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THE BOUNDEDNESS OF BILINEAR SINGULAR INTEGRAL OPERATORS ON SIERPINSKI GASKETS

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Abstract. In the paper we give the boundedness estimate of bilinear singular integral operators on Sierpinski gasket inspired from [1].

Key words: *boundedness, bilinear singular integral, Sierpinski gaskets*

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1 Introduction

In [1], V. Chousionis studied the boundedness of a class of singular integral operators associated with homogeneous Calder*ó*n-Zygmund standard kernels on Sierpinski gaskets. In [1], the author mainly cares about such singular integral operator

$$T_{\lambda,\varepsilon}f(x)=\int_{|x-y|>\varepsilon}\frac{\Omega_\lambda((x-y)/|x-y|)}{h_\lambda(x-y)}f(y)\mathrm{d}\mu_\lambda(y),$$

where μ_λ is the restriction of the d-dimensional Hausdorff measure on λ-Sierpinski gasket E_λ for $d_\lambda=-\dfrac{\log 3}{\log\lambda}$ and $\lambda\in(0,1/3)$, here $\Omega_\lambda(\cdot)$ is an odd function defined on the unit sphere S^1 and $h_\lambda(\cdot)$ satisfies some kind of increasing conditions. In fact, a kind of T(1) theorem[4] on Sierpinski gasket is given in [1]. As we know, in a series of papers, L. Grafakos and R.H. Torres(see[5][6][7][8] etc) gave a version of T(1) theorem for multilinear singular integrals in Euclidean space. Here naturally we have one problem: What is about the boundedness of multilinear singular integrals correponding to [1] on Sierpinski gaskets?The purpose of the paper is to give a complete answer of the problem.

In fact, Sierpinski gaskets considered in the paper with suitable metric can be seen as a space of homogenous type. The multilinear T(1) theorem on a space of homogenous type can be given directly as an extension of the case in Euclidean geometric structure. But in the process it seems that we lose some speciality of fractal structure. Thus it's necessary to study such kind of problems so that we can know deeply the speciality of some fractal structure which leads to some speciality of the boundedness of singular integral operators.

The paper is organized as follows: In section 2, we give the notion of some Sierpinski gaskets and the definition of multilinear singular integral operator, for simplicity, we only consider the bilinear operator. At last we give the statement of main theorem in the paper; In section 3, we give the proof of main theorem.

Through out the paper, the constant "C" and "c" may be different somewhere, but it is not essential.

2 Some Important Lemmas and Main Theorem

The following notations come from the corresponding part in [1]. For $\lambda \in (0,1/3)$, SG_λ can be achieved by the following similitude $s_i^\lambda : \mathbf{R}^2 \to \mathbf{R}^2 (i=1,2,3)$,

(1) $s_1^\lambda(x,y) = \lambda(x,y)$;

(2) $s_2^\lambda(x,y) = \lambda(x,y) + (1-\lambda, 0)$;

(3) $s_3^\lambda(x,y) = \lambda(x,y) + (\frac{1-\lambda}{2}, \frac{\sqrt{3}}{2}(1-\lambda)$.

For $\alpha \in I^n$, say $\alpha = (i_1,\ldots,i_n)$, define $s_\alpha^\lambda : \mathbf{R}^2 \to \mathbf{R}^2$ through the iteration

$$s_\alpha^\lambda = s_{i_1}^\lambda \circ s_{i_2}^\lambda \cdots \circ s_{i_n}^\lambda.$$

Let A be the equilateral triangle with vertices $(0,0),(1,0),(1/2,\sqrt{3}/2)$. Denote $s_\alpha^\lambda(A) = S_\alpha^\lambda$, $I^0 = \{0\}$ and $s_0^\lambda = id$. The limit set of the iteration can be given by

$$E_\lambda = \bigcap_{j\geq 0} \bigcup_{\alpha \in I^j} S_\alpha^\lambda$$

with Hausdorff dimension $d_\lambda = -\dfrac{\log 3}{\log \lambda}$. The measure μ_λ is the restriction of Hausdorff measure to E_λ, which is d_λ-AD regular, that is,

$$\mu(B(x,r)) \sim r^{d_\lambda}, \tag{2.1}$$

where $B(x,r)$ is a ball with center $x \in E_\lambda$ and $0 < r \leq 1$. Also here let $\alpha \in I^n, \beta \in I^k$, set $\beta\lfloor n = \alpha$ to denote the restriction of β in its first n coordinates α.

From the condition (2.1), it's easy to know that $(E_\lambda, \rho, \mu_\lambda)$ is a space of homogeneous type (see [9]), where $\rho(x,y) = \inf\{r > 0 : y \subset B(x,r)\}$ is a quasi-metric function associated to μ_λ. In the following, for simplicity, we denote $\rho(x,y) = |x-y|$. From [9], we know that such quasi-metric ρ is equivalent to another quasi-metric which is Lipschitz continuous with Lipschitz index less than some positive number $\delta_0 > 0$. From [3], we know the test function space on a space of homogeneous type $(E_\lambda, \rho, \mu_\lambda)$ can also be defined as follows.

Definition 2.1. *For fixed $0 < \varepsilon \le \delta_0$ and $x_0 \in E_\lambda$, a function f is said to be a test function of type $\mathcal{M}(E_\lambda)$, if*

$$|f(x)| \le \frac{c}{(1+|x-x_0|)^{d_\lambda+\varepsilon}};$$
$$|f(x)-f(x')| \le c[\frac{|x-x'|}{1+|x-x_0|}]^{\varepsilon}\frac{1}{(1+|x-x_0|)^{d_\lambda+\varepsilon}},$$
$$\text{for}|x-x'| \le (1-2\lambda)(1+|x-x_0|);$$
$$\int_{E_\lambda} f(x)\mathrm{d}\mu_\lambda(x) = 0.$$

Its dual space $\mathcal{M}'(E_\lambda)$ can be given as a replacement of the generalized function space in Euclidean space. Hence we can define $T : \mathcal{M}(E_\lambda) \times \mathcal{M}(E_\lambda) \to \mathcal{M}'(E_\lambda)$ as the following

$$T(f,g)(y_0) = \int_{E_\lambda \times E_\lambda} K_\lambda(y_0,y_1,y_2)f(y_1)g(y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2), y_0,y_1,y_2 \in E_\lambda, \tag{2.2}$$

where

$$K_\lambda(y_0,y_1,y_2) = \frac{\Omega_\lambda(\frac{(y_0-y_1,y_0-y_2)}{|(y_0-y_1,y_0-y_2)|})}{h_\lambda(|(y_0-y_1,y_0-y_2)|)},$$

and the functions Ω and h_λ satisfy some kind of conditions which we'll give later in details; the notation $|(\cdot)|$ denotes the modulo of $(\cdot)$.

Here in the paper, the key to the proof of our main results is how to define the kernel $K_\lambda(y_0,y)$. Let $\theta_{(x,y)}(x,y \in \mathbb{R}^2)$ denote the angle formed by the vector $x-y$ and the vector $(1,0)$. As in [1], for any $x,y \in E_\lambda$ and $x \ne y$, there exists some positive number ε_λ such that $\theta_{(x,y)} \in (\frac{k\pi}{3}-\varepsilon_\lambda, \frac{k\pi}{3}+\varepsilon_\lambda)$ for $k \in \{0,1,\cdots,5\}$ which divideds $[0,\pi]$ into six disjoint parts. Here let $\tilde{\theta}_{(y_0,y_1,y_2,y_3)}(y_0,y_1,y_2,y_3 \in E_\lambda)$ denote the angle $\theta_{(y_0,y_2)} \times \theta_{(y_1,y_3)}$, which means that for any $y_0,y_1,y_2,y_3 \in E_\lambda$ there exists some $\varepsilon_\lambda > 0$ such that $\tilde{\theta}_{(y_0,y_1,y_2,y_3)} \in (\frac{k\pi}{3}-\varepsilon_\lambda, \frac{k\pi}{3}+\varepsilon_\lambda) \times (\frac{j\pi}{3}-\varepsilon_\lambda, \frac{j\pi}{3}+\varepsilon_\lambda)$ for $(k,j) \in \{0,1,\ldots,5\} \times \{0,1,\ldots,5\}$. Here $\theta_{(x,y)}$ satisfies that for $\alpha \ne \beta \ne \gamma \in I^n$ and $\alpha\lfloor n = \beta\lfloor n = \gamma\lfloor n$ (see[1]).

(1) If $x \in S^\lambda_\alpha, y \in S^\lambda_\beta, z \in S^\lambda_\gamma$ and $\theta_{(x,y)} \in (\frac{k\pi}{3}-\varepsilon_\lambda, \frac{k\pi}{3}+\varepsilon_\lambda)$, then $\theta_{(x,z)} \in (\frac{m\pi}{3}-\varepsilon_\lambda, \frac{m\pi}{3}+\varepsilon_\lambda)$ for $m = (k+1)\mathrm{mod}6$ or $m = (k-1)\mathrm{mod}6$.

(2) If $x,z \in S^\lambda_\alpha$, $y \in S^\lambda_\beta$ and $\theta_{(x,y)} \in (\frac{k\pi}{3}-\varepsilon_\lambda, \frac{k\pi}{3}+\varepsilon_\lambda)$, then $\theta_{(z,y)} \in (\frac{k\pi}{3}-\varepsilon_\lambda, \frac{k\pi}{3}+\varepsilon_\lambda)$ too.

Next we give our definitions of the kernel, define C^∞ and the odd function Ω_λ on $S^1 \times S^1$ by

(1) $\Omega_\lambda(z_1,z_2) = (-1)^{k+j}$ for $\tilde{\theta}_{(z_1,z_2,0,0)} \in (\frac{k\pi}{3}-\varepsilon_\lambda, \frac{k\pi}{3}+\varepsilon_\lambda) \times (\frac{j\pi}{3}-\varepsilon_\lambda, \frac{j\pi}{3}+\varepsilon_\lambda)$ for $(k,j) \in \{0,1,\ldots,5\} \times \{0,1,\ldots,5\}$.

(2) $\Omega_\lambda(-z_1,-z_2)=-\Omega_\lambda(z_1,z_2)$ for any $(z_1,z_2)\in S^1\times S^1$.

Notice for the above definition, we conclude that $\Omega_\lambda(z_1,z_2)=\Omega_\lambda(z_2,z_1)$ for any $(z_1,z_2)\in S^1\times S^1$ immediately. Then we can define $h_\lambda(\cdot,\cdot)\in C^\infty:\mathbf{R}^+\times\mathbf{R}^+\to\mathbf{R}^+$ by

$$h_\lambda|_{[(\frac{1}{\lambda}-2)\lambda^k,\lambda^{k-1}]\times[(\frac{1}{\lambda}-2)\lambda^m,\lambda^{m-1}]}=(\lambda^{(k-1)}+\lambda^{(m-1)})^{2d_\lambda}\quad\text{for}\quad k,m\in\mathbb{N}.$$

Thus we have $h(r_1,r_2)\sim(r_1+r_2)^{2d_\lambda}$ for $0<r_1,r_2\le 1$. Then we can define

$$K_\lambda(y_0,y_1,y_2)=\frac{\Omega_\lambda(\frac{(y_0-y_1,y_0-y_2)}{|(y_0-y_1,y_0-y_2)|})}{h_\lambda(|y_0-y_1|,|y_0-y_2|)}.$$

We'll prove that there exist some constants $0<\varepsilon\le 1$ and $c>0$ such that

$$|K_\lambda(y_0,y_1,y_2)|\le\frac{c}{(\sum\limits_{k,l=0}|y_k-y_l|)^{2d_\lambda}};\tag{2.3}$$

$$\begin{aligned}&|K_\lambda(y_0,y_1,y_2)-K_\lambda(y_0',y_1,y_2)|+|K_\lambda(y_1,y_0,y_2)-K_\lambda(y_1,y_0',y_2)|\\&+|K_\lambda(y_2,y_1,y_0)-K_\lambda(y_2,y_1,y_0')|\le\frac{c|y_0-y_0'|^\varepsilon}{(\sum\limits_{k,l=0}|y_k-y_l|)^{2d_\lambda+\varepsilon}},\\&\text{for}\quad|y_0-y_0'|<(1-2\lambda)\max_{0\le j\le 2}|y_i-y_j|.\end{aligned}\tag{2.4}$$

In fact, (2.3) can be got immediately by the definition of Ω_λ and h_λ. (2.4) can be replaced by the following conditions

$$\begin{aligned}&K_\lambda(y_0,y_1,y_2)-K_\lambda(y_0',y_1,y_2)=K_\lambda(y_1,y_0,y_2)-K_\lambda(y_1,y_0',y_2)\\&=K_\lambda(y_2,y_1,y_0)-K_\lambda(y_2,y_1,y_0')=0,\\&\text{for}\quad|y_0-y_0'|<(1-2\lambda)\max_{0\le j\le 2}|y_i-y_j|.\end{aligned}\tag{2.5}$$

Now we prove that

$$\begin{aligned}&K_\lambda(y_0,y_1,y_2)-K_\lambda(y_0',y_1,y_2)=0,\\&\text{for}\quad|y_0-y_0'|<(1-2\lambda)\max_{0\le j\le 2}|y_i-y_j|.\end{aligned}\tag{2.6}$$

In fact, we set $y_0\in S_i^\lambda(S_\alpha^\lambda),y_0\in S_{i'}^\lambda(S_\alpha^\lambda)(i\ne i',\alpha\in I^v)$ and $y_0\in S_j^\lambda(S_\beta^\lambda),y_0\in S_{j'}^\lambda(S_\beta^\lambda)(j\ne j',\beta\in I^m)$, then we deduce that

$$(1/\lambda-2)\lambda^{v+1}\le|y_0-y_1|\le\lambda^v;$$
$$(1/\lambda-2)\lambda^{m+1}\le|y_0-y_2|\le\lambda^m.$$

Since

$$|y_0 - y_0'| \le (1-2\lambda) \max_{0\le j\le 2} |y_i - y_j|,$$

we have

$$|y_0 - y_0'| < (1-2\lambda)\lambda^{\nu} \quad \text{and} \quad |y_0 - y_0'| < (1-2\lambda)\lambda^{m},$$

due to

$$d(S_i^{\lambda}(S_{\alpha}^{\lambda}), S_{\tilde{i}}^{\lambda}(S_{\alpha}^{\lambda})) = (1-2\lambda)\lambda^{\nu} \quad \text{and} \quad d(S_j^{\lambda}(S_{\beta}^{\lambda}), S_{\tilde{j}}^{\lambda}(S_{\beta}^{\lambda})) = (1-2\lambda)\lambda^{m}$$

for any $i \ne \tilde{i}, j \ne \tilde{j}$, $\alpha \in I^{\nu}$ and $\beta \in I^m$. Hence we have

$$y_0' \in S_i^{\lambda}(S_{\alpha}^{\lambda}) \quad \text{and} \quad y_0' \in S_j^{\lambda}(S_{\beta}^{\lambda}).$$

Thus we have

$$\tilde{\theta}_{(y_0,y_0,y_1,y_2)}, \tilde{\theta}_{(y_0',y_0',y_1,y_2)} \in (\frac{k\pi}{3} - \varepsilon_{\lambda}, \frac{k\pi}{3} + \varepsilon_{\lambda}) \times (\frac{j\pi}{3} - \varepsilon_{\lambda}, \frac{j\pi}{3} + \varepsilon_{\lambda})$$

for some $(k, j) \in \{0,1,\dots,5\} \times \{0,1,\dots,5\}$. Therefore

$$\Omega_{\lambda}(\frac{(y_0-y_1, y_0-y_2)}{|(y_0-y_1, y_0-y_2)|}) = \Omega_{\lambda}(\frac{(y_0'-y_1, y_0-y_2)}{|(y_0-y_1, y_0-y_2)|})$$

and

$$h_{\lambda}(|y_0-y_1|, |y_0-y_2|) = h_{\lambda}(|y_0'-y_1|, |y_0'-y_2|) = (\lambda^k + \lambda^j)^{2d_{\lambda}}.$$

Hence we get (2.6) immediately. We can also prove the other equalities in (2.5) in the sameway which is even that for easier than proving (2.6). We omit the details. The proof of (2.5) is completed.

Now we give our main theorem.

Theorem 2.1. *Fixed $1 < q_1, q_2, q < \infty$ and $1/q_1 + 1/q_2 = 1/q$. Let*

$$T : \mathcal{M}(E_{\lambda}) \times \mathcal{M}(E_{\lambda}) \to \mathcal{M}'(E_{\lambda})$$

be a bilinear singular integral operator with the kernel

$$K_{\lambda}(y_0, y_1, y_2) = \frac{\Omega_{\lambda}(\frac{(y_0-y_1, y_0-y_2)}{|(y_0-y_1, y_0-y_2)|})}{h_{\lambda}(|(y_0-y_1, y_0-y_2)|)},$$

where $\Omega_{\lambda}(\cdot,\cdot)$ and $h_{\lambda}(\cdot)$ are defined as above. Then there exists a constant $c > 0$ such that

$$\|T\|_{L^{q_1}\times L^{q_2}\to L^q} \le c.$$

3 The Proof of Main Theorem

Definition 3.1. For $\delta \in (0,1]$, $x \in E_\lambda$ and $r>0$, define $A(\delta,x,r)$ to be the set of all $\phi \in \mathcal{M}$ supported in a ball $B(x,r)$, satisfying $\|\phi\|_{L^\infty} \le 1$ and

$$|\phi(x)-\phi(y)| \le r^{-\delta}|x-y|^{\delta}.$$

A singular integral operator $T:\mathcal{M}(E_\lambda)\times\mathcal{M}(E_\lambda)\to\mathcal{M}'(E_\lambda)$ is bilinear weakly bounded if there exists $\delta \in (0,\delta_0]$ and $c>0$ such that for all $x \in E_\lambda$, $r>0$, and $\phi,\varphi,\psi \in A(\delta,x,r)$,

$$|<T(\phi,\varphi),\psi>| \le c\mu(B(x,r)).$$

To prove our main theorem, we need the following theorem

Theorem 3.1. *Fixed $1<q_1,q_2,q<\infty$ and $1/q_1+1/q_2=1/q$. A bilinear singular integral $T:\mathcal{M}(E_\lambda)\times\mathcal{M}(E_\lambda)\to\mathcal{M}'(E_\lambda)$ with kernel $K_\lambda(y_0,y_1,y_2)$ satisfying (2.3),(2.4), for any $f,g,h \in \mathcal{M}$ such that*

$$\begin{aligned} &<T(1,g),h>=<T(f,1),h>=<T(f,g),1> \\ =\ &<T(1,1),h>=<T(1,g),1>=<T(f,1),1>=0 \end{aligned} \tag{3.1}$$

and T is also bilinear weakly bounded. Then there exists a constant $c>0$ such that

$$\|T\|_{L^{q_1}\times L^{q_2}\to L^q} \le c.$$

Proof. The theorem is a direct extension of the bilinear T(1) theorem in [5]-[8] on the space of homogeneous type. Similar discussion can be found in [10].

Remark 3.1. (3.1) makes sense in the norm of $\mathcal{M}'(E_\lambda)$. Related details about (3.1) can be found in [2].

Proof of Theorem 2.1. From the definition of the kernel K_λ, we see that T satisfies bilinear weakly bounded properties immediately, since for any $f,g,h \in A(\delta,x,r)$

$$\begin{aligned} <T(f,g),h>=\frac{1}{2}\Big(&\int_{E_\lambda}\int_{E_\lambda}\int_{E_\lambda} K_\lambda(y_0,y_1,y_2)f(y_1)g(y_2) \\ &(h(y_0)-h(\tfrac{y_1+y_2}{2}))\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2)\mathrm{d}\mu_\lambda(y_0)\Big). \end{aligned} \tag{3.2}$$

We consider the following truncated singular integral operator for $f,g \in \mathcal{M}$ and $n \in \mathbf{N}$

$$T_\lambda^n(f,g)(y_0)=\int_{|y_0-y_1|>\lambda^n}\int_{|y_0-y_2|>\lambda^n} K_\lambda(y_0,y_1,y_2)f(y_1)g(y_2)d\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2). \tag{3.3}$$

We prove that such truncated integral operator satisfies (3.1). Since the proof of the other equalities in (3.1) are simpler or similar, it's sufficient to verify that

$$T_\lambda^n(1,1)=0 \quad \text{for all} \quad n \in \mathbf{N}. \tag{3.4}$$

Inspired by the idea in [1], we can also prove it by induction.

The first step: for $n=1$, let $y_0 \in S_i^\lambda$ for $i \in I$, then $j \neq k$ and $j,k \in I$, we have

$$\begin{aligned}
T_\lambda^1(1,1)(y_0) &= \int_{|y_0-y_1|>\lambda}\int_{|y_0-y_2|>\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2)\\
&= \int_{S_j^\lambda \cup S_k^\lambda}\int_{S_j^\lambda \cup S_k^\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2)\\
&= \int_{S_j^\lambda \times S_j^\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)d\mu_\lambda(y_2) + \int_{S_j^\lambda \times S_k^\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2)\\
&+ \int_{S_k^\lambda \times S_j^\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2) + \int_{S_k^\lambda \times S_k^\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2).
\end{aligned}$$

From the definition of $K_\lambda(y_0,y_1,y_2)$, we know that

(1) For $(y_0,y_1) \in S_j^\lambda \times S_k^\lambda$, $\Omega_\lambda(\frac{(y_0-y_1,y_0-y_2)}{|(y_0-y_1,y_0-y_2)|}) = (-1)^{m+m+1}$;

(2) For $(y_0,y_1) \in S_j^\lambda \times S_j^\lambda$, $\Omega_\lambda(\frac{(y_0-y_1,y_0-y_2)}{|(y_0-y_1,y_0-y_2)|}) = (-1)^{m+m}$;

(3) For $(y_0,y_1) \in S_k^\lambda \times S_j^\lambda$, $\Omega_\lambda(\frac{(y_0-y_1,y_0-y_2)}{|(y_0-y_1,y_0-y_2)|}) = (-1)^{m+m+1}$;

(4) For $(y_0,y_1) \in S_k^\lambda \times S_k^\lambda$, $\Omega_\lambda(\frac{(y_0-y_1,y_0-y_2)}{|(y_0-y_1,y_0-y_2)|}) = (-1)^{m+m+2}$
for some $m \in \{0,1,\cdots,5\}$.

Moreover, $h_\lambda(|y_0-y_1|,|y_0-y_2|) = 1$ due to $1-2\lambda \leq |y_0-y_1| \leq 1$ and $1-2\lambda \leq |y_0-y_2| \leq 1$.Hence we have

$$\begin{aligned}
&T_\lambda^1(1,1) = (-1)^{2m+1}\mu_\lambda(S_j^\lambda \times S_k^\lambda) + (-1)^{2m}\mu_\lambda(S_j^\lambda \times S_j^\lambda)\\
+\quad &(-1)^{2m+1}\mu_\lambda(S_k^\lambda \times S_j^\lambda) + (-1)^{2m+2}\mu_\lambda(S_k^\lambda \times S_k^\lambda) = 0.
\end{aligned}$$

The second step: we suppose that $T_\lambda^n(1,1) = 0$ firstly. Now it's sufficient to prove that $T_\lambda^{n+1}(1,1) = 0$. Let $y_0 \in S_\alpha^\lambda$ for some $\alpha = (i_1,i_2,\ldots,i_n,i_{n+1}) \in I^{n+1}$. $\beta = (i_1,i_2,\ldots,i_n,j)$ and $\gamma = (i_1,i_2,\ldots,i_n,k)$ for $j,k \in I \backslash \{i_{n+1}\}$, $j \neq k$,

$$T_\lambda^{n+1}(1,1)(y_0) = \int_{|y_0-y_1|>\lambda^{n+1}}\int_{|y_0-y_2|>\lambda^{n+1}} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2).$$

Then we have

$$\begin{aligned}
T_\lambda^{n+1}(1,1)(y_0) &= \int_{|y_0-y_1|>\lambda^n}\int_{|y_0-y_2|>\lambda^n} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2)\\
&+ \int_{|y_0-y_1|>\lambda^n}\int_{y_2\in S_\alpha^\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2) + \int_{|y_0-y_1|>\lambda^n}\int_{y_2\in S_\beta^\lambda}\\
&\quad K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2))\\
&+ \int_{|y_0-y_2|>\lambda^n}\int_{y_1\in S_\alpha^\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2) + \int_{|y_0-y_2|>\lambda^n}\int_{y_1\in S_\beta^\lambda}\\
&\quad K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2))\\
&+ \int_{y_1\in S_\alpha^\lambda}\int_{y_2\in S_\alpha^\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2) + \int_{y_1\in S_\alpha^\lambda}\int_{y_2\in S_\beta^\lambda}\\
&\quad K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2))\\
&+ \int_{y_1\in S_\beta^\lambda}\int_{y_2\in S_\beta^\lambda} K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2) + \int_{y_1\in S_\beta^\lambda}\int_{y_2\in S_\alpha^\lambda}\\
&\quad K_\lambda(y_0,y_1,y_2)\mathrm{d}\mu_\lambda(y_1)\mathrm{d}\mu_\lambda(y_2))\\
&= I+II+III+IV+V+VI+VII+VIII+VIIII.
\end{aligned}$$

$I=0$ can be obtained by the assumption. $II+III=0, IV+V=0, VI+VII+VIII+VIIII=0$ can be got by the similar method in the first step. Thus we obtain

$$T_\lambda^{n+1}(1,1)=0.$$

We can also obtain that $\{T_\lambda^n\}$ is bilinear weakly bounded by using similar argument to (3.2). Meanwhile the kernels of $\{T_\lambda^n\}$ also satisfy (2.3),(2.4). Hence we obtain that $\|T_\lambda^n\|_{L^{q_1}\times L^{q_2}\to L^q}\le c$ uniformly. By using the standard dense argument, we can obtain that for any $\phi,\varphi\in\mathcal{M}(E_\lambda)$, $\{T_\lambda^n(\phi,\varphi)\}$ is a Cauchy sequence in the norm of L^q. Hence by using the standard dense argument, we obtain that for any $f\in L^{q_1}, g\in L^{q_2}$,

$$\lim_{n\to\infty} T_\lambda^n(f,g)\to T(f,g)$$

in the norm of L^q. Then we obtain the following inequality immediately

$$\|T\|_{L^{q_1}\times L^{q_2}\to L^q}\le c.$$

Remark 3.2. In [1], V. Chousionis also gave the proof of the divergence of principle value of the singular integral operator. Thus there's no need to consider such a question in the bilinear case. Moreover, actually in [1], he also considered the boundedness of the maximal singular integral. We can also get similar results by using the bilinear Cotlar inequality as a direct corollary of our main theorem. Hence we have given the answer to our problem completely.

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