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MEASURES OF WEAK NONCOMPACTNESS AND FIXED POINT THEORY FOR 1-SET WEAKLY CONTRACTIVE OPERATORS ON UNBOUNDED DOMAINS

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Abstract. The main purpose of this paper is to prove a collection of new fixed point theorems and existence theorems for the nonlinear operator equation $F(x) = \alpha x \ (\alpha \geq 1)$ for so-called 1-set weakly contractive operators on unbounded domains in Banach spaces. We also introduce the concept of weakly semi-closed operator at the origin and obtain a series of new fixed point theorems and the existence theorems for the nonlinear operator equation $F(x) = \alpha x \ (\alpha \geq 1)$ for such class of operators. As consequences, the main results generalize and improve the relevant results, which are obtained by O'Regan and A. Ben Amar and M. Mnif in 1998 and 2009 respectively. In addition, we get the famous fixed point theorems of Leray-Schauder, Altman, Petryshyn and Rothe type in the case of weakly sequentially continuous, 1-set weakly contractive (μ-nonexpansive) and weakly semi-closed operators at the origin and their generalizations. The main condition in our results is formulated in terms of axiomatic measures of weak compactness.

Key words: *measure of weak noncompactness, weakly condensing and weakly nonexpansive, weakly sequentially continuous, weakly semi-closed at the origin, fixed point theorem*

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1 Introduction

The famous fixed point theorems of Leray-Schauder, Altman, Petryshyn and Rothe type for completely continuous operators take an important role in the study of fixed point theory. Since 1-set contractive mappings are a broader class of operators including completely continuous operators, the study of fixed point problems for 1-set contractive operators especially from a closed convex subset into itself has been one of the main objects of research in nonlinear functional analysis and was started by Sadovskii[23] Petryshyn[21,22], and Nussbaum[18]. Since then, whether a 1-set contractive mapping defined on the closure of bounded open subset of a Banach space has a fixed point, has become an interesting problem[16,17,26,28]. For example, in [16] the author defined the fixed point index of 1-set-contractive operators, introduced the concept of semi-closed 1-set-contractive operator and obtained some fixed point theorems of such a class of operators. These studies are mainly based on the potential tool of degree theory in terms of Kuratowskii measure of noncompactness. Because the weak topology is the convenient and natural setting to investigate the existence problems of fixed points and eigenvectors for operators and solutions of various kinds of nonlinear differential equations and nonlinear integral equations in Banach spaces, the above mentioned results cannot be easily applied. These equations can be transformed into fixed point problems and nonlinear operator equations involving a broader class of nonlinear operators, in which the operators have the property that the image of any set in a certain sense more weakly compact than the original set itself. The major problem to face is that an infinite dimensional Banach space equipped with its weak topology does not admit open bounded sets. That is, a weakly closed and bounded subset has an empty weak interior and thus coincides with its weak boundary which yields very difficult the verification of the boundary conditions. To this interest, we introduce the concept of weakly semi-closed operator at the origin (see Definition 2.6). The notion of the measure of weak noncompactness was introduced by De Blasi in 1977, see [9]). This index has found applications in operator theory (see [14, 15]) and many existence results for weak solutions of differential and integral equations in Banach spaces (see [8, 20, 25] and other). Recall that weak solutions of the Cauchy problem in reflexive Banach spaces were investigated by Szép[25] and weak solutions of nonlinear integral equations in these spaces by O'Regan[20]. But, it is not easy to construct some formulas which allow to express the measure of weak noncompactness in a convenient form. This is the reason for introducing the notion of axiomatic measures of weak noncompactness, see [4]. In this paper, we prove a collection of new fixed point theorems and existence theorems for the nonlinear operator equation $F(x)=\alpha x\ (\alpha\geq 1)$ for so-called 1-set weakly contractive operators on unbounded domains in Banach spaces. We also introduce the concept of weakly semi-closed operator at the origin (see Definition 2.6) and obtain a series of new fixed point theorems and the existence theorems for the nonlinear operator equation $F(x) = \alpha x$ $(\alpha \geq 1)$ for such class of operators in the case of 1-set weakly contractive operators. As consequences, the main results generalize and improve the relevant results, which were obtained by O'Regan and A. Ben Amar and M. Mnif in 1998 and 2009 respectively. In addition, we get the famous fixed point theorems of Leray-Schauder, Altman, Petryshyn and Rothe type in the case of weakly sequentially continuous, 1-set weakly contractive (μ-nonexpansive) and weakly semi-closed operators at the origin and their generalizations. The main condition in our results is formulated in terms of axiomatic measures of weak compactness. In addition, our arguments and methods are elementary in the sense that without any recourse to degree theory or theory of homotopy-extensions.

2 Preliminaries

For a subset Ω of a Banach space E, the weak closure, the convex hull, and the closed convex hull of Ω in E are denoted by, $\overline{\Omega^w}$, convΩ and $\overline{\text{conv}}\Omega$ respectively. If $U \subseteq \Omega \subseteq E$, the weak boundary in the relative topology of Ω is denoted by $\partial_\Omega U$. A set Ω in E is called a wedge if $ax + by \in \Omega$ whenever $a, b \in [0, \infty)$ and $x, y \in \Omega$.

Let E be a Banach space, $\mathcal{B}(E)$ the collection of all nonempty bounded subsets of E and $\mathcal{W}(E)$ the subset of $\mathcal{B}(E)$ consisting of all relatively weakly compact subsets of E. Finally, let B_E denote the closed unit ball of E.

The measure of weak noncompactness of De Blasi [9] is defined in the following way

$$\beta(\Omega) = \inf\{\varepsilon > 0 : \text{ there exists a weakly compact set } D \text{ such that } \Omega \subseteq D + \varepsilon B_E\},$$

where $\Omega \in \mathcal{B}(E)$. This function possesses several useful properties [9]. Foe example, $\beta(B_E) = 1$ whenever E is nonreflexive and $\beta(B_E) = 0$ otherwise.

There exists also an axiomatic approach in defining of measures of noncompactness [4]. Let us recollect this definition.

Definition 2.1. A function $\mu : \mathcal{B}(E) \longrightarrow \mathbf{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions

(1) $\mu(\Omega) = 0 \Longleftrightarrow \Omega \in \mathcal{W}(E)$.

(2) $\mu(\overline{\text{conv}}(\Omega)) = \mu(\Omega)$,

(3) $\Omega_1 \subseteq \Omega_2 \Longrightarrow \mu(\Omega_1) \leq \mu(\Omega_2)$,

(4) $\mu(\Omega_1 \cup \Omega_2) = \max\{\mu(\Omega_1), \mu(\Omega_2)\}$,

(5) $\mu(\Omega_1 + \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$,

(6) $\mu(\lambda\Omega) = |\lambda|\mu(\Omega), \lambda \in \mathbf{R}$.

Let us mention that in the paper [4] a measure of weak non compactness in the above sense is called to be regular. For more examples and properties of measures of weak noncompactness we refer to [1, 4, 5, 14, 15].

Throughout this article we consider μ a measure of weak noncompactness on E.

Definition 2.2. Let D be a nonempty subset of Banach space E. If F maps D into E, we say that

(a) F is μ-condensing if F is bounded and $\mu(F(V)) < \mu(V)$ for all bounded subsets V of D with $\mu(V) > 0$,

(b) F is μ-nonexpansive if F is bounded and $\mu(F(V)) \leq \mu(V)$ for all bounded subsets V of D.

Definition 2.3. Let E be a Banach space. An operator $F : E \longrightarrow E$ is said to be weakly compact if $F(D)$ is relatively weakly compact for every bounded subset $D \subset E$.

Definition 2.4. Let E be a Banach space. An operator $F : E \longrightarrow E$ is said to be weakly sequentially continuous on E if for every sequence $(x_n)_n$ with $x_n \xrightarrow{w} x$, we have $F(x_n) \xrightarrow{w} F(x)$, here $\xrightarrow{w}$ denotes weak convergence.

Remark 2.1. In many situations while it is easy to show that a map between Banach spaces is weakly sequentially continuous, it is considerably more difficult to show that is weakly continuous. One of the reasons for this is the failure in general of the Lebesgue dominated convergence theorem for nets. So it is useful to have fixed point theorems for weakly sequentially continuous maps.

Remark 2.2. One of the advantages of the weak topology of a Banach space E is the fact that if a set D is weakly compact, then every sequentially weakly continuous map $F : D \longrightarrow E$ is weakly continuous. This is an immediately consequence of Eberlein-Šmulian's theorem.

Definition 2.5. A subset D of a Banach space is called weakly sequentially closed if, whenever $x_n \in D$ for all $n \in \mathbf{N}$ and $x_n \xrightarrow{w} x$, then $x \in D$.

Definition 2.6. Let D be a nonempty weakly closed set of a Banach space E and $F : D \longrightarrow E$ a weakly sequentially continuous operator. F is said to be weakly semi-closed operator at θ if the conditions $x_n \in D$, $x_n - F(x_n) \longrightarrow \theta$ imply that there exists $x \in D$ such that $F(x) = x$ (here θ means the zero vector of the space E).

It should be noted that this class of operators, as special cases, includes the weakly sequentially continuous operators which are weakly compact, weakly contractive, μ-condensing, $(I-F)(D)$ is weakly sequentially closed and others.

The following fixed point result stated in [7] as an analogue of Sadovskii's fixed point result[2], will be used throughout this section. The proof follows from O. Arino, S. Gautier and J. P. Penot theorem [3].

Theorem 2.1. *Let Ω be a non-empty, convex closed set in a Banach space E. Assume $F : \Omega \longrightarrow \Omega$ is a weakly sequentially continuous and μ-condensing mapping. In addition, suppose that $F(\Omega)$ is bounded. Then F has a fixed point.*

Remark 2.3. Theorem 2.1 extends and improves many relevant and recent results in [6, 19, 11] and others.

3 Main Results

We start this section by stating some interesting facts of a weakly sequentially and μ-condensing operators which are useful in the sequel.

Lemma 3.1. *Let C be a nonempty weakly closed set of a Banach space E and $F : C \longrightarrow E$ a weakly sequentially continuous and μ-condensing operator with $F(C)$ is bounded, then*

(a) *for all weakly compact subset K of E, $(I-F)^{-1}(K)$ is weakly compact.*

(b) *$I-F$ maps weakly closed subset of C onto weakly sequentially closed sets in E.*

Proof. (a) Let $K \subset E$ be a nonempty weakly compact set and let $D = (I-F)^{-1}(K)$. Since $I-F$ is weakly sequentially continuous, D is weakly sequentially closed. Moreover, we have

$$\mu(D) \leq \mu(K) + \mu(F(D)) = \mu(F(D)).$$

Since F is μ-condensing, it follows that $\mu(D) = 0$. Let $x \in C$, be weakly adherent to D. Since $\overline{D^w}$ is weakly compact, by Eberlein-Šmulian theorem [10, theorem 8.12.4, p. 549], there exists a sequence $(x_n)_n \subset D$ such that $x_n \xrightarrow{w} x$, so $x \in D$. Hence $\overline{D^w} = D$ and D is a weakly closed subset of C. Therefore D is weakly compact.

(b) Let $D \subset C$ be a weakly closed set and consider $x_n \in (I-F)(D)$ such that $x_n \xrightarrow{w} x$ in E. We have $x_n = (I-F)(u_n), \forall n \geq 1$ with $u_n \in D$. The set $K = \overline{\{x_n\}^w}$ is weakly compact and so $(I-F)^{-1}(K)$ is weakly compact. Therefore, we may assume that $u_n \xrightarrow{w} u$ in D, for some $u \in D$. Due to the weak sequential continuity of $I-F$, we have $x = (I-F)(u)$ and so $x \in (I-F)(D)$. Accordingly $(I-F)(D)$ is weakly sequentially closed.

Now, we are ready to investigate a class of operator equations for a broader class of nonlinear weakly sequentially continuous operators, in which the operators have the property that the image of any set is in a certain sense more weakly compact than the original set itself .

Proposition 3.1. Let Ω be a nonempty unbounded closed convex subset of a Banach space E. Suppose that $F : \Omega \longrightarrow \Omega$ is weakly sequentially continuous μ-nonexpansive operator and

$F(\Omega)$ is bounded. In addition, assume that F is weakly semi-closed at θ. Then F has a fixed point in Ω.

Proof. Let x_0 be a fixed element of Ω. Define $F_n = t_nF + (1-t_n)x_0$ $n = 1, 2, \ldots$, where $(t_n)_n$ is a sequence of $(0,1)$ such that $t_n \longrightarrow 1$. Since $x_0 \in \Omega$ and Ω is convex, it follows that F_n maps Ω into itself. Clearly F_n is weakly sequentially continuous and $F_n(\Omega)$ is bounded. Let X be an arbitrary bounded subset of Ω. Then we have

$$\mu(F_n(X)) = \mu(t_nF(X) + \{(1-t_n)x_0\}) \leq t_n\mu(F(X)) + \mu(\{(1-t_n)x_0\}) \leq t_n\mu(X).$$

So, if $\mu(X) \neq 0$ we have

$$\mu(F_n(X)) < \mu(X).$$

Therefore F_n is μ-condensing on Ω. From Theorem 2.1, F_n has a fixed point, say, x_n in Ω. Consequently, $\|x_n - F(x_n)\| = \|(t_n-1)(F(x_n) - x_0)\| \longrightarrow 0$ as $n \longrightarrow \infty$, since $t_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $F(\Omega)$ is bounded. Since F is weakly semi-closed at θ, there exists $x \in \Omega$ such that $F(x) = x$. Accordingly, F has a fixed point in Ω.

Theorem 3.2. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous, μ-condensing operator and $F(\overline{U^w})$ is bounded. Then, either*

(A_1) *the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$, or*

(A_2) *there is a point $x \in \partial_\Omega U$ and $k > \alpha$ with $Fx - \alpha x_0 = k(x - x_0)$.* ♢

Proof. Suppose (A_2) does not hold. If (A_2) is satisfied for $k = \alpha$, then the theorem is trivial. In conclusion, we can consider that the supposition is not satisfied for any $x \in \partial_\Omega U$ and any $k \geq \alpha$. Let D be the set defined by

$$D = \left\{ x \in \overline{U^w} \ : \ x = \frac{\lambda}{\alpha}F(x) + (1-\lambda)x_0, \text{ for some } \lambda \in [0,1] \right\}.$$

D is non-empty and bounded, because $x_0 \in D$ and $F(\overline{U^w})$ is bounded. We have $D \subset \text{conv}(\{x_0\} \cup (\frac{1}{\alpha}F(D)))$. Because the set $\{x_0\}$ is weakly compact and $\alpha \geq 1$, then $\mu(D) \neq 0$ implies

$$\mu(D) \leq \mu(\overline{\text{conv}}(\{x_0\} \cup (\frac{1}{\alpha}F(D))) \leq \frac{1}{\alpha}(F(D)) < \mu(D),$$

which is a contradiction. Hence, $\mu(D) = 0$ and D is relatively weakly compact. Now, we prove that D is weakly closed. The weak sequentially continuity of F implies that D is weakly sequentially closed. For that, let $(x_n)_n$ a sequence of D such that $x_n \stackrel{w}{\longrightarrow} x$, $x \in \overline{U^w}$. For all

$n \in \mathbf{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = \dfrac{\lambda_n}{\alpha}F(x_n) + (1-\lambda_n)x_0$. $\lambda_n \in [0,1]$, we can extract a subsequence $(\lambda_{n_j})_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0,1]$. So, $\dfrac{\lambda_{n_j}}{\alpha}F(x_{n_j}) + (1-\lambda_{n_j})x_0 \stackrel{w}{\longrightarrow} \dfrac{\lambda}{\alpha}F(x) + (1-\lambda)x_0$. Hence $x = \dfrac{\lambda}{\alpha}F(x) + (1-\lambda)x_0$ and $x \in D$. Let $x \in \overline{U^w}$, be weakly adherent to D. Since $\overline{D^w}$ is weakly compact, by Eberlein-Šmulian theorem [10, theorem 8.12.4, p. 549], there exists a sequence $(x_n)_n \subset D$ such that $x_n \stackrel{w}{\longrightarrow} x$, so $x \in D$. Hence $\overline{D^w} = D$ and D is a weakly closed subset of $\overline{U^w}$. Therefore D is weakly compact. We prove that $D \cap (\Omega \setminus U) = \emptyset$. In fact, let $x \in D$, then there exists $\lambda \in (0,1]$ such that $x = \dfrac{\lambda}{\alpha}F(x) + (1-\lambda)x_0$ (if $\lambda = 0$ then $x = x_0 \notin \Omega \setminus U$). So, $F(x) - \alpha x_0 = \dfrac{\alpha}{\lambda}(x - x_0)$ and thus $x \notin \Omega \setminus U$ ($\dfrac{\alpha}{\lambda} \geq \alpha$). Because E endowed with its weak topology is a Hausdorff locally convex space, we have that E is completely regular [24, p. 16]. Since $D \cap (\Omega \setminus U) = \emptyset$, Then by [12, p. 146], there is a weakly continuous function $\varphi : \Omega \longrightarrow [0,1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U$. Since Ω is a wedge, $x_0 \in \Omega$, we can define the operator $F^* : \Omega \longrightarrow \Omega$ by :

$$F^*(x) = \begin{cases} \dfrac{\varphi(x)}{\alpha}F(x) + (1-\varphi(x))x_0, & \text{if } x \in \overline{U^w}, \\ \theta, & \text{if } x \in \Omega \setminus \overline{U^w} \end{cases}$$

Clearly, $F^*(\Omega)$ is bounded. Because $\partial_\Omega U = \partial_\Omega \overline{U^w}$, φ is weakly continuous and F is weakly sequentially continuous, we have that F^* is weakly sequentially continuous. Let $X \subset \Omega$, be bounded. Then, since

$$F^*(X) \subset \text{conv}(\{x_0\} \cup F(X \cap U)),$$

we have

$$\mu(F^*(X)) \leq \mu(\overline{\text{conv}}(\{x_0\} \cup (\frac{1}{\alpha}F(X \cap U))) \leq \mu(F(X)), (\alpha \geq 1)$$

and $\mu(F^*(X)) < \mu(X)$ if $\mu(X) \neq 0$. So, F^* is μ-condensing. Therefore by Theorem 2.1 F^* has a fixed point $x_1 \in \Omega$. If $x_1 \notin U$, $\varphi(x_1) = 0$ and $x_1 = x_0$, which contradicts the hypothesis $x_0 \in U$. Then $x_1 \in U$ and $x_1 = \dfrac{\varphi(x_1)}{\alpha}F(x_1) + (1-\varphi(x_1))x_0$, which implies that $x_1 \in D$. Accordingly, $\varphi(x_1) = 1$ and so $F(x_1) = \alpha x_1$ and the proof is complete.

Remark 3.1. If either $\alpha = 1$ or $x_0 = \theta$, then we obtain the same conclusion by only assuming that Ω is a nonempty unbounded closed convex subset of E.

Remark 3.2. (a) Theorem 3.1 extends and improves Theorem 3.3 in [7].

(b) Theorem 3.1 extends and improves Theorem 2.3 in [19] and shows that the condition $\overline{U^w}$ is weakly compact in the statement of this theorem is redundant.

Corollary 3.3. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous, μ-condensing operator and $F(\overline{U^w})$ is bounded. Assume that*

$$Fx - \alpha x_0 \neq k(x - x_0), \qquad \text{for all } x \in \partial_\Omega U, k > \alpha,$$

then the equation $F(x) = \alpha x$ has at least a solution in $\overline{U^w}$.

Corollary 3.4. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous, weakly compact operator and $F(\overline{U^w})$ is bounded. Assume that*

$$F(x) - \alpha x_0 \neq k(x - x_0), \qquad \text{for all } x \in \partial_\Omega U, k > \alpha,$$

then the equation $F(x) = \alpha x$ has at least a solution in $\overline{U^w}$.

Proof. This is an immediate consequence of Corollary 3.1 since F is clearly μ-condensing.

Remark 3.3. The conditions that F is a weakly compact operator and $F(\overline{U^w})$ is bounded in the statement of Corollary 3.2 can be removed if we assume that $\overline{U^w}$ is weakly compact which improve and extend Theorem 3.1 in [7].

Corollary 3.5. *Let Ω be a nonempty unbounded closed convex set of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous, μ-condensing operator and $F(\overline{U^w})$ is bounded. Assume that*

$$F(x) - x_0 \neq k(x - x_0), \qquad \text{for all } x \in \partial_\Omega U, k > 1,$$

then F has a fixed point in $\overline{U^w}$.

Theorem 3.6. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous μ-nonexpansive operator and $F(\overline{U^w})$ is bounded. In addition, assume that F is weakly semi-closed at θ. Then, either*

(A_1) *the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$, or*

(A_2) *there is a point $x \in \partial_\Omega U$ and $k > \alpha$ with $F(x) - \alpha x_0 = k(x - x_0)$.*

Proof. Suppose (A_2) does not hold. Let $F_n = \dfrac{t_n}{\alpha} F + (1 - t_n)x_0$ $n = 1, 2, \cdots$, where $(t_n)_n$ is a sequence of $(0,1)$ such that $t_n \longrightarrow 1$. Since $x_0 \in \Omega$ and Ω is a wedge, it follows that F_n maps $\overline{U^w}$

into Ω. Clearly $F_n(\overline{U^w})$ is bounded. Suppose that $\lambda_n F_n(y_n) + (1-\lambda_n)x_0 = y_n$ for some $y_n \in \partial_\Omega U$ and for some $\lambda_n \in (0,1)$. So,

$$\begin{aligned} y_n &= \lambda_n F_n(y_n) + (1-\lambda_n)x_0, \\ &= \frac{\lambda_n t_n}{\alpha} F(y_n) + \lambda_n(1-\lambda_n)x_0 + (1-\lambda_n)x_0, \\ &= \frac{\lambda_n t_n}{\alpha} F(y_n) + (1-\lambda_n t_n x_0). \end{aligned}$$

Hence $F(y_n) - \alpha x_0 = \dfrac{\alpha}{\lambda_n t_n}(y_n - x_0)$, a contradiction with the fact that $\dfrac{\alpha}{\lambda_n t_n} > \alpha$. Let X be an arbitrary bounded subset of $\overline{U^w}$. Then we have

$$\mu(F_n(X)) = \mu(\frac{t_n}{\alpha}F(X) + \{(1-t_n)x_0\}) \leq \frac{t_n}{\alpha}\mu(F(X)) + \mu(\{(1-t_n)x_0\}) \leq t_n \mu(X).$$

So, if $\mu(X) \neq 0$ we have

$$\mu(F_n(X)) < \mu(X).$$

Therefore F_n is μ-condensing on $\overline{U^w}$ (note that $\alpha \geq 1$). From Corollary 3.3, F_n has a fixed point, say, x_n in $\overline{U^w}$. Therefore, $\|x_n - \frac{1}{\alpha}F(x_n)\| = (1-t_n)\|\frac{1}{\alpha}F(x_n) - x_0\| \longrightarrow 0$ as $n \longrightarrow \infty$, since $t_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $F(\overline{U^w})$ is bounded. Since $\frac{1}{\alpha}F$ is either μ-condensing (if $\alpha > 1$) or μ-nonexpansive (if $\alpha = 1$), by Lemma 3.1 and condition that F is weakly semi-closed at θ, we obtain that there exists a point x_1 in $\overline{U^w}$ such that $\theta = (I - \frac{1}{\alpha}F)(x_1)$. Thus $F(x_1) = \alpha x_1$.

Corollary 3.7. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous μ-nonexpansive operator weakly semi-closed at θ and $F(\overline{U^w})$ is bounded. In addition we suppose that F satisfies the following condition*

$$F(x) - \alpha x_0 \neq k(x - x_0), \qquad \textit{for all } x \in \partial_\Omega U, k > \alpha, \tag{3.1}$$

then the equation $F(x) = \alpha x$ has at least one solution in $\overline{U^w}$.

Corollary 3.8. *Let Ω be a nonempty unbounded closed convex subset of a Banach space E and U a weakly open subset of Ω. Suppose $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous μ-nonexpansive operator weakly semi-closed at θ and $F(\overline{U^w})$ is bounded. In addition, assume that there exists $x_0 \in U$ such that*

$$x \neq \lambda F(x) + (1-\lambda)x_0 \qquad \textit{for all } x \in \partial_\Omega U, \lambda \in (0,1)$$

then F has a fixed point in $\overline{U^w}$.

Proof. It suffices to apply Corollary 3.4 with $\alpha = 1$.

Corollary 3.9. *Let Ω be a nonempty unbounded closed convex of a Banach space E, U a weakly open subset of Ω such that $\theta \in U$. Suppose $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous μ-nonexpansive operator weakly semi-closed at θ and $F(\overline{U^w})$ is bounded. In addition, assume that F satisfies the Leray-Schauder boundary condition*

$$\lambda F(x) \neq x \qquad \text{for all } x \in \partial_\Omega U, \lambda \in (0,1) \tag{L-S}$$

then F has a fixed point in $\overline{U^w}$.

Proof. It suffices to apply Corollary 3.5 with $x_0 = \theta$.

Remark 3.4. Corollary 3.6 generalizes the famous Leray -Schauder's theorem to the case of weakly sequentially continuous, μ-nonexpansive and semi-weakly closed operator at θ.

Theorem 3.10. *Let E, Ω, U and F be as the same as in Corollary 3.6. In addition, assume that*

$$\|F(x)\| \leq \|x\|, \qquad \text{for all } x \in \partial_\Omega U, \tag{3.2}$$

then F has a fixed point in $\overline{U^w}$.

Proof. It suffices to prove that (3.2) implies the condition $(L-S)$. Suppose the contrary. Then there exists $x_0 \in \partial_\Omega U, \lambda_0 \in (0,1)$ such that $\lambda_0 F(x_0) = x_0$. So, $\|F(x)\| = \frac{1}{\lambda_0}\|x_0\| > \|x_0\|$, contradicting with (3.2).

Theorem 3.11. *Let Ω be a closed wedge of a Banach space E, U a weakly open subset of Ω and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous μ-nonexpansive operator weakly semi-closed at θ and $F(\overline{U^w})$ is bounded. In addition, assume that F satisfies one of the following conditions*

(a) $\theta \in U$, $F(x) \neq \lambda x$, *for* $x \in \partial_\Omega U, \lambda > \alpha$,

(b) $x_0 \in U, \|F(x) - \alpha x_0\| \leq \alpha \|x - x_0\|$ *for all* $x \in \partial_\Omega U$,

then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

Proof. Suppose that (a) is satisfied. We only need to let $x_0 = \theta$ in Corollary 3.4. If (b) is satisfied, we suppose that the operator equation $F(x) = \alpha x$ has no solution in $\partial_\Omega U$ (otherwise we are finished). In order to apply Corollary 3.4, we prove that (3.1) is satisfied. Suppose that (3.1) is not true, that is, there exist $k_0 > \alpha$ and $x_1 \in \partial_\Omega U$ such that $F(x_1) - \alpha x_0 = k_0(x_1 - x_0)$. From (b), we obtain $k_0\|x_1 - x_0\| \leq \alpha\|x_1 - x_0\|$. Since $x_1 \in \partial_\Omega U$ and U is a weakly open subset of Ω, thus $x_1 - x_0 \neq \theta$. Therefore, $\|x_1 - x_0\| \neq 0$ and we obtain $k_0 \leq \alpha$ and this contradicts $k_0 > \alpha$. So (3.1) holds. Accordingly, by Corollary 3.4 the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

Remark 3.5. In Theorem 3.4, if the operator F satisfies the condition (a), then it suffices to take Ω nonempty unbounded closed convex subset of E.

As a consequence we have the following fixed point result.

Corollary 3.12. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous μ-nonexpansive operator, weakly semi-closed at θ and $F(\overline{U^w})$ is bounded. In addition, assume that F satisfies one of the following condition*

$$\|F(x) - x_0\| \leq \|x - x_0\|,$$

for every $x \in \partial_\Omega, U$. Then the operator F has a fixed point in $\overline{U^w}$.

Proof. In fact, from Theorem 3.4 it suffices to set $\alpha = 1$.

The next lemma holds easily.

Lemma 3.13. *When $y > 1$ and $\beta > 0$, the following inequality holds:*

$$(y-1)^{\beta+1} < y^{\beta+1} - 1.$$

Theorem 3.14. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta > 0$ such that*

$$\|F(x) - x\|^{\beta+1} \geq \|F(x)\|^{\beta+1} - \|x\|^{\beta+1} \tag{3.3}$$

for every $x \in \partial_\Omega U$. Then the operator F has a fixed point in $\overline{U^w}$.

We suppose that the operator F has no fixed point in $\partial_\Omega U$ (otherwise we are finished). In order to apply Corollary 3.6, we prove that

$$x \neq \lambda F(x), \lambda \in (0,1), x \in \partial_\Omega U. \tag{3.4}$$

Suppose that (3.4) is not true, that is, there exist $\lambda_0 \in (0,1)$ and $x_0 \in \overline{U^w})$, such that $\lambda_0 F(x_0) = x_0$. That is $F(x_0) = \frac{1}{\lambda_0} x_0$. Inserting $F(x_0) = \frac{1}{\lambda_0} x_0$ into (3.3), we obtain

$$\|\frac{1}{\lambda_0} x_0 - x_0\|^{\beta+1} \geq \|\frac{1}{\lambda_0} x_0\|^{\beta+1} - \|x_0\|^{\beta+1}.$$

This implies

$$(\frac{1}{\lambda_0} - 1)^{\beta+1} \|x_0\|^{\beta+1} \geq (\frac{1}{\lambda_0^{\beta+1}} - 1)\|x_0\|^{\beta+1}. \tag{3.5}$$

Since $x_0 \in \partial_\Omega U$, we see $x_0 \neq \theta$. Therefore, $\|x_0\|^{\beta+1} \neq 0$ and by (3.5), we obtain

$$(\frac{1}{\lambda_0} - 1)^{\beta+1} \geq \frac{1}{\lambda_0^{\beta+1}} - 1,$$

and this contradicts Lemma 3.2, since $\frac{1}{\lambda_0} \in (1,\infty)$. Hence

$$x \neq \lambda F(x), \lambda \in (0,1), x \in \partial_\Omega U.$$

Accordingly, by Corollary 3.6 F has a fixed point in $\overline{U^w}$.

Remark 3.6. Theorem 3.5 is a generalization of the famous Altman fixed point theorem in the case of weakly sequentially, μ-nonexpansive and weakly semi-closed operator at θ.

Corollary 3.15. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta > 0$ and $\alpha \geq 1$ such that*

$$\|F(x) - \alpha x\|^{\beta+1} \geq \|F(x)\|^{\beta+1} - \|\alpha x\|^{\beta+1} \tag{3.6}$$

for every $x \in \partial_\Omega U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

Proof. Using (3.6), we obtain

$$\frac{1}{\alpha^{\beta+1}}\|F(x) - \alpha x\|^{\beta+1} \geq \frac{1}{\alpha^{\beta+1}}\|F(x)\|^{\beta+1} - \frac{1}{\alpha^{\beta+1}}\|\alpha x\|^{\beta+1} \qquad \text{for } x \in \partial_\Omega U.$$

So,

$$\|\frac{1}{\alpha}F(x) - x\|^{\beta+1} \geq \|\frac{1}{\alpha}F(x)\|^{\beta+1} - \|x\|^{\beta+1}.$$

Consequently, the operator $\frac{1}{\alpha}F$, which is weakly sequentially continuous μ-nonexpansive, weakly semi-closed at θ, satisfies the conditions of Theorem 3.5. It follows from Theorem 3.5 that the conclusion of Corollary 3.8 holds true.

Theorem 3.16. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that*

$$\|F(x)\| \leq \|F(x) - x\| \tag{3.7}$$

for every $x \in \partial_\Omega U$. Then the operator F has a fixed point in $\overline{U^w}$.

Proof. It suffices to prove that (3.7) implies (3.6).

Remark 3.7. Theorem 3.6 is a generalization and an analogous of the famous Petryshyn fixed point theorem in the case of weakly sequentially, μ-nonexpansive and weakly semi-closed operator at θ.

From Theorem 3.6 we can easily obtain the following

Corollary 3.17. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\alpha \geq 1$ such that*

$$\|F(x)\| \leq \|F(x) - \alpha x\| \tag{3.8}$$

for every $x \in \partial_\Omega U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

Similarly, we can obtain the following results by using the above mentioned methods and we omit their proofs.

Theorem 3.18. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta \in]-\infty, 0(\cup(0,1)$ such that*

$$\|F(x)-x\|^{\beta} \leq \|F(x)\|^{\beta} - \|x\|^{\beta} \tag{3.9}$$

for every $x \in \partial_{\Omega} U$. Then the operator F has a fixed point in $\overline{U^w}$.

Corollary 3.19. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta \in (-\infty, 0) \cup (0,1)$ and $\alpha \geq 1$ such that*

$$\|F(x)-\alpha x\|^{\beta} \leq \|F(x)\|^{\beta} - \|\alpha x\|^{\beta} \tag{3.10}$$

for every $x \in \partial_{\Omega} U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

Theorem 3.20. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta > 1$ such that*

$$\|F(x)+x\|^{\beta} \leq \|F(x)\|^{\beta} + \|x\|^{\beta} \tag{3.11}$$

for every $x \in \partial_{\Omega} U$. Then the operator F has a fixed point in $\overline{U^w}$.

Corollary 3.21. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta > 1$ and $\alpha \geq 1$ such that*

$$\|F(x)+\alpha x\|^{\beta} \leq \|F(x)\|^{\beta} + \|\alpha x\|^{\beta} \tag{3.12}$$

for every $x \in \partial_{\Omega} U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

Theorem 3.22. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta \in (-\infty, 0) \cup (0,1)$ such that*

$$\|F(x)+x\|^{\beta} \geq \|F(x)\|^{\beta} + \|x\|^{\beta} \tag{3.13}$$

for every $x \in \partial_{\Omega} U$. Then the operator F has a fixed point in $\overline{U^w}$.

Corollary 3.23. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta \in (-\infty, 0) \cup (0,1)$ and $\alpha \geq 1$ such that*

$$\|F(x)+\alpha x\|^{\beta} \geq \|F(x)\|^{\beta} + \|\alpha x\|^{\beta} \tag{3.14}$$

for every $x \in \partial_{\Omega} U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

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