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ON APPROXIMATION AND GENERALIZED TYPE OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. In the present paper, we study the polynomial approximation of analytic functions of several complex variables. The characterizations of generalized type of analytic functions of several complex variables have been obtained in terms of approximation and interpolation errors.

Key words: *analytic function, Siciak extremal function, generalized type, approximation errors, imterpolation errors*

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1 Introduction

The concept of generalized order for analytic functions was given by Seremeta[3] and Janik[2]. Hence, let L^0 denote the class of functions h satisfying the following conditions:

(i) $h(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x\to\infty$;

(ii) $\lim\limits_{x\to\infty}\dfrac{h\{(1+1/\psi(x))x\}}{h(x)}=1$;

for every function $\psi(x)$ such that $\psi(x)\to\infty$ as $x\to\infty$.

Let Λ denote the class of functions h satisfying the condition (i) and

(iii) $\lim\limits_{x\to\infty}\dfrac{h(cx)}{h(x)}=1$,

for every $c>0$, that is $h(x)$ is slowly increasing.

Let K be a compact set in C^N such that the Siciak extremal function of K

$$\Phi_K(z) = \sup[|(p(z)|^{1/n} : p-\text{polynomial}, \deg p \le n, ||p||_K \le 1, n \ge 1], z \in C^N,$$

is continuous, $||.||_K$ being the sup norm on K (see [1] and [2]).

Let $g : C^N \to C, N \ge 1$, be a function analytic in $K_R = \{z \in C^N : \Phi_K(z) < R\}, R > 1$. Now for $1 < r < R$, we put $S(r,g) = \sup\{|g(z)| : \Phi_K(z) = r\}$. For $\alpha \in \Lambda$ and $\beta \in L^0$ Seremeta [4] introduced the concept of generalized order of entire functions. For $\alpha, \beta \in \Lambda$, Janik [2] defined the generalized order of analytic function $g(z)$ as

$$\rho(\alpha,\beta,g) = \limsup_{r\to R} \frac{\alpha\,[\log^+ S(r,g)]}{\beta\,[R/(R-r)]}.$$

He also obtained the characterization of $\rho(\alpha,\beta,g)$ in terms of approximation and interpolation errors.

In this note we define the generalized type of analytic function $g(z)$ and obtain the characterization of $\sigma(\alpha,\beta,\rho,g)$ in terms of approximation and interpolation errors. Thus let the functions α, β and $\gamma \in \Lambda$. Then for $0 < \rho < \infty$, we define the generalized type of analytic function $g(z)$ as

$$\sigma(\alpha,\beta,\rho,g) = \limsup_{r\to R} \frac{\alpha\,[\log^+ S(r,g)]}{\beta\,\{[\gamma\{R/(R-r)\}]^{\rho}\}}.$$

Given a function f defined and bounded on K, we put for $n = 1, 2, \cdots$

$$E_n^1(f,K) = ||f - t_n||_K;$$
$$E_n^2(f,K) = ||f - l_n||_K;$$
$$E_{n+1}^3(f,K) = ||l_{n+1} - l_n||_K;$$

where t_n denotes the n^{th} Chebyshev polynomial of the best approximation to f on K and l_n denotes the n^{th} Lagrange interpolation polynomial for f with nodes at extremal points of K (see [1] and [2]). Before proving the main result we state and prove a lemma.

Lemma 1.1. *Let $\alpha(x)$, $\beta^{-1}(x)$, $\gamma(x) \in \Lambda$ and K be a compact set in C^N such that Φ_K is locally bounded in C^N. Set $F(x,\mu,\rho) = \gamma^{-1}\{[\beta^{-1}(\mu\,\alpha(x))]^{1/\rho}\}$. Assume that for all positive numbers μ and ρ*

$$\lim_{x\to\infty} \frac{d[\log(F(x,\mu,\rho))}{d(\log x)} < 1.$$

Let $(p_n)_{n\in N}$ be a sequence of polynomials in C^N such that

(i) $\deg p_n \le n$, $n \in \mathbf{N}$;

(ii) *there exists $n_0 \in N$ and $R > 1$, such that for all $n \ge n_0$*

$$\log^+(||p_n||R^n) \le n\,\frac{\rho+1}{\rho}\,\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{\mu^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}.$$

Then $\sum_{n=0}^{\infty} p_n$ *is an analytic function and* $\sigma\left(\alpha,\beta,\rho,\sum_{n=0}^{\infty} p_n\right) \leq \mu$.

Proof. By the assumption, for all $n \geq n_0$ and $1 < r < R$, we have

$$\log^+(||p_n||r^n) \ \leq \ n\log(r/R) + n\,\frac{\rho+1}{\rho}\,\frac{1}{F(n/\rho,1/\mu,\rho+1)}.$$

Now let us consider the function

$$\phi(x) = x\log(r/R) + x\,\frac{\rho+1}{\rho}\,\frac{1}{F(x/\rho,1/\mu,\rho+1)}.$$

Now differentiating on both sides and putting $\phi^{'}(x)$ equal to zero, we get

$$\log(r/R) + \frac{\rho+1}{\rho}F^{-1}(x/\rho,1/\mu,\rho+1)$$
$$-\frac{\rho+1}{\rho}F^{-1}(x/\rho,1/\mu,\rho+1)\,\frac{d[\log F(x/\rho,1/\mu,\rho+1)]}{d(\log x)} \ = \ 0$$

or

$$F(x/\rho,1/\mu,\rho+1) = \frac{\rho+1}{\rho}\left(\frac{1-d[\log F(n/\rho,1/\mu,\rho+1)]/d(\log x)}{\log(R/r)}\right).$$

Thus the maximum of $\phi(x)$ is attained for a value of x given by

$$x^*(r) = \rho\alpha^{-1}\left[\mu\beta\left\{\left[\gamma\left\{\frac{\rho+1}{\rho}\left(\frac{1-d[\log F(n/\rho,1/\mu,\rho+1)]/d(\log x)}{\log(R/r)}\right)\right\}\right]^{(\rho+1)}\right\}\right].$$

When $r \to R$, then by using the properties of α,β,γ and the assumption of Lemma, we get

$$x^*(r) = [1+o(1)]\,\rho\alpha^{-1}[\mu\beta\{[\gamma\{R/(R-r)\}]^{\rho}\}].$$

Thus for r sufficiently close to R, we get

$$\log^+(||p_n||r^n) \ \leq \ C_1\,\alpha^{-1}[\mu\beta\{[\gamma\{R/(R-r)\}]^{\rho}\}], \tag{1}$$

where C_1 is a positive constant.

Let us write $K_R = \{z \in C^N : \Phi_K(z) < R\}$, $R > 1$, then for every polynomial p of degree $\leq n$, we have (see [1] and [2])

$$|p_n(z)| \ \leq \ ||p_n||_K \Phi_K^n(z)\ ,\ z \in C^N.$$

Therefore for every $r \in (1,R)$ the series $\sum_{n=0}^{\infty} p_n$ is convergent in every K_r, whence $\sum_{n=0}^{\infty} p_n$ is analytic in K_R. Put

$$M^*(r) = \sup\{||p_n||_K r^n : n \in \mathbf{N}\} \qquad ,\ 1 < r < R$$

and

$$\sigma^* = \lim_{r\to R}\sup \frac{\alpha\left[\log^+ M^*(r)\right]}{\beta\ \{[\gamma\{R/(R-r)\}]^\rho\}}.$$

Now on account of (1), for r sufficiently close to R, we get

$$\log^+ M^*(r) \ \le\ C_1\,\alpha^{-1}[\mu\beta\{[\gamma\{R/(R-r)\}]^\rho\}]$$

or

$$\frac{\alpha\left[C_1^{-1}\log^+ M^*(r)\right]}{\beta\ \{[\gamma\{R/(R-r)\}]^\rho\}} \ \le\ \mu.$$

Now letting $r\to R$ and using properties of α,β and γ, we get

$$\sigma^* \le \mu. \tag{2}$$

Put

$$M(r) = \sup\left\{\sum_{n=0}^{\infty} p_n(z) : n\in\mathbf{N} \text{ and } z\in K_R\right\}, \qquad 1<r<R.$$

Now for every positive $\delta<1$, we have (see [1], Lemma 2.3)

$$\log^+ M(r) \ \le\ \log^+ M^*(r^\delta R^{1-\delta}) - \log\{1-(r/R)^{1-\delta}\}$$

or

$$\frac{\alpha[\log^+ M(r)]}{\beta\ \{[\gamma\{R/(R-r)\}]^\rho\}} \ \le\ \frac{\alpha[\log^+ M^*(r^\delta R^{1-\delta}) - \log\{1-(r/R)^{1-\delta}\}]}{\beta\ \{[\gamma\{R/(R-r)\}]^\rho\}}$$

or

$$\frac{\alpha[\log^+ M(r)]}{\beta\{[\gamma\{R/(R-r)\}]^\rho\}} \ \le\ \frac{\alpha\left(\log^+ M^*(r^\delta R^{1-\delta})\left\{1-\frac{\log\{1-(r/R)^{1-\delta}\}}{\log^+ M^*(r^\delta R^{1-\delta})}\right\}\right)}{\beta\ \left\{\left[\gamma\left\{R/(R-r^\delta R^{1-\delta})\right\}\right]^\rho\right\}} \times$$
$$\times\frac{\beta\ \left\{\left[\gamma\left\{\frac{R}{(R-r^\delta R^{1-\delta})}\right\}\right]^\rho\right\}}{\beta\ \{[\gamma\{R/(R-r)\}]^\rho\}}.$$

Since $\delta<1$ is arbitrary, for r sufficiently close to R, we get

$$\frac{\alpha[\log^+ M(r)]}{\beta\ \{[\gamma\{R/(R-r)\}]^\rho\}} \ \le\ \frac{\alpha[\{1+o(1)\}\log^+ M^*(r)]}{\beta\ \{[\gamma\{R/(R-r)\}]^\rho\}}.$$

Now letting $r\to R$ and using properties of α,β and γ, we get

$$\sigma(\alpha,\beta,\rho,\sum_{n=0}^{\infty} p_n) \ \le\ \sigma^*. \tag{3}$$

Finally from (2) and (3), we get

$$\sigma(\alpha,\beta,\rho,\sum_{n=0}^{\infty} p_n) \ \le\ \mu.$$

Hence the Lemma is proved.

2 Main Result

Here we prove the following:

Theorem 2.1. *Let $\alpha(x)$, $\beta^{-1}(x)$, $\gamma(x) \in \Lambda$ and K be a compact set in C^N such that Φ_K is locally bounded in C^N. Set $F(x,\mu,\rho) = \gamma^{-1}\{[\beta^{-1}(\mu\,\alpha(x))]^{1/\rho}\}$. Assume that for all positive μ and ρ*

$$\lim_{x\to\infty} \frac{d[\log(F(x,\mu,\rho))}{d(\log x)} < 1$$

and

$$\lim_{x\to\infty} \alpha(x)^{-1}\alpha\left[\frac{x(\rho+1)}{\rho F(x/\rho,1/\mu,\rho+1)}\right] = 1.$$

Then the function f defined and bounded on K is the restriction of a function g analytic in K_R and of generalized type $\sigma(\alpha,\beta,\rho,g)$ $(0<\sigma(\alpha,\beta,\rho,g)<\infty)$ if and only if

$$\sigma(\alpha,\beta,\rho,g) = \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\log(E_n^s R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}} \;;\; s=1,2,3.$$

Proof. Let g be a function analytic in K_R. Write $\sigma = \sigma(\alpha,\beta,\rho,g)$ and

$$\eta_s = \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\log(E_n^s R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}} \;;\; s=1,2,3.$$

Here E_n^s stands for $E_n^s(g|_K,K)$, $s=1,2,3$. We claim that $\sigma=\eta_s$, $s=1,2,3$. It is known (see e.g. [5]) that

$$E_n^1 \le E_n^2 \le (n_*+2)E_n^1 \quad,\ n\ge 0, \tag{4}$$

$$E_n^3 \le 2(n_*+2)E_{n-1}^1 \quad,\ n\ge 1, \tag{5}$$

where $n_* = \begin{pmatrix} n+N \\ n \end{pmatrix}$. Using Stirling formula for the approximate value of

$$n! \approx e^{-n}n^{n+1/2}\sqrt{2\pi},$$

we get $n_* \approx \frac{n^N}{N!}$ for all large values of n. Hence for all large values of n, we have

$$E_n^1 \le E_n^2 \le \frac{n^N}{N!}[1+o(1)]E_n^1$$

and

$$E_n^3 \le 2\frac{n^N}{N!}[1+o(1)]E_n^1 .$$

Thus $\eta_3 \le \eta_2 = \eta_1$ and it suffices to prove that $\eta_1 \le \sigma \le \eta_3$. First we prove that $\eta_1 \le \sigma$. Using the definition of the generalized type, for $\mu > \sigma$ and r sufficiently close to R, we have

$$\log^+ M(r) \le \alpha^{-1}[\mu\beta\{[\gamma\{R/(R-r)\}]^\rho\}]$$

Now from Janik (see [1], Lemma 3.4), we have

$$E_n^1 \leq \frac{M(r)}{(r-1)\,r^n} \ , \ 1<r<R.$$

So for every r sufficiently close to R, we get

$$\log^+(E_n^1 R^n) \leq -\log(r-1) - n\log(r/R) + \alpha^{-1}[\mu\beta\{[\gamma\{R/(R-r)\}]^\rho\}].$$

Putting $r = r_n$, where

$$r_n = R\left[1 - \left\{F\left(\frac{n.(\rho+1)}{\rho.F(n/\rho,1/\mu,\rho+1)}, \frac{1}{\mu}, \rho\right)\right\}^{-1}\right],$$

we get

$$\begin{aligned}\log^+(E_n^1 R^n) \quad &\leq \quad -\log(r_n-1) - n\log\left[1 - \left\{F\left(\frac{n.(\rho+1)}{\rho.F(n/\rho,1/\mu,\rho+1)}, \frac{1}{\mu}, \rho\right)\right\}^{-1}\right] \\ &+ n\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{\mu^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}.\end{aligned}$$

Now using the properties of logarithm and assumption of Theorem, we get for sufficiently large value of n

$$\log^+(E_n^1 R^n) \leq C_2\, n\,\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{\mu^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1},$$

where C_2 is a positive constant.

Hence by using the properties of α, β and γ, we get

$$\frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\log(E_n^s R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}} \leq \mu.$$

Now proceeding to limits and taking *sup* on both sides, we get

$$\eta_1 \leq \mu.$$

Since $\mu > \sigma$ is arbitrary, finally we get

$$\eta_1 \leq \sigma.$$

Now we will prove that $\sigma \leq \eta_3$. Suppose that $\eta_3 < \sigma$. Then for every λ_1 , $\eta_3 < \lambda_1 < \sigma$ and n sufficiently large, we have

$$\frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\log(E_n^3 R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}} \leq \lambda_1$$

or

$$\log^+(E_n^3 R^n) \ \leq \ n\,\frac{\rho+1}{\rho}\,\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\lambda_1)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}.$$

So by the previous lemma we get $\eta_3 \leq \lambda_1$. But λ_1 has been chosen less than σ, we get a contradiction. Hence $\sigma \ \leq \ \eta_3$.

Now let f be a function defined and bounded on K and such that for $s = 1,2,3$

$$\eta_s = \lim_{n\to\infty} \sup \frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\log(E_n^s R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}}.$$

So for every $\lambda_2 > \eta_s$ and sufficiently large n, we have

$$\frac{\alpha(n/\rho)}{\beta\left\{\left[\gamma\left\{(\rho+1)\left[\rho\log(E_n^s R^n)^{1/n}\right]^{-1}\right\}\right]^{(\rho+1)}\right\}} \ \leq \ \lambda_2$$

or

$$(E_n^s R^n)^{1/n} \ \leq \ \exp\left\{\frac{\rho+1}{\rho}\,\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\lambda_2)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}\right\}.$$

Now for sufficiently large n, we get

$$(E_n^s R^n)^{1/n} \ \leq \ 1.$$

Proceeding to limits as $n\to\infty$ and taking sup on both sides, we get

$$\lim_{n\to\infty} \sup (E_n^s R^n)^{1/n} \ \leq \ 1.$$

Since $\eta_s > 0$, the sequence $(E_n^3 R^n)_{n\in\mathbf{N}}$ is unbounded, whence

$$\lim_{n\to\infty} \sup (E_n^s R^n)^{1/n} \ \geq \ 1.$$

Hence we get

$$\lim_{n\to\infty} \sup (E_n^s R^n)^{1/n} \ = \ 1.$$

So following Janik[1], Theorem 3.3, we claim that the function f can be continuously extended to an analytic function. Let us put

$$g = l_0 + \sum_{n=1}^{\infty} (l_n - l_{n-1}) \ ,$$

where $\{l_n\}$ is the sequence of Lagrange interpolation polynomials of f as defined earlier. Now we claim that g is the required continuation of f and $\sigma(\alpha,\beta,\rho,g) = \eta_s$. For every $\lambda_2 > \eta_3$ and for sufficiently large n, we have

$$\log^+(E_n^3 R^n) \ \leq \ n\,\frac{\rho+1}{\rho}\,\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\lambda_2)^{-1}\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}$$

or

$$\log^+(||l_n - l_{n-1}||R^n) \;\leq\; n\,\frac{\rho+1}{\rho}\left[\gamma^{-1}\left\{\left[\beta^{-1}\left\{(\lambda_2)^{-1}\,\alpha(n/\rho)\right\}\right]^{1/(\rho+1)}\right\}\right]^{-1}.$$

By using the previous lemma, we get $\sigma(\alpha,\beta,\rho,g) \;\leq\; \lambda_2$. Since $\lambda_2 > \eta_3$ is arbitrary, finally we get

$$\sigma(\alpha,\beta,\rho,g) \;\leq\; \eta_3.$$

Now using the inequalities (4), (5) and the proof of first part given above, we have

$$\sigma(\alpha,\beta,\rho,g) \;=\; \eta_s,$$

as claimed. This completes the proof of the Theorem.

References

[1] Janik, A., A Characterization of the Growth of Analytic Functions by Means of Polynomial Approximation, Univ. Iagel. Acta Math., 24 (1984), 295-319.

[2] Janik, A., On Approximation of Analytic Functions and Generalized Order, Ann. Polon. Math., 55 (1991), 163-167.

[3] Seremeta, M. N., On the Connection Between the Growth of a Function Analytic in a Disc and Modulie of its Taylor Series, Visnik L'viv. Derzh. Univ. Ser. Mekh. Mat., 2 (1965), 101-110.

[4] Seremeta, M. N., On the Connection Between the Growth of the Maximum Modulus of an Entire Function and the Moduli of the Coefficients of its Power Series Expansion, Amer. Math. Soc. Transl., 88:2 (1970), 291-301.

[5] Winiarski, T., Application of Approximation and Interpolation Methods to the Examination of Entire Functions of Complex Variables, Ann. Pol. Math., 28 (1973), 97-121.

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