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UNIFORM MEYER SOLUTION TO THE THREE DIMENSIONAL CAUCHY PROBLEM FOR LAPLACE EQUATION

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Abstract. We consider the three dimensional Cauchy problem for the Laplace equation

$$\begin{cases} u_{xx}(x,y,z)+u_{yy}(x,y,z)+u_{zz}(x,y,z)=0, & x\in\mathbf{R}, y\in\mathbf{R}, 0<z\le 1,\\ u(x,y,0)=g(x,y), & x\in\mathbf{R}, y\in\mathbf{R},\\ u_z(x,y,0)=0, & x\in\mathbf{R}, y\in\mathbf{R}, \end{cases}$$

where the data is given at $z=0$ and a solution is sought in the region $x,y\in\mathbf{R}, 0<z<1$. The problem is ill-posed, the solution (if it exists) doesn't depend continuously on the initial data. Using Galerkin method and Meyer wavelets, we get the uniform stable wavelet approximate solution. Furthermore, we shall give a recipe for choosing the coarse level resolution.

Key words: *Laplace equation, wavelet solution, uniform convergence*

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1 Introduction

Many physical and engineering problems require the solution of the following Cauchy problem for Laplace equation:

$$\begin{cases} u_{xx}(x,y,z)+u_{yy}(x,y,z)+u_{zz}(x,y,z)=0, & x\in\mathbf{R}, y\in\mathbf{R}, 0<z\leq 1, \\ u(x,y,0)=g(x,y), & x\in\mathbf{R}, y\in\mathbf{R}, \\ u_z(x,y,0)=0, & x\in\mathbf{R}, y\in\mathbf{R}. \end{cases} \tag{1.1}$$

Wavelet regularization methods for solving the Cauchy problem for Laplace Equation have been studied by many authors. They used the wavelet method to approximate the Laplace Equation by Meyer wavelets (see [1]-[2]), but most authors concentrated on the two dimensional case. In this paper, we consider the three dimensional Cauchy problem for Laplace Equation.

To the authors' knowledge, so far there are many papers on the Laplace Equation, but theoretically the error estimates of most regularization methods are in L^2-sense. In this paper, we improve the results and get uniform convergent wavelet solution. We also give a rule for choosing an appropriate wavelet subspace depending on the noise level of the data.

For $v(x,y)\in L^2(\mathbf{R}^2)$, define

$$\| v \|_{L^2}=\left(\int_{\mathbf{R}^2}|v(x,y)|^2\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}, \tag{1.2}$$

and for $v(x,y)\in L^1(\mathbf{R}^2)\bigcap L^2(\mathbf{R}^2)$, define

$$\hat{v}(\xi,\tau)=\frac{1}{2\pi}\int_{\mathbf{R}^2}v(x,y)e^{-i(\xi x+\tau y)}\mathrm{d}x\mathrm{d}y. \tag{1.3}$$

In this paper, $g(x,y)\in L^2(\mathbf{R}^2)$ denotes the accurate data, $g_\delta(x,y)$ denotes the measured data satisfying

$$\| g_\delta(x,y)-g(x,y) \|_{L^2}\leq \delta, \tag{1.4}$$

where δ represents a bound on the measurement error.

Applying Fourier transform with respect to x, y to the problem (1.1), we get

$$\begin{cases} \hat{u}_{zz}(\xi,\tau,z)=(\xi^2+\tau^2)\hat{u}(\xi,\tau,z), & \xi\in\mathbf{R}, \tau\in\mathbf{R}, 0<z\leq 1, \\ \hat{u}(\xi,\tau,0)=\hat{g}(\xi,\tau), & \xi\in\mathbf{R}, \tau\in\mathbf{R}, \\ \hat{u}_z(\xi,\tau,0)=0, & \xi\in\mathbf{R}, \tau\in\mathbf{R}, \end{cases} \tag{1.5}$$

The solution of the problem (1.5) can be expressed by

$$\hat{u}(\xi,\tau,z)=\hat{g}(\xi,\tau)\cosh(\sqrt{\xi^2+\tau^2}z), \tag{1.6}$$

or equivalently,

$$u(x,y,z)=\frac{1}{2\pi}\int_{\mathbf{R}^2}\hat{g}(\xi,\tau)\cosh(\sqrt{\xi^2+\tau^2}z)e^{i(\xi x+\tau y)}\mathrm{d}\xi\mathrm{d}\tau. \tag{1.7}$$

Since for $z\in(0,1]$, $\hat{u}(\cdot,\cdot,z)\in L^2(\mathbf{R}^2)$, (1.6) implies that $\hat{g}(\xi,\tau)$, which is the Fourier transform of the exact data $g(x,y)$, must decay rapidly as $|\xi|\to+\infty$ or $|\tau|\to+\infty$. Such a decay is not likely to occur in the Fourier transform of the measured data $\hat{g_\delta}(\xi,\tau)$, hence the problem is ill-posed. In this paper, we shall always assume $\hat{g}(\xi,\tau)$ is continuous and satisfies

$$\hat{f}(\xi,\tau):=\hat{g}(\xi,\tau)e^{5(\xi^2+\tau^2)}\in L^1(\mathbf{R}^2)\cap L^2(\mathbf{R}^2). \tag{1.8}$$

This paper is organized as follows: in section 2 we give some basic properties of Meyer wavelets; in section 3 we prove some auxiliary results and in the last section we present the main conclusion of this paper.

2 Meyer Wavelets

2.1 Multiresolution Analysis

Following [3], A multi-resolution analysis (MRA) of $L^2(\mathbf{R})$ is a set of closed linear subspaces V_j satisfying:

(a) For all $j\in\mathbf{Z}$, $V_j\subseteq V_{j+1}$;

(b) If $f(x)$ is C_c^0 on $\mathbf{R}$, then $f(x)\in\overline{\mathrm{span}}\{V_j\}_{j\in\mathbf{Z}}$. That is, for any given $\varepsilon>0$, exist $j\in\mathbf{Z}$ and $g(x)\in V_j$ such that $\|f-g\|_{L^2}<\varepsilon$;

(c) $\bigcap_{j\in\mathbf{Z}}V_j=\{0\}$;

(d) $f(x)\in V_0$ if and only if $f(2^jx)\in V_j$;

(e) There exists a function $\varphi(x)\in L^2(\mathbf{R})$, called the scaling function, such that $\{\varphi(x-n)\}_{n\in\mathbf{Z}}$ is an orthonormal system of translates and

$$V_0=\overline{\mathrm{span}}\{\varphi(x-n)\}_{n\in\mathbf{Z}}.$$

2.2 One Dimensional Meyer Wavelet

In [4], a Meyer scaling function $\varphi(x)$ is given by its Fourier transform

$$\hat{\varphi}(\xi)=\begin{cases}\dfrac{1}{\sqrt{2\pi}}, & |\xi|\le\dfrac{2}{3}\pi,\\ \dfrac{1}{\sqrt{2\pi}}\cos[\dfrac{\pi}{2}\upsilon(\dfrac{3}{4\pi}|\xi|-1)], & \dfrac{2}{3}\pi\le|\xi|\le\dfrac{4}{3}\pi,\\ 0, & \text{otherwise,}\end{cases}$$

where $\upsilon \in C^\infty(\mathbf{R})$ and satisfies

$$\upsilon(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x \geq 1; \end{cases}$$

$v(x)+v(1-x)=1$, for $0 \leq x \leq 1$.

The corresponding wavelet function $\psi(x)$ is given by

$$\hat{\psi}(\xi) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{\frac{i\xi}{2}} \sin[\frac{\pi}{2}\upsilon(\frac{3}{2\pi}|\xi|-1)], & \frac{2}{3}\pi \leq |\xi| \leq \frac{4}{3}\pi, \\ \frac{1}{\sqrt{2\pi}} e^{\frac{i\xi}{2}} \cos[\frac{\pi}{2}\upsilon(\frac{3}{4\pi}|\xi|-1)], & \frac{4}{3}\pi \leq |\xi| \leq \frac{8}{3}\pi, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\varphi(x)$ be the Meyer scaling function, then $\varphi(x) \in C^\infty(\mathbf{R})$ and

$$|\varphi^{(k)}(x)| \leq \frac{C_{nk}}{(1+|x|)^n}, \tag{2.1}$$

where $k=1,2,3,\cdots;n=2,3,4,\cdots;x\in\mathbf{R}$[5].

Denote $\psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^jx-k)$, $\varphi_{jk}(x) = 2^{\frac{j}{2}}\varphi(2^jx-k)$, $j,k\in\mathbf{Z}$, then for arbitrary $k\in\mathbf{Z}$,

$$\operatorname{supp}(\hat{\psi}_{jk}) = \{\xi : \frac{2}{3}\pi 2^j \leq |\xi| \leq \frac{8}{3}\pi 2^j\}, \quad \operatorname{supp}(\hat{\varphi}_{jk}) = \{\xi : |\xi| \leq \frac{4}{3}\pi 2^j\}. \tag{2.2}$$

2.3 Two Dimensional Tensor-product Wavelets

Lemma 2.1[3] *Let $\varphi(x)$ and $\psi(x)$ be the scaling and wavelet function associated with a MRA respectively and define*

$$\Phi(x,y) = \varphi(x)\varphi(y), \qquad \Psi^{(1)}(x,y) = \varphi(x)\psi(y),$$

$$\Psi^{(2)}(x,y) = \psi(x)\varphi(y), \quad \Psi^{(3)}(x,y) = \psi(x)\psi(y),$$

For each $j,k_1,k_2\in\mathbf{Z}$, define

$$\Phi_{j,k_1,k_2}(x,y) = \varphi_{j,k_1}(x)\varphi_{j,k_2}(y), \qquad \Psi^{(1)}_{j,k_1,k_2}(x,y) = \varphi_{j,k_1}(x)\psi_{j,k_2}(y),$$

$$\Psi^{(2)}_{j,k_1,k_2}(x,y) = \psi_{j,k_1}(x)\varphi_{j,k_2}(y), \quad \Psi^{(3)}_{j,k_1,k_2}(x,y) = \psi_{j,k_1}(x)\psi_{j,k_2}(y),$$

then

(1) *The collection $\{\Phi_{j,k_1,k_2}(x,y)\}_{k_1,k_2\in\mathbf{Z}}$ is an orthonormal basis on $V_j = \overline{span}\{\Phi_{j,k_1,k_2}(x,y)\}_{k_1,k_2\in\mathbf{Z}}$.*

(2) *The collection$\{\Psi^{(m)}_{j,k_1,k_2}(x,y)\}_{1\leq m\leq 3,k_1,k_2\in\mathbf{Z}}$ is an orthonormal basis on $W_j = \overline{span}\{\Psi^{(m)}_{j,k_1,k_2}\}_{1\leq m\leq 3,k_1,k_2\in\mathbf{Z}}$.*

(3) *The collection* $\{\Psi^{(m)}_{j,k_1,k_2}(x,y)\}_{1\le m\le 3, j,k_1,k_2\in\mathbf{Z}}$ *is an orthonormal basis on* $L^2(\mathbf{R}^2)$.

(4) *For each* $J\in\mathbf{Z}$, *the collection* $\{\Phi_{J,k_1,k_2}(x,y)\}_{k_1,k_2\in\mathbf{Z}}\bigcup\{\Psi^{(m)}_{j,k_1,k_2}(x,y)\}_{1\le m\le 3, j\ge J,k_1,k_2\in\mathbf{Z}}$ *is an orthonormal basis on* $L^2(\mathbf{R}^2)$.

From the definition of Fourier transform, we can easily get

$$\hat{\Phi}_{j,k_1,k_2}(\xi,\tau)=\hat{\varphi}_{j,k_1}(\xi)\hat{\varphi}_{j,k_2}(\tau),\ \ \hat{\Psi}^{(1)}_{j,k_1,k_2}(\xi,\tau)=\hat{\varphi}_{j,k_1}(\xi)\hat{\psi}_{j,k_2}(\tau),$$

$$\hat{\Psi}^{(2)}_{j,k_1,k_2}(\xi,\tau)=\hat{\psi}_{j,k_1}(\xi)\hat{\varphi}_{j,k_2}(\tau),\ \ \hat{\Psi}^{(3)}_{j,k_1,k_2}(\xi,\tau)=\hat{\psi}_{j,k_1}(\xi)\hat{\psi}_{j,k_2}(\tau).$$

and for each $k_1,k_2\in\mathbf{Z}$, we have

$$\mathrm{supp}\{\hat{\Phi}_{j,k_1,k_2}(\xi,\tau)\}=\{(\xi,\tau):|\xi|,|\tau|\le\frac{4}{3}\pi 2^j\},$$

$$\mathrm{supp}\{\hat{\Psi}^{(1)}_{j,k_1,k_2}(\xi,\tau)\}=\{(\xi,\tau):|\xi|\le\frac{4}{3}\pi 2^j,\frac{2}{3}\pi 2^j\le|\tau|\le\frac{8}{3}\pi 2^j\},$$

$$\mathrm{supp}\{\hat{\Psi}^{(2)}_{j,k_1,k_2}(\xi,\tau)\}=\{(\xi,\tau):\frac{2}{3}\pi 2^j\le|\xi|\le\frac{8}{3}\pi 2^j,|\tau|\le\frac{4}{3}\pi 2^j\},$$

$$\mathrm{supp}\{\hat{\Psi}^{(3)}_{j,k_1,k_2}(\xi,\tau)\}=\{(\xi,\tau):\frac{2}{3}\pi 2^j\le|\xi|,|\tau|\le\frac{8}{3}\pi 2^j\}.$$

$\forall u\in L^2(\mathbf{R}^2)$, let

$$P_j:L^2(\mathbf{R}^2)\to V_j,\qquad P_ju=\sum_{k_1,k_2\in\mathbf{Z}}\langle u,\Phi_{j,k_1,k_2}\rangle\Phi_{j,k_1,k_2},$$

$$Q_j:L^2(\mathbf{R}^2)\to W_j,\qquad Q_ju=\sum_{m=1}^{3}\sum_{k_1,k_2\in\mathbf{Z}}\langle u,\Psi^{(m)}_{j,k_1,k_2}\rangle\Psi^{(m)}_{j,k_1,k_2},$$

then

$$\hat{u}(\xi,\tau)=\widehat{P_ju}(\xi,\tau),\ \ \xi,\tau\in A_{j-1},\tag{2.3}$$

$$\hat{u}(\xi,\tau)-\widehat{P_ju}(\xi,\tau)=\widehat{Q_ju}(\xi,\tau),\ \ \xi,\tau\in A_j\backslash A_{j-1},\tag{2.4}$$

where $A_j:=[-\frac{4}{3}\pi 2^j,\frac{4}{3}\pi 2^j]\times[-\frac{4}{3}\pi 2^j,\frac{4}{3}\pi 2^j]$.

Let

$$\widehat{V}_j=\overline{\mathrm{span}}\{\hat{\Phi}_{j,k_1,k_2}\}_{k_1,k_2\in\mathbf{Z}},\ \ \widehat{W}_j=\overline{\mathrm{span}}\{\hat{\Psi}^{(m)}_{j,k_1,k_2}\}_{1\le m\le 3,k_1,k_2\in\mathbf{Z}},$$

$\widehat{P}_j$ and $\widehat{Q}_j$ be the orthogonal projection operator from $L^2(\mathbf{R}^2)$ to $\widehat{V}_j$ and $\widehat{W}_j$ respectively, then

$$\widehat{P}_j\hat{f}=\widehat{P_jf},\ \ \widehat{Q}_j\hat{f}=\widehat{Q_jf}.$$

3 Auxiliary Results

Consider the approximate solution of (1.1) in V_j.

$$\begin{cases} P_j u_{xx}(x,y,z) + P_j u_{yy}(x,y,z) + u_{zz}(x,y,z) = 0, & x \in \mathbf{R}, y \in \mathbf{R}, 0 < z \leq 1, \\ u(x,y,0) = P_j g(x,y), & x \in \mathbf{R}, y \in \mathbf{R}, \\ u_z(x,y,0) = 0, & x \in \mathbf{R}, y \in \mathbf{R}, \end{cases}$$

or equivalently,

$$\begin{cases} \langle u_{xx}(x,y,z) + u_{yy}(x,y,z) + u_{zz}(x,y,z), \Phi_{jk_1k_2}(x,y) \rangle = 0, & x \in \mathbf{R}, y \in \mathbf{R}, 0 < z \leq 1, \\ \langle u(x,y,0), \Phi_{jk_1k_2}(x,y) \rangle = \langle g(x,y), \Phi_{jk_1k_2}(x,y) \rangle, & x \in \mathbf{R}, y \in \mathbf{R}, \\ \langle u_z(x,y,0), \Phi_{jk_1k_2}(x,y) \rangle = 0, & x \in \mathbf{R}, y \in \mathbf{R}, \end{cases}$$

where $\Phi_{jk_1k_2}(x,y)$ denotes the two dimensional tensor-product scaling function generated by the one dimensional Meyer scaling function.

Define $u_j(x,y,z) = \sum\limits_{k_1,k_2 \in \mathbf{Z}} \omega_{k_1,k_2}(z) \Phi_{jk_1k_2}(x,y) = \sum\limits_{k_1,k_2 \in \mathbf{Z}} \omega_{k_1,k_2}(z) \varphi_{jk_1}(x) \varphi_{jk_2}(y)$ to be the Meyer wavelet solution in V_j, and the infinite matrix $\omega = \{\omega_{k_1,k_2}(z)\}_{k_1,k_2 \in \mathbf{Z}}$ satisfying the following equation:

$$\begin{cases} \frac{d^2}{d} y^2 \omega(z) = -D_j \omega - \omega D_j, \\ \omega(0) = \gamma, \\ \omega^{'}(0) = 0, \end{cases} \tag{3.1}$$

where the infinite matrix $D_j = \{(D_j)_{kl}\}_{k \in \mathbf{Z}, l \in \mathbf{Z}} = \{\langle \varphi''_{jl}, \varphi_{jk} \rangle\}_{k \in \mathbf{Z}, l \in \mathbf{Z}}$,j and the infinite vector $\gamma = \{\gamma_{k_1k_2}\}_{k_1,k_2 \in \mathbf{Z}} = \{\langle g(x,y), \Phi_{jk_1k_2}(x,y) \rangle\}_{k_1,k_2 \in \mathbf{Z}}$.

Lemma 3.1. *Let $D_j = \{(D_j)_{kl}\}_{k \in \mathbf{Z}, l \in \mathbf{Z}} = \{\langle \varphi''_{jl}, \varphi_{jk} \rangle\}_{k \in \mathbf{Z}, l \in \mathbf{Z}}$, and $\varphi(x)$ be a one dimensional Meyer scaling function,then*

(1) $\{(D_j)_{kl}\}_{k \in \mathbf{Z}, l \in \mathbf{Z}} = \{(D_j)_{lk}\}_{l \in \mathbf{Z}, k \in \mathbf{Z}}$;

(2) $\| D_j \| \leq 3\pi^2 2^{2j}$,

Proof. It is similar to [4], here we omit it.

Lemma 3.2. *Let $\varphi(x)$ be Meyer scaling function, then*

(1) $\sum\limits_{k \in \mathbf{Z}} |\varphi_{jk}(x)|^2 \leq 2^j M_1$;

(2) $\sum\limits_{k \in \mathbf{Z}} |\varphi_{jk}(x)| \leq 2^{\frac{j}{2}} M_2$,

where M_1, M_2 are constants.

Proof. see [10].

Lemma 3.3[6]**.** *Suppose u and v are positive continuous functions, $x \geq a$, $c > 0$. If $u(x) \leq c + \int_a^x \int_a^s v(\tau)u(\tau)d\tau ds$, then*

$$u(x) \leq ce^{\int_a^x \int_a^s v(\tau)d\tau ds}.$$

From Lemma 3.3, we can easily get the following lemmas:

Lemma 3.4. *Suppose $\omega(z)$ is the solution of the equation (3.1), then*

$$\| \omega(z) \|_{l^2} \leq \| \gamma \| e^{6\pi^2 2^{2j} \frac{z^2}{2}}.$$

Lemma 3.5. *Suppose $u(x,y,z)$ is the solution of the equation (1.1), then*

$$|\hat{u}(\xi,\tau,z)| \leq |\hat{g}(\xi,\tau)| e^{(\xi^2+\tau^2)\frac{z^2}{2}}.$$

Lemma 3.6. *Suppose $u(x,y,z)$ is the exact solution of the equation (1.1), and the condition (1.8) holds, then*

(1) $\displaystyle\int\limits_{A_j \backslash A_{j-1}} |\widehat{Q_j u}| \mathrm{d}\xi \mathrm{d}\tau \leq \frac{3}{4\pi^2} e^{-4\pi^2 2^{2j}} \| f \|_{L^1}$,

(2) $\displaystyle\int\limits_{A_j \backslash A_{j-1}} |\widehat{Q_j u}|^2 \mathrm{d}\xi \mathrm{d}\tau \leq \frac{9}{16\pi^4} e^{-8\pi^2 2^{2j}} \| f \|_{L^2}^2$.

Proof. Since

$$\begin{aligned}
\hat{Q}_j \hat{u}(\xi,\tau,z) &= \sum_{i=1}^{3} \sum_{k_1,k_2 \in \mathbf{Z}} \langle \hat{u}(\xi,\tau,z), \hat{\Psi}^{(i)}_{jk_1k_2}(\xi,\tau) \rangle \hat{\Psi}^{(i)}_{jk_1k_2}(\xi,\tau) \\
&= \sum_{i=1}^{3} \hat{\Psi}^{(i)}_{j00} \sum_{k_1,k_2 \in \mathbf{Z}} \langle \hat{u}(\xi,\tau,z), \hat{\Psi}^{(i)}_{jk_1k_2}(\xi,\tau) \rangle e^{-i2^{-j}(k_1\xi + k_2\tau)} \\
&= \sum_{i=1}^{3} \hat{u} \hat{\Psi}^{(i)}_{j00} \overline{\hat{\Psi}^{(i)}_{j00}} = \sum_{i=1}^{3} \hat{u} |\hat{\Psi}^{(i)}_{j00}|^2
\end{aligned}$$

(1)

$$\begin{aligned}
\int\limits_{A_j \backslash A_{j-1}} |\widehat{Q_j u}| d\xi d\tau &\leq \sum_{i=1}^{3} \int\limits_{A_j \backslash A_{j-1}} |\hat{u}| |\hat{\Psi}^{(i)}_{j00}|^2 \mathrm{d}\xi \mathrm{d}\tau \\
&\leq \frac{1}{4\pi^2} \sum_{i=1}^{3} \int\limits_{A_j \backslash A_{j-1}} |\hat{u}| \mathrm{d}\xi \mathrm{d}\tau \\
&\leq \frac{1}{4\pi^2} \sum_{i=1}^{3} \int\limits_{A_j \backslash A_{j-1}} |\hat{g}(\xi,\tau)| e^{\frac{\xi^2+\tau^2}{2}} \mathrm{d}\xi \mathrm{d}\tau \\
&= \frac{3}{4\pi^2} \int\limits_{A_j \backslash A_{j-1}} |\hat{f}(\xi,\tau)| e^{-\frac{9}{2}(\xi^2+\tau^2)} \mathrm{d}\xi \mathrm{d}\tau \\
&\leq \frac{3}{4\pi^2} e^{-4\pi^2 2^{2j}} \| f \|_{L^1}
\end{aligned}$$

(2)

$$\begin{aligned}
\int\limits_{A_j \backslash A_{j-1}} |\widehat{Q_j u}|^2 \mathrm{d}\xi \mathrm{d}\tau &\leq 3\sum_{i=1}^{3} \int\limits_{A_j \backslash A_{j-1}} |\hat{u}|^2 |\hat{\Psi}_{j00}^{(i)}|^4 \mathrm{d}\xi \mathrm{d}\tau \\
&\leq \frac{3}{16\pi^4} \sum_{i=1}^{3} \int\limits_{A_j \backslash A_{j-1}} |\hat{u}|^2 \mathrm{d}\xi \mathrm{d}\tau \\
&\leq \frac{3}{16\pi^4} \sum_{i=1}^{3} \int\limits_{A_j \backslash A_{j-1}} |\hat{g}(\xi,\tau)|^2 e^{\xi^2+\tau^2} \mathrm{d}\xi \mathrm{d}\tau \\
&= \frac{9}{16\pi^4} \int\limits_{A_j \backslash A_{j-1}} |\hat{f}(\xi,\tau)|^2 e^{-9(\xi^2+\tau^2)} \mathrm{d}\xi \mathrm{d}\tau \\
&\leq \frac{9}{16\pi^4} e^{-8\pi^2 2^{2j}} \parallel f \parallel_{L^2}^2 .
\end{aligned}$$

Since

$$P_j u(x,y,z) = \sum_{k_1,k_2} b_{k_1k_2}(z) \Phi_{jk_1k_2}(x,y)$$

satisfies the following equation:

$$\begin{cases}
(P_j u)_{zz} = -P_j(P_j u)_{xx} - P_j(P_j u)_{yy} - P_j[(I-P_j)u]_{xx} - P_j[(I-P_j)u]_{xx}, & x \in \mathbf{R}, y \in \mathbf{R}, 0 < z \leq 1, \\
P_j u(x,y,0) = P_j g(x,y), & x \in \mathbf{R}, y \in \mathbf{R}, \\
P_j u_z(x,y,0) = 0, & x \in \mathbf{R}, y \in \mathbf{R},
\end{cases}$$

then we can get the coefficient matrix

$$b(z) = \{b_{k_1k_2}(z)\}_{k_1,k_2 \in \mathbf{Z}} = \{\langle u(x,y,z), \Phi_{jk_1k_2}(x,y)\rangle\}_{k_1,k_2 \in \mathbf{Z}}$$

satisfies the following equation:

$$\begin{cases}
b_{zz} = -D_j b(z) - b(z) D_j - T(z), \quad 0 < z \leq 1, \\
b(0) = \gamma, \\
b_z(0) = 0.
\end{cases}$$

where

$$\begin{aligned}
&\gamma = \{\gamma_{k_1k_2}\}_{k_1Łk_2 \in \mathbf{Z}} = \{\langle g(x,y), \Phi_{jk_1k_2}(x,y)\rangle\}_{k_1,k_2 \in \mathbf{Z}}, \\
&T(z) = \{T_{k_1k_2}(z)\}_{k_1,k_2 \in \mathbf{Z}} = \{\langle [(I-P_j)u]_{xx} + (I-P_j)u]_{yy}, \Phi_{jk_1k_2}(x,y)\rangle\}_{k_1,k_2 \in \mathbf{Z}}.
\end{aligned}$$

If let $w_j(x,y,z) = P_ju(x,y,z) - u_j(x,y,z)$, and

$$h(z) = \{h_{k_1k_2}(z)\}_{k_1,k_2\in\mathbf{Z}} = \{\langle w_j(x,y,z), \Phi_{jk_1k_2}(x,y)\rangle\}_{k_1,k_2\in\mathbf{Z}},$$

then $h(z) = b(z) - \omega(z)$ satisfies the following equation:

$$\begin{cases} h_{zz} = -D_jh(z) - h(z)D_j - T(z), \quad 0 < z \leq 1, \\ h(0) = 0, \\ h_z(0) = 0. \end{cases}$$

i.e.,

$$h(z) = \int_0^z\int_0^s (-D_j)h(\tau)\mathrm{d}\tau\mathrm{d}s + \int_0^z\int_0^s h(\tau)(-D_j)\mathrm{d}\tau\mathrm{d}s + \int_0^z\int_0^s [-T(\tau)]\mathrm{d}\tau\mathrm{d}s,$$

therefore, we have

$$\| h(z) \| \leq 2\int_0^z\int_0^s \| -D_j \| \| h(\tau) \| \mathrm{d}\tau\mathrm{d}s + \int_0^z\int_0^s \| T(\tau) \| \mathrm{d}\tau\mathrm{d}s.$$

Lemma 3.7. *Suppose $u(x,y,z)$ is the solution of the equation* (1.1), *and $\hat{g}(\xi,\tau)$ satisfies* (1.4), *then*

$$\| h(z) \| \leq \frac{1}{\pi}2^je^{-\pi^22^{2j}} \| f \|_{L^2}.$$

Proof.

$$\begin{aligned} \| T(\tau) \|^2 &= \| P_j(I-P_j)u_{xx} + P_j(I-P_j)u_{yy} \|^2 \\ &= 2(\| \hat{P}_j(I-\hat{P}_j)\hat{u}_{xx} \|^2 + \| \hat{P}_j(I-\hat{P}_j)\hat{u}_{yy} \|^2) \\ &= 2(\| \hat{P}_j(\xi^2(\hat{u}-\widehat{P_ju})) \|^2 + \| \hat{P}_j(\tau^2(\hat{u}-\widehat{P_ju})) \|^2) \\ &= 4 \| \hat{P}_j(\xi^2(\hat{u}-\widehat{P_ju})) \|^2_{A_{j-1}} + 4 \| \hat{P}_j(\xi^2(\hat{u}-\widehat{P_ju})) \|_{A_j\backslash A_{j-1}} + 4 \| \hat{P}_j(\xi^2(\hat{u}-\widehat{P_ju})) \|_{\mathbf{R}^2\backslash A_j} \\ &= 0 + 4 \| \hat{P}_j(\xi^2(\hat{u}-\widehat{P_ju})) \|_{A_j\backslash A_{j-1}} + 0 \\ &\leq 4 \| \xi^2(\hat{u}-\widehat{P_ju}) \|_{A_j\backslash A_{j-1}} \\ &\leq \frac{64}{9}\pi^22^{2j} \| \widehat{P_{j+1}u} - \widehat{P_ju} \|^2_{A_j\backslash A_{j-1}} \end{aligned}$$

Due to Lemma 3.6, we get

$$\| T(\tau) \|^2 \leq \frac{4}{\pi^2}2^{2j}e^{-8\pi^22^{2j}} \| f \|^2_{L^2},$$

therefore,

$$\begin{aligned}
\| h(z) \| &\leq 2\int_0^z\int_0^s \| -D_j \| \| h(\tau) \| \,\mathrm{d}\tau \mathrm{d}s + \int_0^z\int_0^s \| T(\tau) \| \,\mathrm{d}\tau \mathrm{d}s \\
&\leq 2\int_0^z\int_0^s \| -D_j \| \| h(\tau) \| \,\mathrm{d}\tau \mathrm{d}s + \frac{1}{\pi}2^j e^{-4\pi^2 2^{2j}} \| f \|_{L^2} \\
&\leq \frac{1}{\pi}2^j e^{-\pi^2 2^{2j}} \| f \|_{L^2} .
\end{aligned}$$

4 Convergence Results

Theorem 4.1. *Suppose $u_j(x,y,z)$ and $v_j(x,y,z)$ are the Meyer wavelet solution of* (1.1) *satisfying the boundary condition $g(x,y)$ and $g_\delta(x,y)$ respectively. If (1.4) holds and j is chosen satisfying*

$$j \leq \frac{1}{2}\log_2(\frac{1}{6\pi^2}\ln(\frac{1}{\delta})), \tag{4.1}$$

then

$$|u_j(x,y,z) - v_j(x,y,z)| \leq C_1(\ln\frac{1}{\delta})^{\frac{1}{2}}\delta^{\frac{1}{2}},$$

where C_1 is a constant.

Proof. We have

$$\begin{aligned}
|u_j(x,y,z) - v_j(x,y,z)| &= |\sum_{k_1,k_2\in\mathbf{Z}} \omega_{k_1,k_2}(z)\varphi_{jk_1}(x)\varphi_{jk_2}(y) - \sum_{k_1,k_2\in\mathbf{Z}} \tilde{\omega}_{k_1,k_2}(y)\varphi_{jk_1}(x)\varphi_{jk_2}(y)| \\
&\leq (\sum_{k_1,k_2\in\mathbf{Z}} |\omega_{k_1,k_2}(z) - \tilde{\omega_{k_1,k_2}}(z)|^2)^{\frac{1}{2}} (\sum_{k_1,k_2\in\mathbf{Z}} |\varphi_{jk_1}(x)\varphi_{jk_2}(y)|^2)^{\frac{1}{2}} \\
&\leq 2^j M \| \omega(z) - \tilde{\omega}(z) \| \\
&\leq 2^j M \| \gamma - \tilde{\gamma} \| e^{6\pi^2 2^{2j}\frac{z^2}{2}} \\
&\leq 2^j M e^{6\pi^2 2^{2j}\frac{1}{2}}\delta.
\end{aligned}$$

If $j \leq \frac{1}{2}\log_2(\frac{1}{6\pi^2}\ln(\frac{1}{\delta}))$, we have

$$|u_j(x,y,z) - v_j(x,y,z)| \leq C_1(\ln\frac{1}{\delta})^{\frac{1}{2}}\delta^{\frac{1}{2}},$$

where C_1 is a constant.

Theorem 4.2. *Suppose $u(x,y,z)$ is the solution of* (1.1)*, and* (1.4)(1.8) *hold. If we choose j such that satisfies*

$$j \geq \frac{1}{2}\log_2(\frac{1}{24\pi^2}\ln(\frac{1}{\delta})), \tag{4.2}$$

then

$$|u(x,y,z)-P_ju(x,y,z)|\le C_2\delta^{\frac{1}{6}},$$

where C_2 is a constant.

Proof.

$$\begin{aligned}|u(x,y,z)-P_ju(x,y,z)| &\le \frac{1}{2\pi}\int\limits_{R}|\hat{u}(\xi,\tau,z)-\widehat{P_ju}(\xi,\tau,z)|\mathrm{d}\xi\mathrm{d}\tau\\ &= \frac{1}{2\pi}(\int\limits_{A_{j-1}}|\hat{u}-\widehat{P_ju}|\mathrm{d}\xi\mathrm{d}\tau+\int\limits_{A_j\backslash A_{j-1}}|\hat{u}-\widehat{P_ju}|\mathrm{d}\xi\mathrm{d}\tau\\ &\quad+\int\limits_{\mathbf{R}^2\backslash A_j}|\hat{u}-\widehat{P_ju}|\mathrm{d}\xi\mathrm{d}\tau)\\ &= \frac{1}{2\pi}(0+\int\limits_{A_j\backslash A_{j-1}}|\widehat{Q_ju}|\mathrm{d}\xi\mathrm{d}\tau+\int\limits_{\mathbf{R}^2\backslash A_j}|\hat{u}|\mathrm{d}\xi\mathrm{d}\tau)\end{aligned}$$

Due to Lemma 3.6 and (1.8), we get

$$\begin{aligned}|u(x,y,z)-P_ju(x,y,z)| &\le \frac{3}{8\pi^3}e^{-4\pi^2 2^{2j}}\parallel f\parallel_{L^1}+\frac{1}{2\pi}\int\limits_{\mathbf{R}^2\backslash A_j}|\hat{g}(\xi,\tau)|e^{\frac{1}{2}(\xi^2+\tau^2)}\mathrm{d}\xi\mathrm{d}\tau)\\ &\le \frac{3}{8\pi^3}e^{-4\pi^2 2^{2j}}\parallel f\parallel_{L^1}+\frac{1}{2\pi}\int\limits_{\mathbf{R}^2\backslash A_j}|\hat{f}(\xi,\tau)|e^{-\frac{9}{2}(\xi^2+\tau^2)}\mathrm{d}\xi\mathrm{d}\tau)\\ &\le \frac{3}{8\pi^3}e^{-4\pi^2 2^{2j}}\parallel f\parallel_{L^1}+\frac{1}{2\pi}e^{-16\pi^2 2^{2j}}\parallel f\parallel_{L^1}\\ &\le e^{-4\pi^2 2^{2j}}\parallel f\parallel_{L^1}.\end{aligned}$$

From $j\ge\frac{1}{2}\log_2(\frac{1}{24\pi^2}\ln(\frac{1}{\delta}))$, we have

$$|u(\cdot,y)-P_ju(\cdot,y)|\le C_2\delta^{\frac{1}{6}},$$

where C_2 is a constant.

Theorem 4.3. *Suppose $u(x,y,z)$ is the exact solution of the equation* (1.1), *$u_j(x,y,z)$ is the Meyer wavelet solution of* (1.1), *$\hat{g}(\xi,\tau)$ satisfies* (1.4), *and* (1.8) *holds. If we choose j such that* (4.1) *and* (4.2) *holds, then we have*

$$|u_j(x,y,z)-P_ju(x,y,z)|\le C_3(\ln\frac{1}{\delta})\delta^{\frac{1}{24}},$$

where C_3 is a constant.

Proof.

$$
\begin{aligned}
|u_j(x,y,z)-P_ju(x,y,z)| &= |\sum_{l_1,l_2\in\mathbf{Z}} \omega_{l_1,l_2}(z)\Phi_{jl_1l_2}(x,y) \\
&\quad -\sum_{l_1,l_2\in\mathbf{Z}} \langle u(x,y,z),\Phi_{jl_1l_2}(x,y)\rangle\Phi_{jl_1l_2}(x,y)| \\
&\leq (\sum_{l_1,l_2\in\mathbf{Z}} |\omega_{l_1,l_2}(z)-\langle u(x,y,z),\Phi_{jl_1l_2}(x,y)\rangle|^2)^{\frac{1}{2}}(\sum_{l_1,l_2\in\mathbf{Z}} |\Phi_{jl_1l_2}(x,y)|^2)^{\frac{1}{2}} \\
&\leq \| h(z) \| 2^j M.
\end{aligned}
$$

From Lemma 3.7 and

$$\frac{1}{2}\log_2(\frac{1}{24\pi^2}\ln(\frac{1}{\delta})) \leq j \leq \frac{1}{2}\log_2(\frac{1}{6\pi^2}\ln(\frac{1}{\delta}))$$

we have

$$\| h(z) \| \leq M_3(\ln\frac{1}{\delta})^{\frac{1}{2}}\delta^{\frac{1}{24}},$$

therefore,

$$|u_j(x,y,z)-P_ju(x,y,z)| \leq C_3(\ln\frac{1}{\delta})\delta^{\frac{1}{24}},$$

where M_3, C_3 are constants.

From Theorems 4.1, 4.2 and 4.3, we have the main theorem.

Theorem 4.4. *Suppose* $u(x,y,z)$ *is the solution of* (1.1), $v_j(x,y,z)$ *is the Meyer wavelet approximate solution of* (1.1) *satisfying the boundary condition* $g_\delta(x,y)$. *If* (1.8) *holds ,* $\hat{g}(\xi,\tau)$ *satisfies* (1.4), j *satisfies* (4.1) *and* (4.2), *then*

$$|u(x,y,z)-v_j(x,y,z)| \leq C(\ln\frac{1}{\delta})\delta^{\frac{1}{24}},$$

where C is a constant.

Remark: There integer j satisfying (4.1) and (4.2) must exist. In fact, let $M=\frac{1}{\pi^2}\ln(\frac{1}{\delta})$, then the inequality

$$\frac{1}{2}\log_2(\frac{1}{24\pi^2}\ln(\frac{1}{\delta})) \leq j \leq \frac{1}{2}\log_2(\frac{1}{6\pi^2}\ln(\frac{1}{\delta})),$$

means

$$\frac{1}{2}\log_2(\frac{M}{24}) \leq j \leq \frac{1}{2}\log_2(\frac{M}{6}).$$

Since

$$\frac{1}{2}\log_2(\frac{M}{6})-\frac{1}{2}\log_2(\frac{M}{24})=1,$$

therefore, j can be chosen so that (4.1) and (4.2) hold.

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