x}{\sqrt{n}}}^{2x}+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{2}{(t-x)^3}\bigvee_x^t(f_x)\mathrm{d}t\\ &= \frac{1}{x^2}\bigvee_x^{2x}(f_x)-\frac{n}{x^2}\bigvee_x^{x+\frac{x}{\sqrt{n}}}(f_x)\\ &\quad+\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{2}{(x-t)^3}\bigvee_t^x(f_x)\mathrm{d}t.\end{aligned}$$

Now, similarly to the previous case, again setting $t=x+\frac{x}{\sqrt{u}}$ then one has

$$\int_{x+\frac{x}{\sqrt{n}}}^{2x}\frac{2\bigvee_x^t(f_x)}{(t-x)^3}\mathrm{d}t=\frac{1}{x^2}\int_1^n\bigvee_x^{x+\frac{x}{\sqrt{u}}}(f_x)\mathrm{d}u=\frac{1}{x^2}\sum_{l=1}^n\bigvee_x^{x+\frac{x}{\sqrt{l}}}(f_x).$$

Further, we have

$$\begin{aligned}I_{3,1}(n,x) &\le \frac{3x^2}{n-1}\left\{\frac{1}{x^2}\bigvee_x^{2x}(f_x)+\frac{1}{x^2}\sum_{l=1}^{n-1}\bigvee_x^{x+\frac{x}{\sqrt{l}}}(f_x)\right\}\\ &= \frac{6}{n-1}\sum_{l=1}^n\bigvee_x^{x+\frac{x}{\sqrt{l}}}(f_x).\end{aligned}\tag{13}$$

Completing the estimation of $|I_3(n,x)|$, we shall estimate $I_{3,2}(n,x)$. We note that there exists an integer r $(2r\ge\gamma)$ such that

$$f(t)=O(t^{2r}),\quad \text{for every}\quad t>0.$$

Since the function f satisfies the growth condition, i.e., $|f(t)|\le Mt^{\gamma}$, (for some $\gamma>0$, for some constant $M>0$) as $t\to\infty$ and $2(t-x)\ge t$ whenever $t\ge 2x$, we obtain from [6]

$$I_{3,2}(n,x)\le M2^{2r}A(2r,k)\frac{x^{2r}}{n^r}.\tag{14}$$

Combining (13) and (14), we find

$$|I_3(n,x)|\le\frac{6}{n-1}\sum_{l=1}^n\bigvee_x^{x+\frac{x}{\sqrt{l}}}(f_x)+M2^{2r}A(2r,k)\frac{x^{2r}}{n^r}.\tag{15}$$

Putting (11), (12), (15) in (10) and considering Lemma 5 we obtain the required result (8). Thus the proof is completed.

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Abant Izzet Baysal University
Faculty of Science and Arts
Department of Mathematics
14280, Gölköy-Bolu, Turkey

E-mail: karsli_h@ibu.edu.tr