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SEMI INHERITED BIVARIATE INTERPOLATION

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Abstract. The bivariate interpolation in two dimensional space $\mathbf{R}^2$ is more complicated than that in one dimensional space $\mathbf{R}$, because there is no Haar space of continuous functions in $\mathbf{R}^2$. Therefore, the bivariate interpolation has not a unique solution for a set of arbitrary distinct pairwise points. In this work, we suggest a type of basis which depends on the points such that the bivariate interpolation has the unique solution for any set of distinct pairwise points. In this case, the matrix of bivariate interpolation has the semi inherited factorization.

Key words: *inherited factorization, inherited interpolation, semi inherited interpolation, bivariate interpolation, interpolation matrix*

AMS (2010) subject classification: 15A23, 65F05, 65D05

1 Introduction

In recent years, the bivariate and multivariate interpolations have been studied in the papers [3,8,9,12,13,16]. In [6,7] the inherited interpolation of matrices has been introduced by using the LU inherited factorization of a matrix. In this paper, we develop the idea to offer a type of bivariate interpolation which is based on the semi inherited LU factorization of the interpolation matrix. The factorization of matrices is a method to solve the square system of linear equations. One kind of these factorizations is LU factorization which has various types. One of them is the inherited LU factorization which has been described in [1] by M. Arav and et al. In this work, we use a special kind of the inherited LU factorization. We call this special type of LU factorization as semi inherited LU factorization. Then, we constitute the bivariate interpolation matrix such that it has the semi inherited LU factorization. At first, in section 2, we introduce the semi inherited LU factorization[1,10] and present some preliminaries of interpolation briefly[14,15]. In section 3, we describe the semi inherited bivariate interpolation which depends on the semi inherited LU factorization and we prove that the interpolation matrix in this case has the semi inherited LU factorization. In section 4, we illustrate the mentioned method by some numerical examples.

2 Preliminaries

Definition 2.1. Let A be an $n\times n$ matrix that $a_{ii}\neq 0$ for $i=1,\cdots,n$ and write $A=B+D+C$ that B is strictly lower triangular, D is diagonal, and C is strictly upper triangular. Then, A has the semi inherited LU factorization if and if only $A=(I+BD^{-1})(D+C)$. (I is the identity matrix.)

According to the definition 2.1, the following theorem is proved.

Theorem 2.2. *Let $A=B+D+C$ be a decomposition of the matrix A with invertible diagonal entries where B is strictly lower triangular, D is diagonal and C is strictly upper triangular. Then, A has the semi inherited LU factorization if and only if $BD^{-1}C=0$.*

Proof. Suppose the matrix $A_{n\times n}$ has the semi inherited LU factorization. Then,

$$A=B+D+C=(I+BD^{-1})(D+C)=D+C+B+BD^{-1}C\Rightarrow BD^{-1}C=0.$$

Conversely, suppose $BD^{-1}C=0$. Then,

$$A=B+D+C+BD^{-1}C\Rightarrow A=(I+BD^{-1})(D+C).$$

For example,

$$\begin{aligned}A&=\begin{pmatrix}2&0&12&6\\0&6&12&6\\2&-6&1&0\\-4&12&-3&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&-1&1&0\\-2&2&-3&1\end{pmatrix}\begin{pmatrix}2&0&12&6\\0&6&12&6\\0&0&1&0\\0&0&0&1\end{pmatrix}\\&=(I+BD^{-1})(D+C).\end{aligned}$$

Remark 2.3. A useful property of matrices that have semi inherited LU factorization is this fact that the calculation of LU factorization of them is very easy, because the matrix U is inherited entirely and the matrix L is the product of a strictly lower triangular matrix in the diagonal matrix plus the identity matrix[1].

Suppose $X=\{x_1,x_2,\dots,x_n\}$ is a set of n distinct points in $\mathbf{R}^n$ $(n\geq 1)$. We call X the node set. Also, for each x_i the value $z_i\in\mathbf{R}$ is given. The problem of interpolation is to find a suitable function $F:\mathbf{R}^n\longrightarrow\mathbf{R}$ such that $F(x_i)=z_i$ for $i=1,\cdots,n$. F is called the interpolation function of $\{(x_i,z_i)\}_{i=1}^n$.

Let U be a vector space of functions with the basis $\{u_1,u_2,\cdots,u_n\}$. We can consider F in the form of $F=\sum\limits_{j=1}^{n}\lambda_j u_j$. Then,

$$F(x_i)=z_i,\quad \forall i=1,\cdots,n \quad\text{implies}\quad \sum_{j=1}^{n}\lambda_j u_j(x_i)=z_i,\quad \forall i=1,\cdots,n\ .$$

i.e.,

$$\begin{pmatrix}u_1(x_1)&u_2(x_1)&\cdots&u_n(x_1)\\u_1(x_2)&u_2(x_2)&\cdots&u_n(x_2)\\\vdots&\vdots&&\vdots\\u_1(x_n)&u_2(x_n)&\cdots&u_n(x_n)\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\\\vdots\\\lambda_n\end{pmatrix}=\begin{pmatrix}z_1\\z_2\\\vdots\\z_n\end{pmatrix}.$$

The matrix $A = [u_j(x_i)]_{n\times n}$ $(i = 1,\cdots,n\ ,\ j = 1,\cdots,n)$ is called the interpolation matrix. Therefore, the problem of interpolation is to solve the linear system $A\lambda = z$, where $\lambda = (\lambda_1,\lambda_2,\cdots,\lambda_n)^t$ and $z = (z_1,z_2,\cdots,z_n)^t$.

If we consider $u_j(x) = x^{j-1}$ $(j = 1,\cdots,n)$, then the matrix A has the form of Vandermonde matrix that its determinant is $\prod\limits_{1\leq j<i\leq n}(x_i - x_j)$. Hence, the interpolation function has the unique solution if and only if the nodes x_i are distinct. In this case, the interpolation function is a polynomial of degree at most $n-1$.

Definition 2.4. A vector space U of functions is said a Haar space if the only element of U which has more than $n-1$ roots is the element of zero[2].

Theorem 2.5. *Let U be a vector space of functions with basis $\{u_1,\cdots,u_n\}$. Then the following statements are equivalent:*

i*) U is a Haar space.*

ii*)* $\det[u_j(x_i)] \neq 0$ *for any set of distinct points x_i.*

Proof. See [2].

According to Theorem 2.5, we can obtain the unique solution for the problem of interpolation in one dimensional space $\mathbf{R}$. But the following theorem states that this property is not satisfied on $\mathbf{R}^2$.

Theorem 2.6. *There is no Haar space of continuous functions on* $\mathbf{R}^2$.

Proof. See [2].

3 Introducing of Semi Inherited Bivariate Interpolation

Suppose the set $\{(x_1,y_1),\ldots,(x_n,y_n)\}$ is the pairwise node points in $\mathbf{R}^2$. According to the Theorem 2.6 we can not obtain the unique interpolation function with a basis set of continuous functions and any distinct pairwise node points. In order to obtain the unique interpolation matrix, we can consider the conditional points.

For example, with the basis $\{1,x,y,xy\}$, the interpolation is unique for the points of corners of any rectangle in $\mathbf{R}^2$. We consider four points at the corners of the rectangle in $\mathbf{R}^2$. Therefore we have

$$\begin{aligned} &(x_1,y_1) &&,\quad (x_2,y_2) = (x_2,y_1), \\ &(x_3,y_3) = (x_1,y_3) &&,\quad (x_4,y_4) = (x_2,y_3). \end{aligned}$$

The interpolation matrix has the following form:

$$\begin{pmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_1 & x_2y_1 \\ 1 & x_1 & y_3 & x_1y_3 \\ 1 & x_2 & y_3 & x_2y_3 \end{pmatrix}.$$

By using the Gaussian elimination method we have

$$\begin{pmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 0 & (x_2-x_1) & 0 & y_1(x_2-x_1) \\ 0 & 0 & (y_3-y_1) & x_1(y_3-y_1) \\ 0 & 0 & 0 & (y_1-y_3)(x_1-x_2) \end{pmatrix}.$$

We can see that the determinant of the interpolation matrix in this case is

$$(x_1-x_2)^2(y_1-y_3)^2 \neq 0.$$

Now, we consider the basis depended on the pairwise node points and we obtain the unique interpolation function for any set of distinct pairwise nodes. Also, the interpolation matrix in this case has the semi inherited LU factorization.

In order to have such interpolation we consider the basis $\{h_1,\dots,h_n\}$ in $\mathbf{R}$, for $1 \le i \le [\frac{n+1}{2}]$ in the form of

$$h_{2i-1}(x) = (x-x_2)(x-x_4)\cdots(x-x_{2i})(x-x_{2i+1})\cdots(x-x_n), \tag{1}$$

$$h_{2i}(x) = (x-x_2)(x-x_4)\cdots(x-x_{2i-2}), \tag{2}$$

where $h_2(x) \equiv 1$.

The basis $\{k_1,\dots,k_n\}$ in $\mathbf{R}$, for $1 \le i \le [\frac{n+1}{2}]$ in the form of

$$k_{2i-1}(y) = (y-y_2)(y-y_4)\cdots(y-y_{2i})(y-y_{2i+1})\cdots(y-y_n), \tag{3}$$

$$k_{2i}(y) = (y-y_2)(y-y_4)\cdots(y-y_{2i-2}), \tag{4}$$

where $k_2(y) \equiv 1$.

Finally, we consider

$$f_i(x,y) = h_i(x)k_i(y). \qquad 1 \le i \le n \tag{5}$$

In (5), if $h_i(x_i) = 0$ $(\forall\, 1 \le i \le n)$ then we substitute $h_i(x)$ with $(x^\alpha - \beta)$ in $f_i(x,y)$, where α is the degree of the polynomial $h_i(x)$ and β is the arbitrary point only not equal to $\sqrt[\alpha]{x_i}$.

Also, if $k_i(y_i) = 0$ $(\forall\, 1 \le i \le n)$ then we substitute $k_i(y)$ with $(y^\alpha - \beta)$ in $f_i(x,y)$, where α is the degree of the polynomial $k_i(y)$ and β is the arbitrary point only not equal to $\sqrt[\alpha]{y_i}$.

Such basis has this property that the bivariate interpolation with this basis has the unique solution for any distinct pairwise points in $\mathbf{R}^2$ and also the interpolation matrix in this case has the semi inherited LU factorization.

Definition 3.1. The set of pairwise points $\{(x_1,y_1),\dots,(x_n,y_n)\}$ in $\mathbf{R}^2$ is said to be general distinct if and only if $\{x_1,\dots,x_n\}$ is the set of n distinct points in $\mathbf{R}$ and $\{y_1,\dots,y_n\}$ is the set of n distinct points in $\mathbf{R}$.

If the pairwise node points in $\mathbf{R}^2$ are general distinct , then

$$f_i(x,y) = h_i(x)k_i(y).$$

For example, if $\{(x_1,y_1),\ldots,(x_6,y_6)\}$ is a set of general distinct pairwise points, then

$$\begin{aligned}
&f_1(x,y) = (x-x_2)(x-x_3)(x-x_4)(x-x_5)(x-x_6)(y-y_2)(y-y_3)(y-y_4)(y-y_5)(y-y_6),\\
&f_3(x,y) = (x-x_2)(x-x_4)(x-x_5)(x-x_6)(y-y_2)(y-y_4)(y-y_5)(y-y_6),\\
&f_5(x,y) = (x-x_2)(x-x_4)(x-x_6)(y-y_2)(y-y_4)(y-y_6),\\
&f_2(x,y) = 1\\
&f_4(x,y) = (x-x_2)(y-y_2),\\
&f_6(x,y) = (x-x_2)(x-x_4)(y-y_2)(y-y_4).
\end{aligned}$$

In general, $\{f_i(x,y)\}_{i=1}^n$ are the basis functions which we call them semi inherited bivariate polynomials.

The next theorem shows that the matrix $A = [f_j(x_i,y_i)]\ \ (i,j = 1,2,\cdots,n)$ for any distinct pairwise points $\{(x_1,y_1),\cdots,(x_n,y_n)\}$ has the semi inherited LU factorization, and also $A_{n\times n}$ is a nonsingular matrix.

Theorem 3.2. *Let A be the semi inherited bivariate interpolation matrix with distinct pairwise node points in $\mathbf{R}^2$. Then A has the semi inherited LU factorization.*

Proof. Suppose $A = [f_j(x_i,y_i)]_{n\times n}$, is the matrix of semi inherited bivariate interpolation. We know that there is no x_i,y_i such that $h_i(x_i) = 0$ and $k_i(y_i) = 0$, because the nodes are distinct. Therefore, by the definition of semi inherited bivariate polynomials,

$$\begin{aligned}
&f_i(x_i,y_i) \neq 0 && i = 1,\cdots,n && \text{(a)}\\
&f_{2j-1}(x_i,y_i) = 0 && 2j-1 < i, && \text{(b)}\\
&f_j(x_{2i},y_{2i}) = 0 && 2i < j. && \text{(c)}
\end{aligned}$$

We put

$$A = \begin{pmatrix} f_1(x_1,y_1) & f_2(x_1,y_1) & \cdots & f_n(x_1,y_1)\\ f_1(x_2,y_2) & f_2(x_2,y_2) & \cdots & f_n(x_2,y_2)\\ \vdots & \vdots & & \vdots\\ f_1(x_n,y_n) & f_2(x_n,y_n) & \cdots & f_n(x_n,y_n) \end{pmatrix} = B + D + C,$$

where B is strictly lower triangular, D is diagonal and C is strictly upper triangular. By Theorem 2.2, it is sufficient to show $BD^{-1}C = 0$. Under the condition (a), $D = \mathrm{diag}(f_1(x_1,y_1),\ldots,f_n(x_n,y_n))$ is a nonsingular matrix and $D^{-1} = \mathrm{diag}(f_1(x_1,y_1)^{-1},\cdots,f_n(x_n,y_n)^{-1})$[9]. Let $E = BD^{-1}$, hence

E has the following form:

$$E=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{f_1(x_2,y_2)}{f_1(x_1,y_1)} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{f_1(x_n,y_n)}{f_1(x_1,y_1)} & \frac{f_2(x_n,y_n)}{f_2(x_2,y_2)} & \cdots & \frac{f_{n-1}(x_n,y_n)}{f_{n-1}(x_{n-1},y_{n-1})} & 0 \end{pmatrix}.$$

We consider the $i-$th row of E and $j-$th column of C $(1<i,j\leq n)$. Then,

$$e^i c_j=\begin{pmatrix} \frac{f_1(x_i,y_i)}{f_1(x_1,y_1)} & \frac{f_2(x_i,y_i)}{f_2(x_2,y_2)} & \cdots & \frac{f_{i-1}(x_i,y_i)}{f_{i-1}(x_{i-1},y_{i-1})} & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} f_j(x_1,y_1) \\ f_j(x_2,y_2) \\ \vdots \\ f_j(x_{j-1},y_{j-1}) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Now, it is sufficient to show

$$\frac{f_k(x_i,y_i)}{f_k(x_k,y_k)}f_j(x_k,y_k)=0 \quad (k<\min(i,j), \quad 1<i,j\leq n).$$

If k is odd, under the condition (b) $f_k(x_i,y_i)=0$, and if k is even, under the condition (c) $f_j(x_k,y_k)=0$. Then, A has the semi inherited LU factorization.

According to Theorem 3.2, $A=[f_j(x_i,y_i)]$ for any distinct pairwise points $(x_1,y_1),\ldots,(x_n,y_n)$ has the semi inherited LU factorization. Then,

$$A=(I+BD^{-1})(D+C).$$

Therefore, we conclude $\det(A)\neq 0$ for any set of distinct pairwise points $\{(x_1,y_1)\cdots,(x_n,y_n)\}$[4,5]. Then, the interpolation is unique. In this case, the interpolation function is a bivariate polynomial of degree at most $2n-2$.

Remark 3.3. If we consider $y_i=0$ $(\forall i=1,\cdots n)$, then we can conclude the values x_i $(\forall i=1,\cdots n)$ are distinct, because the pairwise nodes are distinct. In this case, we obtain the bivariate interpolation function. Now, if we eliminate the monomials that has the variable y, then we obtain the polynomial in one dimensional case that interpolates $\{(x_i,z_i)\}_{i=1}^n$. We know that the polynomial interpolation in one dimensional space is unique. This result is obtainable with consideration the basis polynomial $\{h_1,h_2,\cdots,h_n\}$ and interpolation in one dimension by the nodes $x_1,\cdots,x_n$ and values $z_1,\cdots,z_n$. This type of interpolation in one dimensional case is called the semi inherited interpolation, because in this case the interpolation matrix similar to Theorem 3.2 has the semi inherited LU factorization.

Remark 3.4. In order to avoid reiteration of calculations, we can calculate the semi inherited polynomials $\{h_i\}_{i=1}^n$ and also $\{k_i\}_{i=1}^n$ in the following form:

For odd indices with backward substitution,

$$\vdots$$

$$h_{2i-1}(x) = h_{2i+1}(x)(x - x_{2i+1}),$$

$$\vdots$$

$$h_1(x) = h_3(x)(x - x_3),$$

and for even indices with forward substitution,

$$h_2(x) = 1,$$

$$h_4(x) = h_2(x)(x - x_2),$$

$$\vdots$$

$$h_{2i}(x) = h_{2i-2}(x)(x - x_{2i}),$$

$$\vdots$$

4 Numerical Examples

In this section, we present some numerical examples to illustrate the use of the semi inherited LU factorization in the bivariate interpolation. The results have been provided by Maple software.

Example 4.1. Find the semi inherited bivariate interpolation polynomial for the pairwise nodes $\{(-2,3),(1,-5),(4,1)\}$ and the values $z = \{2,4,5\}$.

In this case,

$$\begin{aligned} f_1(x,y) &= (x-x_2)(x-x_3)(y-y_2)(y-y_3), \\ f_2(x,y) &= 1, \\ f_3(x,y) &= (x-x_2)(y-y_2). \end{aligned}$$

$$A = \begin{pmatrix} 288 & 1 & -24 \\ 0 & 1 & 0 \\ 0 & 1 & 18 \end{pmatrix} = (I + BD^{-1})(D + C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 288 & 1 & -24 \\ 0 & 1 & 0 \\ 0 & 0 & 18 \end{pmatrix}.$$

By solving the system $A\lambda = z$,

$$\lambda_1 = \frac{-1}{432} \quad , \quad \lambda_2 = 4 \quad , \quad \lambda_3 = \frac{1}{18}.$$

Consequently,

$$\begin{aligned} F(x,y) &= \sum_{j=1}^{3} \lambda_j f_j(x,y) \\ &= -\frac{1}{432}x^2y^2 - \frac{1}{108}x^2y + \frac{5}{432}x^2 \\ &\quad + \frac{5}{432}xy^2 + \frac{11}{108}xy + \frac{95}{432}x - \frac{1}{108}y^2 - \frac{5}{54}y + \frac{407}{108}. \end{aligned}$$

We can see

$$F(-2,3) = 2 \quad , \quad F(1,-5) = 4 \quad , \quad F(4,1) = 5.$$

Example 4.2. Find the semi inherited interpolation polynomial for the pairwise nodes $\{(1,0),(-1,1),(0,-2),(2,-1),(4,2)\}$ and the values $z = \{3,-4,0,2,2\}$.

In this example,

$$\begin{aligned} &f_1(x,y) = (x-x_2)(x-x_3)(x-x_4)(x-x_5)(y-y_2)(y-y_3)(y-y_4)(y-y_5), \\ &f_2(x,y) = 1, \\ &f_3(x,y) = (x-x_2)(x-x_4)(x-x_5)(y-y_2)(y-y_4)(y-y_5), \\ &f_4(x,y) = (x-x_2)(y-y_2), \\ &f_5(x,y) = (x-x_2)(x-x_4)(y-y_2)(y-y_4). \end{aligned}$$

$$A = \begin{pmatrix} 24 & 1 & 12 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -96 & -3 & -6 \\ 0 & 1 & 0 & -6 & 0 \\ 0 & 1 & 0 & 5 & 30 \end{pmatrix} = (I + BD^{-1})(D + C)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{-5}{6} & 1 \end{pmatrix} \begin{pmatrix} 24 & 1 & 12 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -96 & -3 & -6 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 30 \end{pmatrix}.$$

By solving the system $A\lambda = z$,

$$\lambda_1 = \frac{7}{36} \ , \ \lambda_2 = -4 \ , \ \lambda_3 = \frac{-1}{30} \ , \ \lambda_4 = -1 \ , \ \lambda_5 = \frac{11}{30}.$$

Therefore,

$$\begin{aligned}
F(x,y) &= \sum_{j=1}^{5} \lambda_j f_j(x,y) \\
&= -\frac{11}{5}y + \frac{671}{90}x + \frac{137}{90}x^2 + \frac{1}{30}x^3y - \frac{1}{6}x^2y \\
&\quad -\frac{14}{15}xy + \frac{887}{180}x^3y^2 - \frac{86}{45}x^2y^2 - \frac{721}{90}xy^2 - \frac{1}{5}y^2 - \frac{1}{30}x^3y^3 + \frac{1}{6}x^2y^3 \\
&\quad -\frac{1}{15}xy^3 - \frac{35}{36}x^3y^4 + \frac{7}{18}x^2y^4 + \frac{14}{9}xy^4 - \frac{35}{36}x^4y^2 + \frac{7}{36}x^4y^4 - \frac{178}{45}x^3 - \frac{4}{15}y^3 + \frac{7}{9}y^4 - \frac{14}{5}.
\end{aligned}$$

We can see

$$F(1,0) = 3 \ , \ F(-1,1) = -4 \ , \ F(0,-2) = 0 \ , \ F(2,-1) = 2 \ , \ F(4,2) = 2.$$

Example 4.3. Find the semi inherited interpolation polynomial for the pairwise nodes $\{(1,5),(1,-4),(0,0),(-2,2),(3,2)\}$ and the values $z = \{1,-4,8,2,-2\}$.

$$\begin{aligned}
&f_1(x,y) = x^4(y-y_2)(y-y_3)(y-y_4)(y-y_5), \\
&f_2(x,y) = 1, \\
&f_3(x,y) = (x-x_2)(x-x_4)(x-x_5)(y-y_2)(y-y_4)(y-y_5), \\
&f_4(x,y) = (x-x_2)(y-y_2), \\
&f_5(x,y) = (x-x_2)(x-x_4)y^2.
\end{aligned}$$

$$\begin{aligned}
A &= \begin{pmatrix} 405 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 96 & -4 & 0 \\ 0 & 1 & 0 & -18 & 0 \\ 0 & 1 & 0 & 12 & 40 \end{pmatrix} = (I + BD^{-1})(D + C) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{-2}{3} & 1 \end{pmatrix} \begin{pmatrix} 405 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 96 & -4 & 0 \\ 0 & 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 0 & 40. \end{pmatrix}.
\end{aligned}$$

By solving the system $A\lambda = z$,

$$\lambda_1 = \frac{1}{81} \ , \ \lambda_2 = -4 \ , \ \lambda_3 = \frac{1}{9} \ , \ \lambda_4 = \frac{-1}{3} \ , \ \lambda_5 = \frac{3}{20}.$$

Consequently,

$$\begin{aligned} F(x,y) &= \sum_{j=1}^{5} \lambda_j f_j(x,y) \\ &= \frac{1}{81}x^4y^4 - \frac{4}{27}x^4y^2 + \frac{16}{81}x^4y + \frac{1}{9}x^3y^3 - \frac{4}{3}x^3y + \frac{16}{9}x^3 \\ &\quad -\frac{2}{9}x^2y^3 + \frac{8}{3}x^2y - \frac{32}{9}x^2 - \frac{5}{9}xy^3 + \frac{19}{3}xy \\ &\quad -\frac{92}{9}x + \frac{2}{3}y^3 - \frac{23}{3}y + \frac{3}{20}x^2y^2 + \frac{3}{20}xy^2 - \frac{3}{10}y^2 + 8. \end{aligned}$$

We can see

$$F(1,5) = 1 \ , \ F(1,-4) = -4 \ , \ F(0,0) = 8 \ , \ F(-2,2) = 2 \ , \ F(3,2) = -2.$$

Example 4.4. Find the semi inherited bivariate interpolation polynomial for pairwise nodes $\{(-1,0),(3,0),(-5,0)\}$ and the values $z = \{1,5,7\}$.

$$\begin{aligned} &f_1(x,y) = (x-x_2)(x-x_3)(y^2-1), \\ &f_2(x,y) = 1, \\ &f_3(x,y) = (x-x_2)(y-1). \\ &A = \begin{pmatrix} 16 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & 8 \end{pmatrix} = (I+BD^{-1})(D+C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 16 & 1 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & 8. \end{pmatrix}. \end{aligned}$$

By solving the system $A\lambda = z$,

$$\lambda_1 = \frac{-5}{16} \quad , \quad \lambda_2 = 5 \quad , \quad \lambda_3 = \frac{1}{4}.$$

Hence,

$$\begin{aligned} F(x,y) &= \sum_{j=1}^{3} \lambda_j f_j(x,y) \\ &= -\frac{5}{16}x^2y^2 + \frac{5}{16}x^2 - \frac{5}{8}xy^2 + \frac{1}{4}xy + \frac{3}{8}x - \frac{75}{16}y^2 - \frac{3}{4}y + \frac{17}{16}. \end{aligned}$$

We can see

$$F(-1,0) = 1 \quad , \quad F(3,0) = 5 \quad , \quad F(-5,0) = 7.$$

This polynomial interpolates the nodes that lies in the x-axis by the bivariate function. If we eliminate the monomials that have the variable y we have

$$F(x) = \frac{5}{16}x^2 + \frac{3}{8}x + \frac{17}{16},$$

where

$$F(-1)=1 \quad , \quad F(3)=5 \quad , \quad F(-5)=7.$$

Now if we consider the basis $\{h_1,h_2,h_3\}$ in one dimensional space $\mathbf{R}$, we have

$$h_1(x)=(x-x_2)(x-x_3),$$
$$h_2(x)=1,$$
$$h_3(x)=(x-x_2).$$

$$A=\begin{pmatrix} -16 & 1 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & -8 \end{pmatrix}=(I+BD^{-1})(D+C)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} -16 & 1 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & -8. \end{pmatrix}.$$

By solving the system $A\lambda = z$,

$$\lambda_1=\frac{5}{16}, \qquad \lambda_2=5, \qquad \lambda_3=\frac{-1}{4}.$$

So the interpolation polynomial is

$$F(x)=\sum_{j=1}^{3}\lambda_j h_j(x)=\frac{5}{16}x^2+\frac{3}{8}x+\frac{17}{16}\ .$$

5 Conclusion

In this paper we offer a new method to find the unique bivariate interpolation polynomial for any set of distinct pairwise node points. We obtain the semi inherited bivariate interpolation polynomial where the bivariate interpolation matrix has the semi inherited LU factorization. Also, we notice that the semi inherited LU factorization of interpolation matrix helps us to find the solution of interpolation problem easily.

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