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BOUNDEDNESS OF PARABOLIC SINGULAR INTEGRALS AND MARCINKIEWICZ INTEGRALS ON TRIEBEL-LIZORKIN SPACES

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Abstract. In this paper, we obtain the boundedness of the parabolic singular integral operator T with kernel in $L(\log L)^{1/\gamma}(S^{n-1})$ on Triebel-Lizorkin spaces. Moreover, we prove the boundedness of a class of Marcinkiewicz integrals $\mu_{\Omega,q}(f)$ from $\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}$ into $L^p(\mathbf{R}^n)$.

Key words: *parabolic singular integral, Triebel-Lizorkin space, Marcinkiewica integral, rough kernel*

AMS (2010) subject classification: 42B25, 42B35

1 Introduction

Let S^{n-1} denote the unit sphere on the n-dimension Euclidean space $\mathbf{R}^n$ and $\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq 1$ be fixed real numbers. For each fixed $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$, the function

$$F(x,\rho) = \sum_{i=1}^{n} \frac{x_i^2}{\rho^{2\beta_i}}$$

is strictly decreasing of $\rho > 0$. Therefore, there exists a unique $\rho = \rho(x)$ such that $F(x,\rho) = 1$. Define $\rho(x) = t$ and $\rho(0) = 0$. It is proved in [10] that ρ is a metric on $\mathbf{R}^n$ and $(\mathbf{R}^n, \rho)$ is called the mixed homogeneity space related to $\{\beta_i\}_{i=1}^n$. For any $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$, let

$$x_1 = \rho^{\beta_1} \cos\varphi_1 \ldots \cos\varphi_{n-2} \cos\varphi_{n-1},$$

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$$x_2 = \rho^{\beta_2} \cos\varphi_1 \ldots \cos\varphi_{n-2} \sin\varphi_{n-1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_{n-1} = \rho^{\beta_{n}-1} \cos\varphi_1 \sin\varphi_2,$$
$$x_n = \rho^{\beta_n} \sin\varphi_1.$$

Then $dx = \rho^{\beta-1}J(x')d\rho d\sigma$, where $\beta = \sum_{i=1}^{n}\beta_i, x' \in S^{n-1}$, and $\rho^{\beta-1}J(x')$ is the Jacobian of the above transform. In [10] Fabes and Rivière pointed out that $J(x')$ is a C^∞ function on S^{n-1}, and $1 \le J(x') \le M$. For $\lambda > 0$, let $B_\lambda = \mathrm{diag}[\lambda^{\beta_1}, \cdots, \lambda^{\beta_n}]$ be a diagonal matrix. We say a real valued measurable function $\Omega(x)$ is homogeneous of degree zero with respect to B_λ if for any $\lambda > 0$ and $x \in \mathbf{R}^n$

$$\Omega(B_\lambda x) = \Omega(x). \tag{1.1}$$

Moreover, we assume that $\Omega(x)$ satisfies the condition

$$\int_{S^{n-1}} \Omega(x')J(x')\mathrm{d}\sigma(x') = 0. \tag{1.2}$$

Let $\alpha > 0$ and

$$L(\log L)^\alpha(S^{n-1}) = \left\{\Omega : \int_{S^{n-1}} |\Omega(y')| \log^\alpha(2+|\Omega(y')|)\mathrm{d}\sigma(y') < \infty\right\}.$$

It is well known that the following relations hold:

$$L^q(S^{n-1})(q>1) \subseteq L\log^+ L(S^{n-1}) \subseteq H^1(S^{n-1}) \subseteq L^1(S^{n-1}),$$

$$L(\log L)^\beta(S^{n-1}) \subseteq L(\log L)^\alpha(S^{n-1}), 0 < \alpha < \beta,$$
$$L(\log L)^\alpha(S^{n-1}) \subseteq H^1(S^{n-1}), \ \ \alpha \ge 1,$$

where $H^1(S^{n-1})$ is the Hardy space on the unit sphere. While

$$L(\log L)^\alpha(S^{n-1}) \not\subseteq H^1(S^{n-1}) \not\subseteq L(\log L)^\alpha(S^{n-1}), \qquad 0 < \alpha < 1.$$

For $\gamma \ge 1$, let $\Delta_\gamma(\mathbf{R}^+)$ be the set of all measurable functions h on $\mathbf{R}^+$ satisfying the condition

$$\sup_{R>0}\left(R^{-1}\int_0^R |h(t)|^\gamma \mathrm{d}t\right)^{1/\gamma} < \infty,$$

and $\Delta_\infty(\mathbf{R}^+) = L^\infty(\mathbf{R}^+)$. Also, define $H_\gamma(\mathbf{R}^+)$ to be the set of all measurable functions h on $\mathbf{R}^+$ satisfying the condition

$$\|h\|_{L^\gamma(\mathbf{R}^+, \frac{\mathrm{d}t}{t})} = \left(\int_{\mathbf{R}^+} |h(t)|^\gamma \frac{\mathrm{d}t}{t}\right)^{1/\gamma} \le 1,$$

and define $H_\infty(\mathbf{R}^+) = L^\infty(\mathbf{R}^+, \frac{\mathrm{d}t}{t})$. It is easy to verify that $H_\gamma(\mathbf{R}^+) \subseteq \Delta_\gamma(\mathbf{R}^+)$ and $H_\infty(\mathbf{R}^+) = \Delta_\infty(\mathbf{R}^+)$ when $1 < \gamma < \infty$. The parabolic singular integral operators are defined by

$$\tilde{T}f(x) = \mathrm{p.v.} \int_{\mathbf{R}^n} \frac{\Omega(y)}{\rho(y)^\beta} f(x-y)\mathrm{d}y. \tag{1.3}$$

$$Tf(x) = \mathrm{p.v.} \int_{\mathbf{R}^n} \frac{h(\rho(y))\Omega(y)}{\rho(y)^\beta} f(x-y)\mathrm{d}y. \tag{1.4}$$

Notice that if $\beta_1 = \beta_2 = \cdots = \beta_n = 1$, then $\rho(x) = |x|$ and $(\mathbf{R}^n, \rho) = (\mathbf{R}^n, |\cdot|)$. In this case, the operator $\tilde{T}$ is the classical singular integral operator of convolution type and whose boundedness in various function spaces has been well-studied by many authors, see [3,6,8,11,13,15,18]. Nagel and Rivière proved in [10] that if $\Omega \in C^1(S^{n-1})$ and $h \equiv 1$, then the parabolic singular integral operator T is bounded on $L^p(\mathbf{R}^n)$. Later in [14], Nagel and Rivière further improved their result by assuming a weaker condition $\Omega \in L\log^+ L(S^{n-1})$. Recently, Hung Viet Le showed in [12] that T defined in (1.4) is bounded on the homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ if $\Omega \in L(\log L)^+(S^{n-1})$ and $h \in \Delta_{\tilde{q}}(\mathbf{R}^+)$. The following theorem was proved in [12].

Theorem A. *Let* $\tilde{q} = \max\{2, q'\}$ *and* $h \in \Delta_{\tilde{q}}(\mathbf{R}^+)$. *Suppose* $\Omega \in L(\log L)^+(S^{n-1})$ *with conditions* (1.1) *and* (1.2). *Then*

$$\|Tf\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)} \le C\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)}, \quad \text{for} \ \ \alpha \in \mathbf{R}, 1 < p, q < \infty.$$

The aim of this paper is to give the boundedness on $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ for the parabolic singular integral operator T under an other weaker condition. Our main results can be stated as follows.

Theorem 1.1. *Let* $h \in H_{\gamma'}(\mathbf{R}^+)$ *and* T *be defined as in (1.4). Suppose* $\Omega \in L(\log L)^{1/\gamma}(S^{n-1})$ *with conditions* (1.1) *and* (1.2). *Then*

(i) *If* $1 < \gamma < 2$, *then*

$$\|Tf\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)} \le C\|\Omega\|_{L(\log L)^{1/\gamma}(S^{n-1})}\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)}, \text{ for } \alpha \in \mathbf{R}, 1 < p < \infty, \gamma < q < \gamma'.$$

In particular, if $\gamma = 2$, *then*

$$\|Tf\|_{\dot{F}_p^{\alpha,2}(\mathbf{R}^n)} \le C\|\Omega\|_{L(\log L)^{1/2}(S^{n-1})}\|f\|_{\dot{F}_p^{\alpha,2}(\mathbf{R}^n)}, \text{ for } \alpha \in \mathbf{R}, 1 < p < \infty.$$

(ii) *If* $2 < \gamma < \infty$, *then*

$$\|Tf\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)} \le C\|\Omega\|_{L(\log L)^{1/\gamma}(S^{n-1})}\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)}, \text{ for } \alpha \in \mathbf{R}, 1 < p < \infty, \gamma' < q < \gamma.$$

Remark 1. It is not hard to verify $H_{\gamma'}(\mathbf{R}^+) \not\subseteq \Delta_2(\mathbf{R}^+)$ as $2 < \gamma < \infty$, therefore our result is new in the case $2 < \gamma < \infty$. It must be pointed out that some ideas for proving Theorem 1.1

follow from [12]. However, the reader will find that some techniques and estimates in our proof are different from those in [12].

Finally, we also obtain the boundedness for a class of Marcinkiewicz functions $\mu_{\Omega,q}(f)$ under a much weaker condition on Ω, where $\mu_{\Omega,q}(f)$ is defined by

$$\mu_{\Omega,q}(f)(x)=\left(\int_0^\infty |F_\Omega(x,t)|^q\frac{\mathrm{d}t}{t^{q+1}}\right)^{1/q}, \tag{1.5}$$

and

$$F_\Omega(x,t)=\int_{|y|\le t}\frac{h(|y|)\Omega(y)}{|y|^{n-1}}f(x-y)\mathrm{d}y.$$

Moreover, we know if $q=2$ and $h\equiv 1$ then $\mu_\Omega(f)\equiv\mu_{\Omega,q}(f)$ is the classical Marcinkiewicz integral which is first defined by Stein in [17] and the boundedness of $\mu_\Omega(f)$ has been well-studied by many authors in various function spaces in the literature, see [2,4,9,19]. We have the following result.

Theorem 1.2. *Let $\alpha\in\mathbf{R}$ and $\mu_{\Omega,q}(f)$ be defined as in (1.5). Suppose $\Omega\in L(\log L)^{1/\gamma}(S^{n-1})$ and satisfies conditions (1.1) with ($\beta_1=\ldots=\beta_n=1$) and (1.2) with($J(x')\equiv 1$). Then*

(i) *If $1<\gamma<2$ and $h\in\Delta_{\gamma'}(\mathbf{R}^+)$, we have*

$$\|\mu_{\Omega,q}(f)\|_p\le C\|\Omega\|_{L(\log L)^{1/\gamma}(S^{n-1})}\|f\|_{\dot F_p^{0,q}(\mathbf{R}^n)},\ \text{for } 1<p<\infty,\gamma<q<\gamma'.$$

In particular, if $\gamma=2$ and $h\in\Delta_2(\mathbf{R}^+)$, we have

$$\|\mu_{\Omega,2}(f)\|_p\le C\|\Omega\|_{L(\log L)^{1/2}(S^{n-1})}\|f\|_p\ ,\ \text{for } 1<p<\infty.$$

(ii) *If $2<\gamma<\infty$ and $h\in H_{\gamma'}(\mathbf{R}^+)$, we have*

$$\|\mu_{\Omega,q}(f)\|_p\le C\|\Omega\|_{L(\log L)^{1/\gamma}(S^{n-1})}\|f\|_{\dot F_p^{0,q}(\mathbf{R}^n)},\ \text{for } 1<p<\infty,\gamma'<q<\gamma.$$

Remark 2. As $h\equiv 1$ and $\gamma=2$, our results is consistent with the results in [2].

Throughout this paper, C denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. For $1\le\gamma\le\infty$, γ' denotes the conjugate index of γ, i.e., $1/\gamma+1/\gamma'=1$.

2 Proofs of Theorems

Before proving the theorems, we recall the definition of the Triebel-Lizorkin space. Fix a radial Schwartz function $\phi(\xi)\in S(\mathbf{R}^n)$ such that supp $\hat\phi\subset\{\xi\in\mathbf{R}^n:1/2\le|\xi|\le 2\}$, $\hat\phi(\xi)\ge 0$, $\hat\phi(\xi)\ge c>0$ if $3/5\le|\xi|\le 5/3$. Denote $\hat\phi_t(\xi)=\hat\phi(t\xi),t\in\mathbf{R}$, so that $\phi_t(x)=t^{-n}\phi(\frac{x}{t}),x\in\mathbf{R}^n$.

For $\alpha \in \mathbf{R}$, $0 \le p < \infty$, and $1 \le q \le \infty$, the homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ is defined by

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)} = \left\| \left(\int_0^\infty |t^{-\alpha}\phi_t * f(x)|^q \frac{\mathrm{d}t}{t} \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)}, \tag{1.7}$$

with the usual modification if $q = \infty$. It is well known that $\dot{F}_p^{\alpha,q}(\mathbf{R}^n)$ is a unified setting of many well-known function spaces including Lebesgue spaces $L^p(\mathbf{R}^n)$, Hardy spaces $H^p(\mathbf{R}^n)$, and Sobolev spaces $L^p_\alpha(\mathbf{R}^n)$. Thus considering the boundedness of the operators on Triebel-Lizorkin space is of great meaningful.

Proof of Theorem 1.1. Let $\Omega \in L(\log L)^{1/\gamma}(S^{n-1})$. Following the idea in [1], we proceed as follows. For $m \in \mathbf{N}$, let $E_0 = \{x' \in S^{n-1} : |\Omega(x')| < 2\}$, $E_m = \{x' \in S^{n-1} : 2^m \le |\Omega(x')| < 2^{m+1}\}$, and $\tilde{\Omega}_m(x') = \Omega(x')\chi_{E_m}(x')$. Denote $\lambda_0 = 1$, $\lambda_m = \|\tilde{\Omega}_m\|_1$, $\mathbf{I} = \{m \in \mathbf{N} : \lambda_m \ge 2^{-2m}\}$, and $\delta = \int_{S^{n-1}} J(y')d\sigma(y')$. Define the sequence of functions $\{\Omega_m\}_{m\in \mathbf{I}\bigcup\{0\}}$ by

$$\Omega_0(x') = \sum_{m\in\{0\}\bigcup(\mathbf{N}-\mathbf{I})} \tilde{\Omega}_m(x') - \frac{1}{\delta} \sum_{m\in\{0\}\bigcup(\mathbf{N}-\mathbf{I})} \left(\int_{S^{n-1}} \tilde{\Omega}_m(x')J(x')\mathrm{d}\sigma(x') \right),$$

$$\Omega_m(x') = (\lambda_m)^{-1}\left(\tilde{\Omega}_m(x') - \frac{\int_{S^{n-1}} \tilde{\Omega}_m(x')J(x')d\sigma(x')}{\delta} \right) \text{ for } m \in \mathbf{I}.$$

Denote $a_m = 2^m$, then it is not hard to verify that

$$\|\Omega_m\|_{L^2(S^{n-1})} \le Ca_m^2, \quad \|\Omega_m\|_{L^1(S^{n-1})} \le C, \quad \int_{S^{n-1}} \Omega_m(x')J(x')\mathrm{d}\sigma(x') = 0,$$

$$\Omega(x') = \sum_{m\in I\bigcup\{0\}} \lambda_m \Omega_m(x'), \tag{2.1}$$

and

$$\sum_{m\in I\bigcup\{0\}} (m+1)^{1/\gamma}\lambda_m \le C\|\Omega\|_{L(\log L)^{1/\gamma}(S^{n-1})}. \tag{2.2}$$

Therefore, we have

$$Tf(x) = \sum_{m\in I\bigcup\{0\}} \lambda_m \int_{\mathbf{R}^n} \frac{h(\rho(y))\Omega_m(y')}{\rho(y)^\beta} f(x-y)\mathrm{d}y := \sum_{m\in I\bigcup\{0\}} \lambda_m T_m f(x), \tag{2.3}$$

where

$$T_m f(x) = p.v. \int_{\mathbf{R}^n} \frac{h(\rho(y))\Omega_m(y')}{\rho(y)^\beta} f(x-y)\mathrm{d}y.$$

Now, we choose a real-valued function $\phi(\xi) \in S(\mathbf{R}^n)$ such that $\mathrm{supp}\hat{\phi} \subset \{\xi \in \mathbf{R}^n : 1/2 \le \rho(\xi) \le 2\}$, $0 \le \hat{\phi}(\rho(\xi)) \le 1$, $\hat{\phi}(\rho(\xi)) \ge c > 0$ as $3/5 \le \rho(\xi) \le 5/3$, and $\int_{-\infty}^{\infty} |\hat{\phi}^2(2^t(\rho(\xi))dt = 1$ for

$\xi \neq 0$. Define ψ on $\mathbf{R}^n$ by $\hat{\psi}_{2^t}(\xi) = \hat{\phi}(2^t\rho(\xi))$. Denote $S_{2^t} = \psi_{2^t} * f$, then for $f \in S(\mathbf{R}^n)$, $f = m\int_{\mathbf{R}} S_{2^{mt}}(S_{2^{mt}} f)dt$ for any fixed $m \in \mathbf{N}$. Thus for $f \in S(\mathbf{R}^n)$ and each fixed $x \in \mathbf{R}^n$, we have

$$\begin{aligned} T_m f(x) &= \int_{\mathbf{R}^n} \frac{h(\rho(y))\Omega_m(y')}{\rho(y)^\beta} f(x-y)\mathrm{d}y \\ &= \int_{\mathbf{R}}\int_{\mathbf{R}^n} \frac{h(\rho(y))\Omega_m(y')}{\rho(y)^\beta} \chi_{2^{mt}}(\rho(y)) f(x-y)\mathrm{d}y\mathrm{d}t \\ &:= \int_{\mathbf{R}} \sigma_{2^{mt}} * f(x)\mathrm{d}t, \end{aligned}$$

where $\sigma_{2^{mt}}(y) = \dfrac{h(\rho(y))\Omega_m(y')}{\rho(y)^\beta}\chi_{2^{mt}}(\rho(y))$ and $\chi_{2^{mt}}(\rho(y))$ is the characteristic function of the set $\{y \in \mathbf{R}^n : 2^{mt} \leq \rho(y) < 2^{m(t+1)}\}$. Thus we write

$$\begin{aligned} T_m f(x) &= m\int_{\mathbf{R}} \sigma_{2^{mt}} * \left(\int_{\mathbf{R}} S_{2^{m(t+s)}} S_{2^{m(t+s)}} f \mathrm{d}s\right)\mathrm{d}t \\ &= m\int_{\mathbf{R}}\int_{\mathbf{R}} S_{2^{m(t+s)}}(\sigma_{2^{mt}} * S_{2^{m(t+s)}} f)\mathrm{d}t\mathrm{d}s \\ &:= \int_{\mathbf{R}} T_{m,s} f \mathrm{d}s, \end{aligned}$$

where

$$T_{m,s} f = m\int_{\mathbf{R}} S_{2^{m(t+s)}}(\sigma_{2^{mt}} * S_{2^{m(t+s)}} f)\mathrm{d}t. \tag{2.4}$$

We claim that if $m \in \mathbf{I}$, then

$$\|T_m(f)\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)} \leq C m^{1/\gamma} \|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)}. \tag{2.5}$$

On the other hand, we can also prove

$$\|T_0(f)\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)} \leq C \|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)}. \tag{2.6}$$

In fact, the proof of the inequality (2.6) is essentially the same as (2.5) only fixed $m = 0$ and replace Ω_m by Ω_0 in the proof of (2.5). Therefore we only need to prove (2.5). To do this, we need the following lemmas.

Lemma 2.1[6]. Suppose $\lambda_j' s$ and $\alpha_j' s$ are fixed numbers and $\Gamma(t) = (\lambda_1 t^{\alpha_1}, \ldots \lambda_n t^{\alpha_n})$ is a function from $\mathbf{R}_+$ to $\mathbf{R}^n$. The maximal function associated to the homogeneous curve Γ is defined by

$$\mathrm{M}_\Gamma(f)(x) = \sup_{h>0} \frac{1}{h}\left|\int_0^h f(x-\Gamma(t))\mathrm{d}t\right|.$$

Then, for $1 < p < \infty$, there is a constant C independent of $\lambda_j' s$, $\alpha_j' s$, and f such that

$$\|\mathrm{M}_\Gamma(f)\|_{L^p} \leq C\|f\|_{L^p}.$$

Lemma 2.2[7]. Let $0\le\delta\le 1$ and L denote the distinct numbers of $\{\beta_i\}$. Then

$$\left|\int_1^2 e^{-i<B_\lambda x,y>}\frac{\mathrm{d}\lambda}{\lambda}\right|\le C|<x,y>|^{-\delta/L},\ \text{for any}\ \ x,\ y\in\mathbf{R}^n,$$

where $C>0$ is independent of x and y.

Lemma 2.3. Let $\sigma_{2^{mt}}(y)=\dfrac{h(\rho(y))\Omega_m(y')}{\rho(y)^\beta}\chi_{2^{mt}}(\rho(y))$, $h\in H_{\gamma}(\mathbf{R}^+)$, $1<\gamma<\infty$. Then

$$|\hat{\sigma}_{2^{mt}}(\xi)|\le Cm^{1/\gamma}\|\Omega_m\|_1|B_{2^{m(t+1)}}\xi|, \tag{2.7}$$

$$|\hat{\sigma}_{2^{mt}}(\xi)|\le Cm^{1/\gamma}|B_{2^{mt}}\xi|^{-\varepsilon/m}, \tag{2.8}$$

where $\varepsilon=2\delta/\gamma L$ if $2\le\gamma<\infty$ and $\varepsilon=\delta/L$ if $1<\gamma<2$.

Proof. By the vanishing moment of Ω_m and $J(x')\in C_0^\infty$, we have

$$\begin{aligned}
|\hat{\sigma}_{2^{mt}}(\xi)| &= \left|\int_{\{y\in\mathbf{R}^n:2^{mt}\le\rho(y)<2^{m(t+1)}\}}\frac{e^{-2\pi i\xi\cdot y}h(\rho(y))\Omega_m(y')}{)}\rho(y)^\beta\mathrm{d}y\right|\\
&\le \int_{2^{mt}}^{2^{m(t+1)}}\int_{S^{n-1}}\left|(e^{-2\pi i\xi\cdot B_\rho y'}-1)\Omega_m(y')J(y')\right|\mathrm{d}\sigma(y')|h(\rho)|\frac{\mathrm{d}\rho}{\rho}\\
&\le C\|J\|_\infty|B_{2^{m(t+1)}}\xi|\|\Omega_m\|_1\int_{2^{mt}}^{2^{m(t+1)}}|h(\rho)|\frac{\mathrm{d}\rho}{\rho}\\
&\le C|B_{2^{m(t+1)}}\xi|\|\Omega_m\|_1\left(\int_{2^{mt}}^{2^{m(t+1)}}|h(\rho)|^{\gamma'}\frac{\mathrm{d}\rho}{\rho}\right)^{1/\gamma'}\left(\int_{2^{mt}}^{2^{m(t+1)}}\frac{d\rho}{\rho}\right)^{1/\gamma}\\
&\le Cm^{1/\gamma}\|\Omega_m\|_1|B_{2^{m(t+1)}}\xi|.
\end{aligned}$$

Thus (2.7) holds. On the other hand, noticing that

$$\left|\int_{S^{n-1}}e^{-2\pi i\xi\cdot B_\rho y'}\Omega_m(y')J(y')\mathrm{d}\sigma(y')\right|\le\|J\|_\infty\|\Omega_m\|_1,$$

we have

$$\begin{aligned}
|\hat{\sigma}_{2^{mt}}(\xi)| &\le \left(\int_{2^{mt}}^{2^{m(t+1)}}|h(\rho)|^{\gamma'}\frac{\mathrm{d}\rho}{\rho}\right)^{1/\gamma'}\left(\int_{2^{mt}}^{2^{m(t+1)}}\left|\int_{S^{n-1}}e^{-2\pi i\xi\cdot B_\rho y'}\Omega_m(y')J(y')\mathrm{d}\sigma(y')\right|^\gamma\frac{\mathrm{d}\rho}{\rho}\right)^{1/\gamma}\\
&\le C\|J\|_\infty\|\Omega_m\|_1\left(\int_{2^{mt}}^{2^{m(t+1)}}\frac{\mathrm{d}\rho}{\rho}\right)^{1/\gamma}.
\end{aligned}$$

Thus,

$$|\hat{\sigma}_{2^{mt}}(\xi)|\le Cm^{1/\gamma}.\ \text{for}\ \ 1<\gamma<\infty. \tag{2.9}$$

Set

$$R_{m,t}(\xi)=\int_{2^{mt}}^{2^{m(t+1)}}\left|\int_{S^{n-1}}e^{-2\pi i<B_\rho y',\xi>}\Omega_m(y')J(y')\mathrm{d}\sigma(y')\right|^2\frac{\mathrm{d}\rho}{\rho},$$

then we have

$$R_{m,t}(\xi)=\int\int_{S^{n-1}\times S^{n-1}}\Omega_m(y')\overline{\Omega_m(x')}|I_{m,t}(\xi)|J(y')J(x')\mathrm{d}\sigma(y')\mathrm{d}\sigma(x'), \tag{2.10}$$

where

$$I_{m,t}(\xi)=\int_{2^{mt}}^{2^{m(t+1)}}e^{-2\pi i<B_\rho(y'-x'),\xi>}\frac{\mathrm{d}\rho}{\rho}.$$

Let $0<\delta<\min\{1/2,L/4\}$, we obtain by Lemma 2.2

$$\begin{aligned}|I_{m,t}(\xi)| &\le \sum_{j=0}^{m-1}\left|\int_{2^{mt+j}}^{2^{mt+j+1}}e^{-2\pi i<B_\rho(y'-x'),\xi>}\frac{\mathrm{d}\rho}{\rho}\right|\\ &\le \sum_{j=0}^{m-1}\left|\int_1^2 e^{-2\pi i<B_{2^{mt+j}\lambda}(y'-x'),\xi>}\frac{\mathrm{d}\lambda}{\lambda}\right|\\ &\le C\sum_{j=0}^{m-1}|<B_{2^{mt+j}}(y'-x'),\xi>|^{-2\delta/L}\\ &\le C\sum_{j=0}^{m-1}(|<(y'-x'),\eta'>||B_{2^{mt+j}}\xi|)^{-2\delta/L}\\ &\le C\sum_{j=0}^{m-1}2^{-2j\delta\beta_1/L}|B_{2^{mt}}\xi|^{-2\delta/L}|<(y'-x'),\eta'>|^{-2\delta/L}\\ &\le Cm|B_{2^{mt}}\xi|^{-2\delta/L}|<(y'-x'),\eta'>|^{-2\delta/L},\end{aligned}$$

where $\eta'=\dfrac{B_{2^{mt}}\xi}{|B_{2^{mt}}\xi|}$. Therefore, we have by (2.10)

$$\begin{aligned}R_{m,t}(\xi) &\le C\|J\|_\infty^2\left(\int\int_{S^{n-1}\times S^{n-1}}|\Omega_m(x')\overline{\Omega_m(y')}|^2\mathrm{d}\sigma(x')\mathrm{d}\sigma(y')\right)^{1/2}\\ &\quad\times\left(\int_{S^{n-1}\times S^{n-1}}|I_{m,t}(\xi)|^2\mathrm{d}\sigma(x')\mathrm{d}\sigma(y')\right)^{1/2}\\ &\le Cm\|\Omega_m\|_2^2|B_{2^{mt}}\xi|^{-2\delta/L}\left(\int\int_{S^{n-1}\times S^{n-1}}|<(y'-x'),\eta'>|^{-4\delta/L}\mathrm{d}\sigma(y')\right)^{1/2}\\ &\le Cm\|\Omega_m\|_2^2|B_{2^{mt}}\xi|^{-2\delta/L}.\end{aligned}$$

It follows

$$(R_{m,t}(\xi))^{1/2}\le Cm^{1/2}2^{2m}\|B_{2^{mt}}\xi|^{-\delta/L}. \tag{2.11}$$

Thus,

$$\begin{aligned}|\hat{\sigma}_{2^{mt}}(\xi)| &\le \left(\int_{2^{mt}}^{2^{m(t+1)}}|h(\rho)|^{\gamma'}\frac{d\rho}{\rho}\right)^{1/\gamma'}\left(\int_{2^{mt}}^{2^{m(t+1)}}\left|\int_{S^{n-1}}e^{-2\pi i\xi\cdot B_\rho y'}\Omega_m(y')J(y')\mathrm{d}\sigma(y')\right|^\gamma\frac{\mathrm{d}\rho}{\rho}\right)^{1/\gamma}\\ &\le \|J\|_\infty\|\Omega_m\|_1\left(\int_{2^{mt}}^{2^{m(t+1)}}\left|\frac{1}{\|J\|_\infty\|\Omega_m\|_1}\int_{S^{n-1}}e^{-2\pi i\xi\cdot B_\rho y'}\Omega_m(y')J(y')\mathrm{d}\sigma(y')\right|^\gamma\frac{\mathrm{d}\rho}{\rho}\right)^{1/\gamma}.\end{aligned}$$

If $2 \le \gamma < \infty$, noticing that

$$\left| \frac{1}{\|J\|_\infty \|\Omega_m\|_1} \int_{S^{n-1}} e^{-2\pi i \xi \cdot B_\rho y'} \Omega_m(y') J(y') \mathrm{d}\sigma(y') \right| \le 1,$$

Then we have

$$\begin{aligned} |\hat{\sigma}_{2^{mt}}(\xi)| &\le \|J\|_\infty \|\Omega_m\|_1 \left(\int_{2^{mt}}^{2^{m(t+1)}} \left| \frac{1}{\|J\|_\infty \|\Omega_m\|_1} \int_{S^{n-1}} e^{-2\pi i \xi \cdot B_\rho y'} \Omega_m(y') J(y') \mathrm{d}\sigma(y') \right|^2 \frac{\mathrm{d}\rho}{\rho} \right)^{1/\gamma} \\ &\le C \|\Omega_m\|^{-2/\gamma} \left(\int_{2^{mt}}^{2^{m(t+1)}} \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot B_\rho y'} \Omega_m(y') J(y') \mathrm{d}\sigma(y') \right|^2 \frac{\mathrm{d}\rho}{\rho} \right)^{1/\gamma} \\ &\le C m^{1/\gamma} \|\Omega_m\|_2^2 |B_{2^{mt}} \xi|^{-2\delta/\gamma L}. \end{aligned}$$

Thus,

$$|\hat{\sigma}_{2^{mt}}(\xi)| \le C m^{1/\gamma} 2^{4m} |B_{2^{mt}} \xi|^{-2\delta/\gamma L}, \ \text{ for } \ 2 \le \gamma < \infty. \tag{2.12}$$

If $1 < \gamma < 2$, we get by Hölder's inequality and (2.11)

$$\begin{aligned} |\hat{\sigma}_{2^{mt}}(\xi)| &\le \left(\int_{2^{mt}}^{2^{m(t+1)}} \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot B_\rho y'} \Omega_m(y') J(y') \mathrm{d}\sigma(y') \right|^\gamma \frac{\mathrm{d}\rho}{\rho} \right)^{1/\gamma} \\ &\le C \left(\int_{2^{mt}}^{2^{m(t+1)}} \frac{d\rho}{\rho} \right)^{1/\gamma - 1/2} \left(\int_{2^{mt}}^{2^{m(t+1)}} \left| \int_{S^{n-1}} e^{-2\pi i \xi \cdot B_\rho y'} \Omega_m(y') J(y') \mathrm{d}\sigma(y') \right|^2 \frac{\mathrm{d}\rho}{\rho} \right)^{1/2} \\ &\le C m^{1/\gamma - 1/2} m^{1/2} 2^{2m} |B_{2^{mt}} \xi|^{-\delta/L}. \end{aligned}$$

Thus, we have

$$|\hat{\sigma}_{2^{mt}}(\xi)| \le C m^{1/\gamma} 2^{2m} |B_{2^{mt}} \xi|^{-\delta/L}, \ \text{ for } \ 1 < \gamma < 2. \tag{2.13}$$

Therefore, (2.8) follows immediately from (2.12), (2.13), and (2.9) respectively. This finishes the proof of Lemma 2.3.

Lemma 2.4. *Let* $\alpha \in \mathbf{R}$, $1 < p, q < \infty$. *Then*

$$\|T_{m,s}(f)\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)} \le C m^{1/q} \left\| \left(\int_{\mathbf{R}} |2^{-m(t+s)\alpha} \sigma_{2^{mt}} * S_{2^{m(t+s)}} f|^q \mathrm{d}t \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)}. \tag{2.14}$$

Proof. Noticing that

$$\|f\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)} \simeq m^{1/q} \left\| \left(\int_{\mathbf{R}} |2^{-mt\alpha} S_{2^{mt}} f|^q \mathrm{d}t \right)^{1/q} \right\|_{L^p(\mathbf{R}^n)}, \tag{2.15}$$

Therefore, for any $g\in \dot{F}_{p'}^{-\alpha,q'}(\mathbf{R}^n)$ with $\|g\|_{\dot{F}_{p'}^{-\alpha,q'}(\mathbf{R}^n)}\le 1$, we have

$$\begin{aligned}
|<T_{m,s}(f),g>| &= m\left|\int_{\mathbf{R}^n}\int_{\mathbf{R}} S_{2^{m(t+s)}}(\sigma_{2^{mt}}*S_{2^{m(t+s)}}f)(x)g(x)\mathrm{d}t\mathrm{d}x\right| \\
&\le m\int_{\mathbf{R}^n}\left|\int_{\mathbf{R}}(\sigma_{2^{mt}}*S_{2^{m(t+s)}}f)(x)\tilde{S}_{2^{m(t+s)}}g(x)\mathrm{d}t\right|\mathrm{d}x \\
&\le m\left\|\left(\int_{\mathbf{R}}|2^{-m(t+s)\alpha}\sigma_{2^{mt}}*S_{2^{m(t+s)}}f|^q dt\right)^{1/q}\right\|_{L^p(\mathbf{R}^n)} \\
&\quad\times\left\|\left(\int_{\mathbf{R}}|2^{m(t+s)\alpha}\tilde{S}_{2^{m(t+s)}}g|^q dt\right)^{1/q}\right\|_{L^{p'}(\mathbf{R}^n)} \\
&\le m^{1/q}\|g\|_{\dot{F}_{p'}^{-\alpha,q}(\mathbf{R}^n)}\left\|\left(\int_{\mathbf{R}}|2^{-m(t+s)\alpha}\sigma_{2^{mt}}*S_{2^{m(t+s)}}f|^q dt\right)^{1/q}\right\|_{L^p(\mathbf{R}^n)},
\end{aligned}$$

where $\tilde{S}_{2^{m(t+s)}}$ is the dual operator of $S_{2^{m(t+s)}}$, i.e.,

$$\tilde{S}_{2^{m(t+s)}}g(x)=S_{2^{m(t+s)}}(\tilde{g})(x)$$

and $\tilde{g}(x)=g(-x)$. Thus, we have

$$\|T_{m,s}(f)\|_{\dot{F}_p^{\alpha,q}(\mathbf{R}^n)}\le Cm^{1/q}\left\|\left(\int_{\mathbf{R}}|2^{-m(t+s)\alpha}\sigma_{2^{mt}}*S_{2^{m(t+s)}}f|^q\mathrm{d}t\right)^{1/q}\right\|_{L^p(\mathbf{R}^n)}.$$

The proof of Lemma 2.4 is completed.

Next, we estimate the norm $\|T_{m,s}(f)\|_{\dot{F}_2^{\alpha,2}(\mathbf{R}^n)}$. By taking $p=q=2$ in (2.14), we have

$$\begin{aligned}
&\|T_{m,s}(f)\|^2_{\dot{F}_2^{\alpha,2}}\le Cm\int_{\mathbf{R}}\int_{\mathbf{R}^n}|2^{-m(t+s)\alpha}\hat{\sigma}_{2^{mt}}(\xi)\hat{\psi}(2^{m(t+s)}\xi)\hat{f}(\xi)|^2\mathrm{d}\xi\mathrm{d}t \\
&=Cm\int_{\mathbf{R}}\int_{D_{t+s}}|2^{-m(t+s)\alpha}\hat{\sigma}_{2^{mt}}(\xi)\hat{\phi}(2^{m(t+s)}\rho(\xi))\hat{f}(\xi)|^2\mathrm{d}\xi\mathrm{d}t \\
&=Cm\int_{\mathbf{R}}\int_{S^{n-1}}\int_{E_{t+s}}|2^{-m(t+s)\alpha}\hat{\sigma}_{2^{mt}}(B_\rho\xi')\hat{\phi}(2^{m(t+s)}\rho)\hat{f}(B_\rho\xi')|^2\rho^{\beta-1}J(\xi')\mathrm{d}\sigma(\xi')\mathrm{d}\rho\mathrm{d}t,
\end{aligned}$$

where $D_{t+s}=\{\xi\in\mathbf{R}^n:1/2\le 2^{m(t+s)}\rho(\xi)\le 2\}$ and $E_{t+s}=\{\rho\in\mathbf{R}^+:1/2\le 2^{m(t+s)}\rho\le 2\}$. Therefore, we have by (2.7) as $s\ge 1$,

$$\begin{aligned}
\|T_{m,s}(f)\|_{\dot{F}_2^{\alpha,2}(\mathbf{R}^n)} &\le Cm^{1/\gamma}\|\Omega_m\|_1 2^{-m(s-1)\beta_1}m^{1/2}\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}}|2^{-m(t+s)\alpha}\psi_{2^{m(t+s)}}*f(x)|^2\mathrm{d}t\mathrm{d}x\right)^{1/2} \\
&\le Cm^{1/\gamma}\|\Omega_m\|_1 2^{-m(s-1)\beta_1}\|f\|_{\dot{F}_2^{\alpha,2}(\mathbf{R}^n)}. \qquad (2.16)
\end{aligned}$$

Similar to (2.16), we have by (2.8)

$$\|T_{m,s}(f)\|_{\dot{F}_2^{\alpha,2}(\mathbf{R}^n)}\le Cm^{1/\gamma}2^{s\beta_1\varepsilon}\|f\|_{\dot{F}_2^{\alpha,2}(\mathbf{R}^n)},\ \text{ for }\ s<1. \qquad (2.17)$$

Lemma 2.5. *Let* $1<\gamma,\ p<\infty$ *and*

$$L_t(f)(x)=\int_{\mathbf{R}^n}\frac{\Omega_m(y')}{\rho(y)^\beta}\chi_{2^{mt}}(\rho(y))f(x-y)\mathrm{d}y.$$

$\tilde{L}_t$ *denotes the dual operator of* L_t, *i.e.,* $\tilde{L}_t(f)(x)=L_t(\tilde{f})(-x)$, *where* $\tilde{f}(x)=f(-x)$. *Then*

$$|\sigma_{2^{mt}}*S_{2^{m(t+s)}}(f)(x)|\le C\|\Omega_m\|_1^{1/\gamma'}(L_t(|S_{2^{m(t+s)}}f|^\gamma)(x))^{1/\gamma},$$

$$\|\sigma_{2^{mt}}*S_{2^{m(t+s)}}(f)(x)\|_{L^\gamma(\mathbf{R}^n)}\le C\|\Omega_m\|_1\|S_{2^{m(t+s)}}f\|_{L^\gamma(\mathbf{R}^n)},$$

$$\big\|\sup_{t\in\mathbf{R}}L_t(|f|)\big\|_{L^p(\mathbf{R}^n)}\le Cm\|\Omega_m\|_1\|f\|_{L^p(\mathbf{R}^n)}.$$

Proof. Noticing that

$$\begin{aligned}|\sigma_{2^{mt}}*S_{2^{m(t+s)}}(f)(x)| &\le \left|\int_{\mathbf{R}^n}\frac{h(\rho(y))\Omega_m(y')}{\rho(y)^\beta}\chi_{2^{mt}}(\rho(y))S_{2^{m(t+s)}}f(x-y)\mathrm{d}y\right|\\ &\le \left(\int_{\mathbf{R}^n}\frac{|h(\rho(y))|^{\gamma'}\Omega_m(y')}{\rho(y)^\beta}\chi_{2^{mt}}(\rho(y))\mathrm{d}y\right)^{1/\gamma'}\\ &\quad\times\left(\int_{\mathbf{R}^n}\frac{|\Omega_m(y')|}{\rho(y)^\beta}|S_{2^{m(t+s)}}f(x-y)|^\gamma\chi_{2^{mt}}(\rho(y))dy\right)^{1/\gamma}\\ &\le \|\Omega_m\|_1^{\gamma'}(L_t(|S_{2^{m(t+s)}}f|^\gamma)(x))^{1/\gamma},\end{aligned}$$

Then we have

$$\|\sigma_{2^{mt}}*S_{2^{m(t+s)}}(f)(x)\|_{L^\gamma(\mathbf{R}^n)}\le C\|\Omega_m\|_1^{\gamma'}\|S_{2^{m(t+s)}}f\|_{L^\gamma(\mathbf{R}^n)}.$$

Since

$$L_t(|f|)=\int_{S^{n-1}}|\Omega_m(y')|J(y')\left(\int_{2^{mt}}^{2^{m(t+1)}}|f(x-B_\rho y')|\frac{\mathrm{d}\rho}{\rho}\right)\mathrm{d}\sigma(y')$$

and

$$\begin{aligned}\int_{2^{mt}}^{2^{m(t+1)}}|f(x-B_\rho y')|\frac{\mathrm{d}\rho}{\rho} &= \sum_{j=1}^m\int_{2^{mt+j-1}}^{2^{mt+j}}|f(x-B_\rho y')|\frac{\mathrm{d}\rho}{\rho}\\ &\le 2m\frac{1}{2^{mt+j}}\int_{2^{mt+j-1}}^{2^{mt+j}}|f(x-B_\rho y')|\mathrm{d}\rho\\ &\le 2m\sup_{h>0}\left\{\frac{1}{h}\int_0^h|f(x-B_\rho y')|\mathrm{d}\rho\right\}\\ &:= 2m\mathrm{M}_\Gamma(f)(x),\end{aligned}$$

we get by Minkowski's inequality and Lemma 2.3

$$\big\|\sup_{t\in\mathbf{R}}L_t(|f|)\big\|_{L^p(\mathbf{R}^n)}\le Cm\|\Omega_m\|_1\|f\|_{L^p(\mathbf{R}^n)}.$$

This completes the proof of Lemma 2.5.

Now, we turn to give the proof of (2.5). By taking $q=\gamma$ in (2.14), there exists a non-negative function $g(x)\in L^{r'}(\mathbf{R}^n)$ with $\|g\|_{L^{r'}(\mathbf{R}^n)}\le 1$, where $r=p/\gamma$, such that

$$\begin{aligned}\|T_{m,s}(f)\|^{\gamma}_{\dot F_p^{\alpha,\gamma}(\mathbf{R}^n)} &\le Cm\int_{\mathbf{R}}\int_{\mathbf{R}^n}|2^{-m(t+s)\alpha}\sigma_{2^{mt}}*S_{2^{m(t+s)}}f|^{\gamma}g(x)\mathrm{d}x\mathrm{d}t\\ &\le Cm\int_{\mathbf{R}}\int_{\mathbf{R}^n}2^{-m(t+s)\alpha\gamma}L_t(|S_{2^{m(t+s)}}f|^{\gamma})(x)g(x)\mathrm{d}x\mathrm{d}t\\ &\le Cm\int_{\mathbf{R}}\int_{\mathbf{R}^n}|2^{-m(t+s)\alpha}S_{2^{m(t+s)}}f(x)|^{\gamma}\tilde L_t g(x)\mathrm{d}x\mathrm{d}t\\ &\le Cm\int_{\mathbf{R}}\left(\int_{\mathbf{R}^n}|2^{-m(t+s)\alpha}S_{2^{m(t+s)}}f(x)|^{\gamma}dt\right)\sup_{t\in\mathbf{R}}\tilde L_t g(x)\mathrm{d}x\\ &\le Cm\left(\int_{\mathbf{R}^n}(\int_{\mathbf{R}}|2^{-m(t+s)\alpha}S_{2^{m(t+s)}}f|^{\gamma}dt)^r dx\right)^{1/r}\left(\int_{\mathbf{R}^n}|\sup_{t\in\mathbf{R}}\tilde L_t g(x)|^{r'}dx\right)^{1/r'}\\ &\le Cm\|f\|^{\gamma}_{\dot F_p^{\alpha,\gamma}(\mathbf{R}^n)}\|g\|_{L^{r'}(\mathbf{R}^n)}.\end{aligned}$$

It follows

$$\|T_{m,s}(f)\|_{\dot F_p^{\alpha,\gamma}(\mathbf{R}^n)}\le Cm^{1/\gamma}\|f\|_{\dot F_p^{\alpha,\gamma}(\mathbf{R}^n)}. \tag{2.18}$$

Using an interpolation between (2.16), (2.17), and (2.18) respectively (see [5]) and by a standard duality argument, we obtain that there exist θ_1 and θ_2: $0<\theta_1,\ \theta_2\le 1$, such that for all q lying in γ' and γ

$$\|T_{m,s}(f)\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)}\le Cm^{1/\gamma}\|\Omega_m\|_1 2^{-m(s-1)\beta_1\theta_1}\|f\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)},\ \text{ for } s\ge 1. \tag{2.19}$$

$$\|T_{m,s}(f)\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)}\le Cm^{1/\gamma}2^{s\beta_1\theta_2\varepsilon}\|f\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)},\ \text{ for } s<1. \tag{2.20}$$

Thus, we have

$$\|T_m(f)\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)}\le\int_{\mathbf{R}}\|T_{m,s}(f)\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)}\mathrm{d}s\le Cm^{1/\gamma}\|f\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)}.$$

It follows that (2.5) holds. Therefore, for all q lying in between γ' and γ, we have from (2.2), (2.3), (2.5), and (2.6)

$$\begin{aligned}\|T(f)(x)\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)} &\le \sum_{m\in I\bigcup\{0\}}\lambda_m\|T_m(f)(x)\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)}\\ &\le C\sum_{m\in I\bigcup\{0\}}\lambda_m m^{1/\gamma}\|f\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)}\\ &\le C\|\Omega\|_{L(\log L)^{1/\gamma}(S^{n-1})}\|f\|_{\dot F_p^{\alpha,q}(\mathbf{R}^n)}.\end{aligned}$$

This finishes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let $\Omega \in L(\log L)^{1/\gamma}(S^{n-1})$. We decompose Ω as that in the proof of Theorem 1.1, except for a slight modification by taking $J(x') \equiv 1$. Then we get by (2.1) and Minkowski's inequality

$$\|\mu_{\Omega,q}(f)(x)\|_{L^p(\mathbf{R}^n)} \le \sum_{m\in I\bigcup\{0\}} \lambda_m \|\mu_{\Omega_m,q}(f)(x)\|_{L^p(\mathbf{R}^n)}, \tag{2.21}$$

where

$$\mu_{\Omega_m,q}(f)(x) = \left(\int_0^\infty |F_{\Omega_m}(x,t)|^q \frac{\mathrm{d}t}{t^{q+1}}\right)^{1/q}, \tag{2.22}$$

$$F_{\Omega_m}(x,t) = \int_{|y|\le t} \frac{h(|y|)\Omega_m(y)}{|y|^{n-1}} f(x-y)\mathrm{d}y.$$

For a fixed $m \in \mathbf{N}$, let us define the measures $\{\sigma_{m,t}\}_{t\in\mathbf{R}}$ by setting

$$\sigma_{m,t} * f(x) = 2^{-t} \int_{|y|\le 2^t} \frac{h(|y|)\Omega_m(y)}{|y|^{n-1}} f(x-y)\mathrm{d}y.$$

Then

$$\mu_{\Omega_m,q} \sim \left(\int_{\mathbf{R}} |\sigma_{m,t} * f(x)|^q \mathrm{d}t\right)^{1/q}. \tag{2.23}$$

We choose a real-valued radical function $\phi(\xi) \in S(\mathbf{R}^n)$ such that supp $\hat{\phi} \subset \{\xi \in \mathbf{R}^n : 1/2 \le |\xi| \le 2\}$, $\hat{\phi}(\xi) \ge 0$, $\hat{\phi}(\xi) \ge c > 0$ as $3/5 \le |\xi| \le 5/3$ and $\int_{\mathbf{R}} |\hat{\phi}_{2^t}(\xi)|\mathrm{d}t = 1$ for $\xi \neq 0$, where $\hat{\phi}_{2^t}(\xi) = \hat{\phi}(2^t\xi)$. Set $S_{2^t} = \phi_{2^t} * f$. Then we get

$$\sigma_{m,t} * f = \int_{\mathbf{R}} \sigma_{m,t} * S_{2^{(t+s)}} f\mathrm{d}s. \tag{2.24}$$

According to (2.24) and Minkowski's inequality, we obtain

$$\|\sigma_{m,t} * f\|_{L^q(\mathbf{R})} \le \int_{\mathbf{R}} \|\sigma_{m,t} * S_{2^{m(t+s)}} f\|_{L^q(\mathbf{R})}\mathrm{d}s := m^{-1/q} \int_{\mathbf{R}} I_{q,s} f\mathrm{d}s, \tag{2.25}$$

where

$$I_{q,s}f(x) = m^{1/q}\left(\int_{\mathbf{R}} |\sigma_{m,t} * S_{2^{m(t+s)}} f|^q \mathrm{d}t\right)^{1/q}.$$

First we consider the case $1 < \gamma \le 2$.

Lemma 2.6. *Assume that $h \in \Delta_\gamma(\mathbf{R}^+)$, $1 < \gamma \le 2$, and $\sigma_{m,t} = \dfrac{h(|y|)\Omega_m(y)}{2^t|y|^{n-1}}\chi_{\{|y|\le 2^t\}}$, then*

$$|\hat{\sigma}_{m,t}(\xi)| \le C\min\{|2^t\xi|, ||2^t\xi|^{-1/8m}\}.$$

Proof. We have by the vanishing moment of Ω_m

$$\begin{aligned}
|\hat{\sigma}_{m,t}(\xi)| &= \left|\int_0^{2^t}\int_{S^{n-1}} e^{-2\pi i r\xi\cdot y'}\Omega_m(y'))\mathrm{d}\sigma(y')2^{-t}h(r)\mathrm{d}r\right| \\
&\le \int_0^{2^t}\int_{S^{n-1}} \left|(e^{-2\pi i r\xi\cdot y'}-1)\Omega_m(y'))\right|\mathrm{d}\sigma(y')2^{-t}|h(r)|\mathrm{d}r \\
&\le C\|\Omega_m\|_1|2^t\xi|2^{-t}\int_0^{2^t}|h(r)|\mathrm{d}r \\
&\le C\|\Omega_m\|_1|2^t\xi|\left(2^{-t}\int_0^{2^t}|h(r)|^{\gamma'}\mathrm{d}r\right)^{1/\gamma'}\left(2^{-t}\int_0^{2^t}dr\right)^{1/\gamma} \\
&\le C\|\Omega_m\|_1|2^t\xi|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\hat{\sigma}_{m,t}(\xi)| &\le \left(2^{-t}\int_0^{2^t}|h(r)|^2\mathrm{d}r\right)^{1/2}\left(2^{-t}\int_0^{2^t}\left|\int_{S^{n-1}} e^{-2\pi i r\xi\cdot y'}\Omega_m(y'))\mathrm{d}\sigma(y')\right|^2\mathrm{d}r\right)^{1/2} \\
&\le C\left(\int_0^{2^t}\left|\int_{S^{n-1}} e^{-2\pi i r\xi\cdot y'}\Omega_m(y'))\mathrm{d}\sigma(y')\right|^2 dr\right)^{1/2}.
\end{aligned}$$

Denote

$$R_{m,t}(\xi) = 2^{-t}\int_0^{2^t}\left|\int_{S^{n-1}} e^{-2\pi i r\xi\cdot y'}\Omega_m(y'))\mathrm{d}\sigma(y')\right|^2\mathrm{d}r.$$

Noticing that

$$\left|\int_{S^{n-1}} e^{-2\pi i\xi\cdot y'}\Omega_m(y'))\mathrm{d}\sigma(y')\right| \le C\|\Omega_m\|_1,$$

then we obtain

$$|R_{m,t}(\xi)| \le C\|\Omega_m\|_1^2.$$

It follows that

$$|\hat{\sigma}_{m,t}(\xi)| \le C\|\Omega_m\|_1. \tag{2.26}$$

Rewrite

$$R_{m,t}(\xi) = \int\int_{S^{n-1}\times S^{n-1}} \Omega_m(y')\overline{\Omega_m(x')}|I_{m,t}(\xi)|\mathrm{d}r\mathrm{d}\sigma(y')\mathrm{d}\sigma(x'), \tag{2.27}$$

where

$$I_{m,t}(\xi) = 2^{-t}\int_0^{2^t} e^{-2\pi i r\xi\cdot(y'-x')}\mathrm{d}r.$$

By integrating by parts, we have

$$I_{m,t}(\xi) \le C\min\{1, |2^t\xi|^{-1}(y'-x')\} \le C|2^t\xi|^{-\nu}|\xi'\cdot(y'-x')|^{-\nu}$$

with $0<\nu<1$ and $2\nu<1$. Taking $\nu=1/4$, we get from (2.27)

$$\begin{aligned} R_{m,t}(\xi) &\le C\Big(\int\int_{S^{n-1}\times S^{n-1}}|\Omega_m(x')\overline{\Omega_m(y')}|^2\mathrm{d}\sigma(x')\mathrm{d}\sigma(y')\Big)^{1/2} \\ &\quad\times\Big(\int_{S^{n-1}\times S^{n-1}}|I_{m,t}(\xi)|^2\mathrm{d}\sigma(x')\mathrm{d}\sigma(y')\Big)^{1/2} \\ &\le C\|\Omega_m\|_2^2|2^t\xi|^{-1/4}\Big(\int\int_{S^{n-1}\times S^{n-1}}|<(y'-x',\eta'>|^{-1/8}\mathrm{d}\sigma(y')\Big)^{1/2} \\ &\le C\|\Omega_m\|_2^2|2^t\xi|^{-1/4}. \end{aligned}$$

It follows that

$$|\hat{\sigma}_{m,t}(\xi)|\le C2^{2m}|2^t\xi|^{-1/8}. \tag{2.28}$$

From (2.26) and (2.28), we obtain

$$|\hat{\sigma}_{m,t}(\xi)|\le C|2^t\xi|^{-1/8m}.$$

This finishes the proof of Lemma 2.6.

By the similar argument as in the proof of Lemma 2.5, we have

Lemma 2.7. *Let*

$$N_t(f)(x)=2^{-t}\int_{|y|\le 2^t}\frac{\Omega_m(y)}{|y|^{n-1}}f(x-y)\mathrm{d}y.$$

$\tilde{N}_t$ denotes the dual operator of N_t, i.e., $\tilde{N}_t(f)(x)=N_t(\tilde{f})(-x)$, where $\tilde{f}(x)=f(-x)$. Then

$$|\sigma_{m,t}*S_{2^{m(t+s)}}(f)(x)|\le C\|\Omega_m\|_1^{1/\gamma'}(N_t(|S_{2^{m(t+s)}}f|^\gamma)(x))^{1/\gamma},$$

$$\|\sigma_{m,t}*S_{2^{m(t+s)}}(f)(x)\|_{L^\gamma(\mathbf{R}^n)}\le C\|\Omega_m\|_1\|S_{2^{m(t+s)}}f\|_{L^\gamma(\mathbf{R}^n)},$$

$$\big\|\sup_{t\in\mathbf{R}}N_t(|f|)\big\|_{L^p(\mathbf{R}^n)}\le C\|\Omega_m\|_1\|f\|_{L^p(\mathbf{R}^n)}.$$

Therefore, from Lemmas 2.6 and 2.7, we obtain by applying the similar argument as that in getting (2.16) and (2.17), there exists a positive constant τ such that

$$\|I_{q,s}f(x)\|_{L^p}\le C2^{-|s|\tau}\|\Omega_m\|_1\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)},\ \gamma<q<\gamma'. \tag{2.29}$$

We denote the mixed norm

$$\|f\|_{L^p(L^q)}=\left\|\Big(\int_R|f(x,t)|^q\mathrm{d}t\Big)^{1/q}\right\|_{L^p(\mathbf{R}^n)}.$$

Then we have by (2.25) and (2.29)

$$\|\sigma_{m,t}*f\|_{L^p(L^q)}\le m^{1/\gamma}\|\Omega_m\|_1\int_{\mathbf{R}}\|I_{q,s}f\|_{L^p}\mathrm{d}s\le Cm^{1/\gamma}\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}.$$

$$\|\mu_{\Omega_m,q}f\|_{L^p} \le \|\sigma_{m,t} * f\|_{L^p} \le Cm^{1/\gamma}\|\Omega_m\|_1\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}. \tag{2.30}$$

Thus, according to the estimates (2.21), (2.23), and (2.30), we obtain

$$\begin{aligned}\|\mu_{\Omega,q}(f)(x)\|_{L^p(\mathbf{R}^n)} &\le \sum_{m\in I\bigcup\{0\}} \lambda_m\|\mu_{\Omega_m,q}(f)(x)\|_{L^p(\mathbf{R}^n)}\\ &\le C\sum_{m\in I\bigcup\{0\}} \lambda_m m^{1/\gamma}\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}\\ &\le C\|\Omega\|_{L(\log L)^{1/\gamma}(S^{n-1})}\|f\|_{\dot{F}_p^{0,q}(\mathbf{R}^n)}.\end{aligned}$$

Finally, by the similar argument as that in the proof of Lemma 2.3 and the case $1 < \gamma \le 2$, we can prove Theorem 1.2 in the case $2 < \gamma < \infty$. We omit the details here. So the proof of Theorem 1.2 is finished.

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