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ON APPROXIMATION OF SMOOTH FUNCTIONS FROM NULL SPACES OF OPTIMAL LINEAR DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

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Abstract. For a real valued function f defined on a finite interval I we consider the problem of approximating f from null spaces of differential operators of the form $L_n(\psi) = \sum_{k=0}^{n} a_k\psi^{(k)}$, where the constant coefficients $a_k \in \mathbf{R}$ may be adapted to f.

We prove that for each $f \in C^{(n)}(I)$, there is a selection of coefficients $\{a_1, \cdots, a_n\}$ and a corresponding linear combination

$$S_n(f,t) = \sum_{k=1}^{n} b_k e^{\lambda_k t}$$

of functions $\psi_k(t) = e^{\lambda_k t}$ in the nullity of L which satisfies the following Jackson's type inequality:

$$\|f^{(m)} - S_n^{(m)}(f,t)\|_\infty \le \frac{|I|^{1/q} e^{|\lambda_n||I|}}{|a_n| 2^{n-m-1/p} |\lambda_n|^{n-m-1}} \|L_n(f)\|_p,$$

where $|\lambda_n| = \max_k |\lambda_k|$, $0 \le m \le n-1$, $p, q \ge 1$, and $\frac{1}{p} + \frac{1}{q} = 1$.

For the particular operator $M_n(f) = f + 1/(2n)!f^{(2n)}$ the rate of approximation by the eigenvalues of M_n for non-periodic analytic functions on intervals of restricted length is established to be exponential. Applications in algorithms and numerical examples are discussed.

Key words: *approximation of analytic function, differential operator, fundamental set of solutions*

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1 Introduction

The problem of approximating a real valued function f on a finite interval I by linear combinations from subsets of the set $E = \{e^{\lambda t} | \lambda - \text{complex number}\}$ is studied extensively. The most classical approach is the Fourier series expansion when one uses expanding subspaces $S_N = \text{span}\{1, e^{\pm it}, \dots, e^{\pm iNt}\}$ and the linear combination is the solution of the extremal problem

$$ET_N(f) = \min_{c_k \in \mathbf{R}} \|f(t) - \sum_{k=-N}^{N} c_k e^{ikt}\|_2(I),$$

where

$$\|f\|_p(I) = \left(\int_I |f(t)|^p \mathrm{d}t\right)^{1/p}, \qquad 0 < p < \infty$$

and $\|f\|_\infty = \inf_{t \in I} |f(t)|$ are the norms of f in the spaces $L_p(I), 0 < p \leq \infty$. The exponents ik lie on the imaginary axis and are predetermined for any function f. The "goodness" of approximation is measured by the rate at which $ET_N(f)$ approaches 0 as $N \to \infty$. To characterise the classes of functions with the same rate of approximation for smooth functions it is important to establish Jackson's type estimates of the form $ET_N(f) \leq \dfrac{\|f^{(m)}\|_2}{N^\alpha}$, in L_2 for example. The maximum α determines the rate of approximation, for more details see [6]. One way to generalize the classical Fourier series is to replace S_N by the set of eigenfunctions of predetermined differential operators, Sturm-Liouville and self-adjoint operators, and etc., for details see [1].

The goal of the present paper is to study approximation by finite linear combinations of elements of E with exponents $\lambda_k, k = 1, \dots, n$ adapted to f. For a given function f on a finite interval I we determine λ's as the characteristic roots of the linear differential operator $L_n(\psi, t) = \sum_{k=0}^{n} a_k \psi^{(k)}(t), a_k \in \mathbf{R}$, that minimizes $\|L_n(f, t)\|_2(I)$. For a particular operator L_n the space of solutions to the equation $L_n(\psi, t) = 0$ is called null space of L_n or fundamental set of solutions. That space is spanned by n functions, $\psi_k(t) = e^{\lambda_k t}, k = 1, \dots, n$, where λ_k are the roots of the corresponding characteristic polynomial $P_n(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$. Once the optimal operator L_n is determined we approximate f in $L_p, p > 0$ by linear combinations from the finite dimensional null space of L_n. Fourier and polynomial approximations could be considered in that setting with predetermined operators. For the particular sequence of operators $L_n(f) = f^{(n)}$ the fundamental set of solutions are the power functions $\psi_k(t) = t^k, k = 0, \dots, n-1$. For the operators $S_N = \prod_{k=-N}^{N} (D + ikId)$, where $i^2 = -1, D^k f = f^{(k)}$ and $Id(f) = f$ is the identity operator the fundamental set of solutions is the set of the Fourier modes $e^{ikt}, k = -N, \cdots, N$.

It is well known, see [6], that if the function f is non-periodic on I the rate of convergence of the Fourier series is $1/n$. In Section 2 we establish exponential decay of the rate of approximation of analytic non-periodic functions in uniform norm from the Null Space of the operator $M_n = D^n/n! + Id$ when the interval I on which the approximation takes place is of a length less than $\log 4$. That result is related to the result of a method due to D.Gottlieb, C.W. Shu, A. Solomonoff, and H. Vendeven[4] about approximation of an analytic non-periodic function, defined on a finite interval I, by algebraic polynomials with exponential decay in uniform norm.

In Section 2 we also establish Jackson's type estimates for the error of approximation form Null Spaces of optimal operators and for particular types of operators the rate of approximation is discussed. In Section 3 we consider explicit formulas, algorithms and examples to illustrate the use of the results obtained in Section 2.

2 Approximation from Null Spaces of Linear Differential Operators

In the current section we study the error of approximation of a smooth function by fundamental set of solutions of prescribed differential operators with constant coefficients and obtain Jackson's type estimates. The operators considered are of the type $L_n(f,t) = \sum_{j=0}^{n} a_j f^{(j)}(t)$, for real $a_j,\ j = 0,\dots,n$, with the requirement that $a_n \neq 0$. The characteristic polynomial of L_n is denoted by $P_n(\lambda) = \sum_{j=0}^{n} a_j \lambda^j$. The roots of P_n are the complex conjugated numbers $\lambda_j,\ j = 1,\cdots,n$. Throughout the paper we consider only operators with simple roots. The case of repeated roots can be treated similarly with slight modifications. The collection of functions $\psi_j(t) = e^{\lambda_j t}, j = 1,\dots,n$ is the fundamental set of solutions to the equation $L_n(g,t) = 0$ i.e. $L_n(\psi_j,t) = 0$. The determinant of the Wronskian of ψ's is $W(\psi_1,\cdots,\psi_n)(s) = \det V(\lambda_1,\cdots,\lambda_n) e^{\sum_{k=1}^{n} \lambda_k t}$, where $V(\lambda_1,\cdots,\lambda_n)$ is the Van der Monde matrix of λ's.

Let

$$Q_{n-1}(\lambda) = \sum_{j=1}^{n} f^{(j-1)}(t_0) \left(\sum_{m=j}^{n} a_m \lambda^{m-j} \right),$$

C be a smooth closed curve in the complex plane containing in its interior all λ's, and $\int_C$ indicates integration over C. By $B_r(0)$ we denote the circle in the complex plane with radius r centred at the origin and $C^n(I)$ is the class of functions with continuous n-th derivative on I. The following theorem holds true on any I.

Theorem 2.1. *For any $f \in C^{(n)}(I)$ and any $t_0 \in I$ the solution $g^*(t)$ of the initial value problem* (IVP)

$$L_n(g,t) = L_n(f,t), \quad g^{(j)}(t_0) = 0, \qquad j = 0,\cdots,n-1, \tag{1}$$

is given by each of the next two formulas

$$g^*(t) = \frac{1}{2\pi i}\int_{t_0}^{t}\left(\int_C \frac{e^{z(t-s)}}{P_n(z)}\mathrm{d}z\right)L_n(f,s)\mathrm{d}s, \tag{2}$$

and

$$g^*(t) = f(t) - \sum_{k=1}^{n}\frac{Q_{n-1}(\lambda_k)}{P_n'(\lambda_k)}e^{\lambda_k(t-t_0)}. \tag{3}$$

Proof. First we prove the representation (2). Let $W_k(\psi_1,\ldots,\psi_n)(s)$ be the determinant of the Wronskian with k-th column replaced by the column $(0,\ldots,0,1)$. It is well known, see for example [1], that the solution of IVP (1) is given by the formula

$$g^*(t) = \frac{1}{a_n}\sum_{k=1}^{n}\psi_k(t)\int_{t_0}^{t}\frac{W_k(\psi_1,\ldots,\psi_n)(s)}{W(\psi_1,\ldots,\psi_n)(s)}L_n(f,s)\mathrm{d}s.$$

From the explicit expressions for W_k and W and the fact that V is Van der Monde matrix we get $W(s) = \prod_{j<m}(\lambda_m-\lambda_j)e^{\Sigma\lambda_j s}$, $W_k(s) = (-1)^{n+k}\prod_{j<m,j,m\neq k}(\lambda_m-\lambda_j)e^{(\Sigma\lambda_j s)-\lambda_k s}$, and hance

$$\frac{W_k(s)}{W(s)} = a_n\frac{e^{-\lambda_k s}}{P_n'(\lambda_k)}.$$

Multiplying by $\psi_k(t)$ and summing the terms for $k = 1,\ldots,n$ we get

$$\sum_{k=1}^{n}\psi_k(t)\frac{W_k(\psi_1,\ldots,\psi_n)(s)}{W(\psi_1,\ldots,\psi_n)(s)} = a_n\sum_{k=1}^{n}\frac{e^{(t-s)\lambda_k}}{P_n'(\lambda_k)}.$$

The expression on the right is the divided difference of the complex value function $H(z) = e^{z(t-s)}$ evaluated at the complex nodes λ_k, $k = 1,\ldots,n$. The closed curve C encircles all λ's and by using the results from [2] it follows that

$$a_n\sum_{k=1}^{n}\frac{e^{(t-s)\lambda_k}}{P_n'(\lambda_k)} = \frac{a_n}{2\pi i}\int_C\frac{e^{z(t-s)}}{P_n(z)}\mathrm{d}z.$$

Summing up we get the identity

$$\begin{aligned}\frac{1}{a_n}\sum_{k=1}^{n}\psi_k(t)\int_{t_0}^{t}\frac{W_k(\psi_1,\ldots,\psi_n)(s)}{W(\psi_1,\ldots,\psi_n)(s)}L_n(f,s)\mathrm{d}s \\ = \frac{1}{2\pi i}\int_{t_0}^{t}\left(\int_C\frac{e^{z(t-s)}}{P_n(z)}\,dz\right)L_n(f,s)\mathrm{d}s,\end{aligned}$$

and (2) follows immediately.

Next we derive the form (3) of the solution. Since f is a particular solution to $L_n(g,t) = L_n(f,t)$ from the general theory, see for example [1], it follows that the solution of IVP (1) is of the form $f-\psi_h$. Here ψ_h is the solution of the homogeneous IVP $L_n(\psi_h,t)=0$ with the same initial conditions as in (1), and hence $\psi_h(t)=\sum_{j=1}^{n} b_j e^{\lambda_j t}$ where b_j are solutions of the linear system $\psi_h^{(k)}(t_0)=\sum_{j=1}^{n} b_j\lambda_j^k e^{\lambda_j t_0}=f^{(k)}(t_0), k=0,\ldots,n-1$. Let $W_k(f,t_0)$ be the Wronskian W with the $k-$th column replaced by the column vector $(f(t_0),\ldots,f^{(n-1)}(t_0))$ and evaluated at t_0. If $A_{j,k}$ denotes the algebraic compliment of the element λ_k^{j-1} in V then by using the Cramer's rule we get

$$b_k=\frac{W_k(f,t_0)}{W(t_0)}=e^{-\lambda_k t_0}\sum_{j=1}^{n} f^{(j-1)}(t_0)(-1)^{k+j}\frac{A_{j,k}}{\det V}.$$

Let $\sigma_j(\lambda_1,\ldots,\lambda_n)$ be the $j-$th symmetric polynomial of all of λ's and $\sigma_{j,k}(\lambda_1,\ldots,\lambda_n)$ be the $j-$th symmetric polynomial of all of λ's but λ_k. Using the results obtained in [4], it follows that

$$b_k=\frac{a_n e^{-\lambda_k t_0}}{P_n'(\lambda_k)}\sum_{j=1}^{n}(-1)^{n+j}\sigma_{n-j,k}f^{(j-1)}(t_0).$$

From the relation $\sigma_{n-j,k}=\sigma_{n-j}-\lambda_k\sigma_{n-j-1,k}$ we get

$$\begin{aligned}\psi_h(t) &= a_n\sum_{k=1}^{n}\left(\sum_{j=1}^{n} f^{(j-1)}(t_0)(-1)^{n+j}\sigma_{n-j,k}\right)\frac{e^{\lambda_k(t-t_0)}}{P_n'(\lambda_k)}\\ &= \sum_{k=1}^{n}\frac{e^{\lambda_k(t-t_0)}Q_{n-1}(\lambda_k)}{P_n'(\lambda_k)},\end{aligned}$$

and since $g^*=f-\psi_h$ the proof is completed.

By identity (2) we obtain representations for the derivatives of the function $g(t)=f(t)-\psi_h(t)$ in terms of the contour integral over C.

Corollary 2.1. *Let ψ_h be as in Theorem 2.1 and $g(t)=f(t)-\psi_h(t)$, then for any non-negative integer $m\le n-1$ we have*

$$g^{(m)}(t)=\frac{1}{2\pi i}\int_{t_0}^{t}\left(\int_C\frac{z^m e^{z(t-s)}}{P_n(z)}\mathrm{d}z\right)L_n(f,s)\mathrm{d}s. \tag{4}$$

Proof. The result can be shown by using the companion system of first order differential equations but we consider a more direct approach. From Theorem 2.1 it follows

$$g(t)=\sum_{k=1}^{n}\frac{e^{\lambda_k t}}{P_n'(\lambda_k)}\int_{t_0}^{t}e^{-\lambda_k s}L_n(f,s)\mathrm{d}s, \tag{5}$$

and after differentiating once we get that

$$g'(t)=\sum_{k=1}^{n}\frac{\lambda_k e^{\lambda_k t}}{P_n'(\lambda_k)}\int_{t_0}^{t}e^{-\lambda_k s}L_n(f,s)\,ds+L_n(f,t)\sum_{k=1}^{n}\frac{1}{P_n'(\lambda_k)}.$$

Considering the divided differences with nodes λ's, it is clear that the second term in the expression above is the divided difference of the constant function 1, and hence zero. Differentiating $m-1$ more times and taking into account that $\sum_{k=1}^{n} \frac{\lambda_k^r}{P_n'(\lambda_k)} = 0, r \in N$, since it is the divided difference of z^r, we get (4).

Combining (2) and (3) we obtain a representation formula for f by a linear combination, $S_n(f,t) = \sum_{k=1}^{n} b_k e^{\lambda_k t}$, of the set of fundamental solutions to $L_n(g,t) = 0$. The choice of b_k ensures that $f^{(k)}(t_0) = S_n^{(k)}(t_0)$ for $k = 0,\dots,n-1$ and an error estimate in $C(I)$ can be obtained. In the rest of the section we investigate different operators L_n and estimate the integral error representation in $L_p, p \geq 1$. Let λ_j be enumerated in such a way, that $|\lambda_1| \leq \dots, \leq |\lambda_n|$, then the following theorem holds true.

Theorem 2.2. *Let the fundamental set of the solutions to* $L_n(g,t) = \sum_{j=0}^{n} a_j g^{(j)}(t) = 0$ *be* $e^{\lambda_k t}, k = 1,\dots n$, *then for any* $f \in C^{(n)}(I)$ *we have the following estimate of approximation of* $f^{(m)}$ *by functions from the set* $\left\{\sum_{k=1}^{n} c_k e^{\lambda_k t} \mid c_k \in \mathbf{R}\right\}$,

$$\begin{aligned} &\min_{S_n \in \{\sum_{k=1}^{n} c_k e^{\lambda_k t} \mid c_k \in \mathbf{R}\}} \|f^{(m)} - S_n^{(m)}(f,t)\|_\infty \\ &\leq \frac{|I|^{1/q} e^{|\lambda_n||I|}}{|a_n| 2^{n-m-1/p} |\lambda_n|^{n-m-1}} \|L_n(f)\|_p, \end{aligned} \tag{6}$$

where $m \leq n-1$ *is an integer,* $p,q \geq 1$, *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $R = 2|\lambda_n|$ and $C = B_R(0)$, then $|z - \lambda_k| \geq R$ for any k and any z on C. The following estimate for the contour integral in (4) holds

$$\left| \frac{1}{2\pi i} \int_C \frac{z^m e^{z(t-s)}}{P_n(z)} \mathrm{d}z \right| \leq \frac{1}{2\pi |a_n|} \int_C \frac{|z|^m e^{|z||t-s|}}{\prod_{k=1}^{n} |z - \lambda_k|} |\mathrm{d}z| \leq \frac{R^m e^{R|t-s|}}{|a_n| R^n} R,$$

and hence for $c_k = b_k$, as considered in Theorem 2.1, we get the estimate

$$|f^{(m)}(t) - S_n^{(m)}(f,t)| \leq \frac{1}{|a_n| R^{n-m-1}} \left| \int_{t_0}^{t} e^{R|t-s|} |L_n(f,s)| \mathrm{d}s \right|.$$

By choosing t_0 to be the middle point of I and applying Hölder's inequality we get the following uniform estimate for the approximation of f by linear combinations of ψ's on I

$$\|f^{(m)} - S_n^{(m)}(f,t)\|_\infty \leq \frac{e^{R|I|/2}}{2^{1/q} \pi |a_n| R^{n-m-1}} |I|^{1/q} \|L_n(f)\|_p.$$

Finally, from the relations $R = 2|\lambda_n|$ and the fact that the right-hand side is independent of the choice of c_k we obtain (6).

A few remarks are in order. The estimate (6) depends exponentially on the maximum modulus of the characteristic roots of L_n and the length of the interval I. The dependence on the interval is similar to the situation with Fourier approximation, for example the function $e^{-1/(1-x^2)}$ has exponential rate of convergence on the interval $[-1,1]$ but any expansion of that interval forces the rate to drop to linear i.e $1/n$(see [5]).

The error estimate also depends on the amplitude of the largest characteristic root $|\lambda_n|$ and there are no suitable(known to the author) a priori estimates for the magnitude of $|\lambda_n|$ based on the function and its derivatives. In order to develop an efficient method for approximation form the null spaces we need to consider an intermediate approximation step. From (6) it is clear that if the operator L_n has all of its characteristic roots in $B_{R/2}(0)$, then the right-hand side in (6) decreases by decreasing the norm of L_n. The intermediate step is to approximate the function by the fundamental set of solutions of the operator $M_n(f,t) = f(t) + \frac{1}{n!}f^{(n)}(t)$. In the rest of the section we consider real analytic functions f and study the convergence of approximation error for $L_n = M_n$, when $n \to \infty$. First we provide an explicit formula for approximation by using M_n.

The characteristic polynomial of M_n is $P_n(\lambda) = \frac{1}{n!}\lambda^n + 1$, and hence

$$Q_{n-1}(\lambda_k) = \sum_{j=1}^{n} f^{(j-1)}(t_0)\frac{\lambda_k^{n-j}}{n!}.$$

Let

$$D_n(f,t_0,z) = \sum_{j=1}^{n} f^{(j-1)}(t_0)z^{1-j}$$

be the truncated $z-$ transform of the sequence $\{f^{(j)}(t_0)\}_{j=0}^{\infty}$, then

$$\begin{aligned} b_k &= \frac{Q_{n-1}(\lambda_k)}{P_n'(\lambda_k)}e^{-t_0\lambda_k} = \frac{\frac{1}{n!}\sum_{j=1}^{n} f^{(j-1)}(t_0)\lambda_k^{n-j}}{\frac{\lambda_k^{n-1}}{(n-1)!}}e^{-t_0\lambda_k} \\ &= \frac{1}{n}D_n(f,t_0,\lambda_k)e^{-t_0\lambda_k}, \end{aligned}$$

and the approximant defined by (3) is

$$\Sigma_n(f,t) = \frac{1}{n}\sum_{k=1}^{n} D_n(f,t_0,\lambda_k)e^{(t-t_0)\lambda_k}.$$

Let $C = B_{2(n!^{1/n})}(0)$, then by differentiating m times and taking into account the identities for divided differences with nodes λ's we get

$$\begin{aligned} \Sigma_n^{(m)}(f,t) &= \frac{1}{n}\sum_{k=1}^{n} \lambda_k^m D_n(f,t_0,\lambda_k)e^{(t-t_0)\lambda_k} \qquad (7) \\ &= \frac{1}{2\pi i n!}\int_C \frac{z^m D_n(f,t_0,z)e^{(t-t_0)z}}{P_n(z)}\mathrm{d}z. \end{aligned}$$

The zeros of P_n are $\lambda_k = n!^{1/n} e^{\frac{2k+1}{n} i\pi}, k = 0, \ldots, n-1$ and all of them belong to $B_{n!^{1/n}}(0)$. The approximants Σ_n are linear combinations of $e^{\lambda_k t}$ and represent an approximation of the function f on I. For comparison the Fourier partial sums have their modes λ on the imaginary axis on the segment $[-in, in]$. Next we show that if the interval I has a length less than $\log 4$, then letting $n \to \infty$, i.e. when expanding the circles $B_{2n!^{1/n}}(0)$ the functions Σ_n approach f uniformly.

Theorem 2.3. *For any real function f, analytic on $I = [t_0 - \eta, t_0 + \eta]$, we have*

$$\|f^{(m)} - \Sigma_n^{(m)}(f)\|_\infty(I) \le Ce^{(\eta - \log 2)n} n^{m+1}, \tag{8}$$

for $n \ge 5$, $m \le M \le n-1$, and a positive constant C. Furthermore, if M does not depend on n and $\eta < \log 2$, then the right hand side converges to zero.

Proof. Since f is analytic on I, there exists a positive constant C such that $|f^{(k)}(t)| \le Ck!$ for any integer k and any $t \in I$. Then $\|M_n(f)\|_p \le 2C|I|^{1/p}$ and the inequality (6) becomes

$$\|f^{(m)} - \Sigma_n^{(m)}(f)\|_\infty(I) \le C2^{1/p}|I| \frac{e^{n!^{1/n}|I|}}{2^{n-m}} n!^{\frac{m+1}{n}}.$$

By using the estimate $n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}$, given by the Stirling's approximation of $n!$, we get $n!^{1/n} \le (2\pi n)^{1/2n} e^{1/12n^2 - 1} n < n/2$ for $n \ge 5$. Substituting in the above expression, we obtain the estimate

$$\begin{aligned} \|f^{(m)} - \Sigma_n^{(m)}(f)\|_\infty(I) &\le \frac{C|I|e^{|I|/2n} n^{m+1}}{2^n} \\ &= C\exp\left(|I|/2 - \log 2 + (m+1)\frac{\log n}{n}\right) n^{m+1}. \end{aligned}$$

Since $m < M$ and M does not depend on n, it is clear that the expression on the right converges to zero if the exponent $\eta - \log 2$ is negative i.e. $\eta < \log 2$. The convergence is geometric for large n and bounded M if $|I|/2 - \log 2 + (m+1)\dfrac{\log n}{n} < 0$ i.e. $\eta < \log 2 - (M+1)\dfrac{\log n}{n} \to \log 2$.

From Theorem 2.3, it follows that any real analytic function f on I $|I| < 2\log 2$, is uniformly approximated by $\Sigma_n(f)$, thus it avoids Gibbs phenomenon at the end points. Similar result about approximation with algebraic polynomials and using as an intermediate step the Fourier partial sums is obtained in [3]. The similar question about approximating periodic functions is solved in [7].

From the proof of Theorem 2.3 it clear that any sequence of operators L_n with roots of their characteristic polynomials in bounded and slowly expanding domains C_n, and such that $\|L_n(f)\|_p < Cn$, also provide approximations to f with an exponential decay.

In the next section we consider two step algorithm to find a sequence of operators that provide convergent sequence of errors. Numerical implementation and examples of the result in Theorem 2.1 are also presented.

3 Applications and Examples

In the previous section it was discussed that the error estimate (6) depends on the magnitude of the largest root of the characteristic polynomial of L_n as well, as the magnitude of $\|L_n(f,t)\|_p(I)$. In this section we suggest an algorithm for constructing operators L_n that provides convergent error estimates in (6). At the end of the section we consider two numerical examples.

First we consider the problem of minimizing L_2 norm of operators with roots of their characteristic polynomials restricted to the interior of expanding circles. If the characteristic polynomial P_n is of an odd degree, than it has at least one real root which corresponds to $\psi(t)=e^{rt}$ with $r\in\mathbf{R}$. To ensure that the characteristic polynomial has only complex conjugated roots we consider only the case of even n. First we approximate f by the set of fundamental solutions of $M_{2n}(f)=f+1/(2n)!f^{(2n)}$. According to Theorem 2.3 the error of approximation has exponential decay on small intervals. The second step is to optimize the approximation of $\Sigma_{2n}(f)$ over differential operators with characteristics roots in $B_{(2n)!^{1/2n}}(0)$.

For large n, from Theorem 2.3, it follows that $\Sigma_{2n}(f)$, the approximant corresponding to M_{2n}, approaches f exponentially fast. If we fix n large enough to achieve a prescribed precision, then instead of seeking operators $L_v(g)=\sum_{j=0}^{v}c_jg^{(j)}$, $c_j\in\mathbf{R}$, with the normalization condition $c_v=1$, that minimize $\|L_v(f,t)\|_p$ we can seek L_v to minimize $\|L_v(\Sigma_{2n}(f),t)\|_p$. In the case $p=2$ the space $L_2(I)$ is a Hilbert space and after some weakening of the inequalities the problem of characterizing L_v can be considered as a problem of finding weighted orthogonal polynomials, for an overview source of orthogonal polynomials on the unit disc see [6]. Let $R=(2n)!^{1/2n}$ and $C=B_{2R}(0)$, then from (7) it follows that

$$\sum_{j=0}^{v}c_j\Sigma_{2n}^{(j)}(f,t)=\frac{1}{2\pi i(2n)!}\int_C\frac{\left(\sum_{j=0}^{v}c_jz^j\right)D_{2n}(f,t_0,z)e^{(t-t_0)z}}{P_{2n}(z)}\mathrm{d}z. \tag{9}$$

For $P_{2n}(z)$ on C we have that $|P_{2n}(z)|=|z^{2n}/(2n)!+1|\geq 2^{2n}-1$ and from the inequality $\left|\int_C F(z)\mathrm{d}z\right|\leq\int_C|F(z)||\mathrm{d}z|$, we obtain the following estimate

$$\left|\sum_{j=0}^{v}c_j\Sigma_{2n}^{(j)}(f,t)\right|\leq\frac{1}{2(2^{2n}-1)\pi(2n)!}\int_C|D_{2n}(f,t_0,z)||e^{z(t-t_0)}||\sum_{j=0}^{v}c_jz^j|\,|\mathrm{d}z|.$$

By applying the Schwartz inequality and estimating $|t-t_0|\leq I$ we get that

$$\|\sum_{j=0}^{v}c_j\Sigma_{2n}^{(j)}(f,t)\|_\infty(I)\leq C(n)\left(\int_0^{2\pi}|D_{2n}(f,t_0,z)|^2|\sum_{j=0}^{v}c_jz^j|^2 2R\mathrm{d}\theta\right)^{1/2},$$

where

$$C(n)=\frac{e^{2(2n)!^{1/2n}|I|}}{\pi^{1/2}(2^{2n}-1)(2n!)^{1-1/4n}}\to 0,\quad \text{as}\quad n\to\infty.$$

It is clear that, in order to minimize the norm on the left, over $c_0,\dots,c_{\nu-1}$ it suffices to find the minimizer of the following extremal problem

$$\min_{T_\nu(z)=(2Re^{i\theta})^\nu+\dots}\int_0^{2\pi}|D_{2n}(f,t_0,2Re^{i\theta})|^2|T_\nu(2Re^{i\theta})|^2 2R\mathrm{d}\theta.$$

The weight function $|D_{2n}(f,t_0,z)|^2$ is uniformly bounded for any n,z and $t_0\in I$. Indeed, by using the fact that f is analytic and the Stirling's estimate we get the inequalities

$$\begin{aligned}||D_{2n}(f,t_0,z)|-|f(t_0)|| &\le \sum_{j=1}^{2n-1}\frac{|f^{(j)}(t_0)|}{|z|^j}\le C\sum_{j=1}^{2n-1}\frac{j!}{2^j(2n)!^{j/2n}}\\ &\le C\sum_{j=1}^{2n-1}\sqrt{j}\left(\frac{je^{-\frac{11}{12}}}{2n}\right)^j\le C\sum_{j=1}^{2n-1}\frac{\sqrt{j}}{e^{\frac{11}{12}j}}.\end{aligned}$$

The last inequality follows from the fact that $j<2n$. Since $e^{\frac{11}{12}}>1$, then taking the limit when $n\to\infty$ of the convergent series on the very right we get that $|D_{2n}(f,t_0,z)|^2\le K<\infty$. Going back to the extremal problem we see that the minimum is attained when $T_\nu(z)$ is the $\nu-$th orthogonal polynomial on C with a weight function $|D_{2n}(f,t_0,z)|^2$. It is well known, see [6], that all of the zeros of the ν-th orthogonal polynomial T_ν are inside C. On the other hand the zeros of T_ν are the roots of the characteristic polynomial of $L_\nu=T_\nu(D)$ and to ensure decay of the error they have to belong to $B_R(0)$. To formulate the problem on $B_R(0)$ we use the following inequality

$$\int_0^{2\pi}|D_{2n}(f,t_0,2Re^{i\theta})T_\nu(2Re^{i\theta})|^2\mathrm{d}\theta\le 2^{2\nu}\int_0^{2\pi}|D_{2n}(f,t_0,Re^{i\theta})T_\nu(Re^{i\theta})|^2\mathrm{d}\theta.$$

Indeed,

$$D_{2n}(f,t_0,2Re^{i\theta})=\sum_{j=0}^{2n-1}\frac{f^{(j)}(t_0)}{2^jR^j}e^{-ij\theta},$$

$$T_\nu(2Re^{i\theta})=\sum_{k=0}^{\nu}c_k2^kR^ke^{ik\theta},$$

and for L_2 the norm of their product it follows that

$$\|\sum_{s=-2n+1}^{\nu}\left(\sum_{k-j=s}\frac{f^{(j)}(t_0)}{R^j}R^kc_k\right)2^se^{is\theta}\|^2=\sum_{s=-2n+1}^{\nu}\left(\sum_{k-j=s}\frac{f^{(j)}(t_0)}{R^j}R^kc_k\right)^2 2^{2s}$$

since $e^{is\theta}$ are orthogonal on $[0,2\pi]$. Similar calculations show that

$$\int_0^{2\pi}|D_{2n}(f,t_0,Re^{i\theta})T_\nu(Re^{i\theta})|^2\mathrm{d}\theta=\sum_{s=-2n+1}^{\nu}\left(\sum_{k-j=s}\frac{f^{(j)}(t_0)}{R^j}R^kc_k\right)^2,$$

and hence the inequality holds. Let

$$O_{2n}(f,v) = 2^{2v+1} \min_{T_v(z)=(Re^{i\theta})^v+\dots} \int_0^{2\pi} |D_{2n}(f,t_0,Re^{i\theta})|^2 |T_v(Re^{i\theta})|^2 R\mathrm{d}\theta, \tag{10}$$

then summing up we can formulate a Jackson's type estimate for approximation of an analytic function f by linear combinations of complex exponents $e^{\zeta_1 t},\dots,e^{\zeta_{2v}t}$ with $\zeta_j, j=1,\dots,2v \le 2n-2$ in the disk $D_R = D_{n!^{1/n}}(0) = \{z||z| \le n!^{1/n}\}$.

Lemma 3.1. *For a function f analytic on D_{2R} the following estimate holds*

$$\min_{d_j\in\mathbf{R},\zeta_j\in D_R} \|f - \sum_{j=1}^{v} d_j e^{\zeta_j t}\|_\infty(I) \le Ce^{(|I|/2-\log 2)v} v O_{2n}^{1/2}(f,v) + \|f - \Sigma_{2n}(f,t)\|_\infty, \tag{11}$$

where C is a real constant.

Proof. Let $L_v = T_v(D)$ be define by (10), then the characteristic roots $\xi_k, k=1,\cdots,v$ of L_v, are inside the disk D_R. From (6) and using similar arguments as in the proof of Theorem 2.3 we get that,

$$\|\Sigma_{2n}(f,t) - \sum_{j=1}^{m} d_j e^{\xi_j t}\|_\infty \le Ce^{(|I|/2-\log 2)v} v O_{2n}^{1/2}(f,v).$$

From the inequality

$$\|f - \sum_{j=1}^{v} d_j e^{\xi_j t}\|_\infty(I) \le \|f - \Sigma_{2n}(f)\|_\infty(I) + \|\Sigma_{2n}(f,t) - \sum_{j=1}^{v} d_j e^{\zeta_j t}\|_\infty(I)$$

and the above estimate we get (11) for any $\zeta_j \in D_R$.

We conclude the paper with two numerical examples illustrating the roles of n and the length of the interval I in the error estimate in Theorem 2.2. In the first example we consider the single frequency function $f(t) = 2\cos\mu t$, $\mu \in \mathbf{R}$ on an arbitrary interval containing 0 with length less than $\log 4$. Since $f'' + \frac{1}{\mu^2} f = 0$ we study the solution of the extremal problem (10) for $v=2$ and increasing n. The approximation procedure is as follows: For a fixed n, first approximate f by the operator M_{2n} and then find the zeros of T_2 that minimizes (10). It turns out that the zeros of T_2 are purely imaginary and complex conjugated with absolute value $x^*(n)$ and $|x^*(n) - |\mu|| \le Cn^{-3}, C \in \mathbf{R}$.

Example 1. Let $f(t) = 2\cos\mu t = e^{i\mu t} + e^{-i\mu t}$, $R = n!^{1/n}$, and $t_0 = 0$, then $D_{2n}(e^{i\mu t},0,z) = \dfrac{1-(i\mu/z)^{2n}}{1-\mu/z}$. Since the weight is a symmetric function and the domain is a circle, it follows that $T_2(z) = z^2 + x$, for some real x. The extremal problem (10) for T_2 on $B_R(0)$ is equivalent to

$$\begin{aligned} &\min_{x\in\mathbf{R}} \int_{B_R(0)} \left| \frac{z^{2n}-(i\mu)^{2n}}{z-i\mu} + \frac{z^{2n}-(-i\mu)^{2n}}{z-(-i\mu)} \right|^2 |z^2+x|^2\, |\mathrm{d}z| \\ &= C(R) \min_{x\in\mathbf{R}} \int_{-\pi}^{\pi} \left| \frac{R^{2n}e^{i2n\theta} + (-1)^{(n+1)}|\mu|^{2n}}{R^2 e^{i2\theta} + |\mu|^2} \right|^2 |R^2 e^{i2\theta} + x|^2\, \mathrm{d}\theta, \end{aligned}$$

where $C(R)$ depends only on R. Since the interval is symmetric and the integrand even, it follows that the extremal problem to be considered is

$$\min_{x\in\mathbf{R}} \int_0^{\pi} \left| \frac{R^{2n}e^{in\theta} + (-1)^{n+1}|\mu|^{2n}}{R^2 e^{i\theta} + |\mu|^2} \right|^2 \left((R^2\cos\theta + x)^2 + R^4 \sin^2\theta \right) \mathrm{d}\theta.$$

Let $r = \mu/R^2$ and $u_n(r,\theta) = \dfrac{r^{2n} + 2(-1)^{n+1} r^n \cos n\theta + 1}{r^2 + 2(-1)^{n+1} r \cos n\theta + 1}$, then the extremal $x^*(r)$ is given by the formula

$$x^*(n) = -R^2 \frac{\int_0^{\pi} \cos\theta \left| \frac{R^{2n}e^{in\theta} + (-1)^{n+1}|\mu|^{2n}}{R^2 e^{i\theta} + |\mu|^2} \right|^2 \mathrm{d}\theta}{\int_0^{\pi} \left| \frac{R^{2n}e^{in\theta} + (-1)^{n+1}|\mu|^{2n}}{R^2 e^{i\theta} + |\mu|^2} \right|^2 \mathrm{d}\theta} = -R^2 \frac{\int_0^{\pi} \cos\theta |u_n(r,\theta)| \mathrm{d}\theta}{\int_0^{\pi} |u_n(r,\theta)| \mathrm{d}\theta}.$$

For $n \to \infty$ we have that $r(n) \to 0$, $u_n(r,\theta) \to 1$, and $\int_0^{\pi} |u_n(r,\theta)| d\theta \to \pi$. For large n the function u_n is positive and the integral $\int_0^{\pi} \cos\theta u_n(r,\theta) d\theta$, as a function of r, has a Taylor's series expansion about $r = 0$ of the form $-\pi r + Cr^2$, where C is a constant. By substituting in the formula for $x^*(n)$ we get $|x_n^* - |\mu|| < 1/R^4$. In other words, the zero of the orthogonal polynomial T_2 extracts the frequency μ with a rate of convergence faster than $1/n^3$. We want to stress once again that the interval on which f is defined could be smaller than half of the period $\mu/2$.

In the second example we keep the order of the operator L_n fixed to 4 and to achieve better approximation we partition the interval $[-1,1]$.

Example 2. Let $f(t) = \exp(-t^2)\sin 2(t+2)^{2.2} + \exp(2t)\sin 20(t+1)$ and $I = [-1,1]$. The function f is infinitely many times differentiable on I. We consider minimization of $\|f^{(4)} + c_3 f^{(3)} + c_2 f^{(2)} + c_1 f^{(1)} + c_0 f\|_2$ over the real constants c's. According to the estimate (6), in order to decrease the error, we need to consider small intervals. For that we generate the partition with break points at -1, 1, and the extrema of f. On each of the subintervals we solve (10) with $n = 4$ and get four complex conjugated roots of the characteristic polynomial $P(\lambda) = \lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$. The collection of all of the approximants on the partition is a piecewise continuous function with discontinuities at the break points. The jumps are relatively small, since the method provides uniform approximation including at the end points, and we apply a simple smoothing procedure to get continuous approximant. The function f(the continuous line) and the approximant(the dotted line) are plotted in Fig. 1. The dashed line represents the error function.

On each subinterval the real and the imaginary parts of the roots of P are constants. In Fig 2. each of the four numbers is plotted at the right end of the corresponding subinterval. The upper continuous line is the function $4.4(t+2)^{1.2}$ which is the instantaneous frequency of the first term

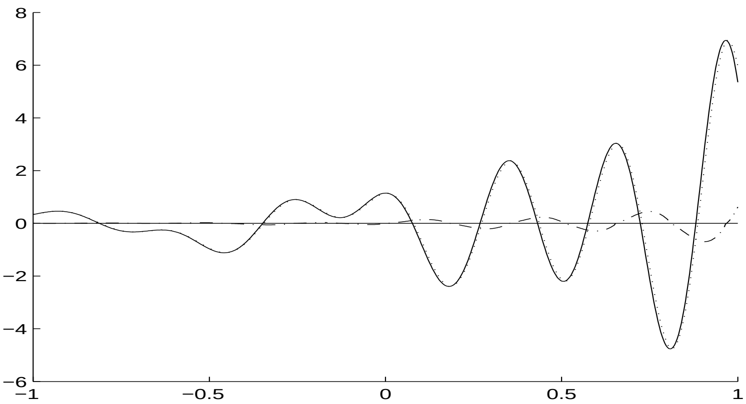

Fig. 1 Piece-wise continuous approximation to f

of f. The lower line is the function $-2t$ which is the instantaneous bandwidth of the first term of f. The two, clearly recognizable horizontal, lines are at levels 20 and 2 and correspond to the instantaneous frequency and the instantaneous bandwidth of the second term of f.

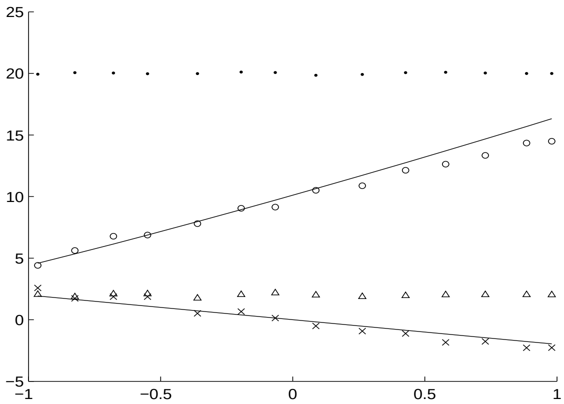

Fig. 2 Discrete approximation to the instantaneous frequencies and bandwidths

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