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APPROXIMATION PROPERTIES OF rth ORDER GENERALIZED BERNSTEIN POLYNOMIALS BASED ON $q-$CALCULUS

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Abstract. In this paper we introduce a generalization of Bernstein polynomials based on q calculus. With the help of Bohman-Korovkin type theorem, we obtain $A-$statistical approximation properties of these operators. Also, by using the Modulus of continuity and Lipschitz class, the statistical rate of convergence is established. We also gives the rate of $A-$statistical convergence by means of Peetre's type $K-$functional. At last, approximation properties of a rth order generalization of these operators is discussed.

Key words: *$q-$integers, $q-$Bernstein polynomials, $A-$statistical convergence, modulus of continuity, Lipschitz class, Peetre's type $K-$functional*

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1 Introduction

Phillips[7] in 1997 proposed $q-$Bernstein polynomials based on q calculus as

$$B_{n,q}(f;x)=\sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix} n \\ k \end{bmatrix} x^k(1-x)_q^{n-k-1}.$$

Very recently Heping[12] obtained Voronovaskaya type asymptotic formula for q-Bernstein operator. In 2002 Ostrovska S.[9], studied the convergence of generalized Bernstein Polynomials. Study of $A-$statistical approximation by positive linear operators is attempted by O.Duman, C.Orhan in [8].

First, we recall the concept of A-statistical convergence.

Let $A = (a_{jn})_{j,n}$ be a non-negative infinite summability matrix. For a sequence $x := (x_n)_n$, $A-$transform of the sequence x, denoted by $Ax := (Ax)_j$, is given by

$$(Ax)_j := \sum_{n=1}^{\infty} a_{jn} x_n,$$

provided that the series on the right hand side converges for each j. We say that A is regular (see [8]) if $\lim Ax = L$ whenever $\lim x = L$. Let A be a non-negative summability matrix. The sequence $x := (x_n)_n$ is said to be $A-$statistically convergent to a number L, if for any given $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0,$$

and we denote this limit by $st_A - \lim_n x_n = L$.

We also know that

1. (see [1],[4]) For $A := C_1$, the Cesàro matrix of order one defined as

$$c_{jn} := \begin{cases} \dfrac{1}{j}, & 1 \leq n \leq j, \\ 0, & n > j, \end{cases}$$

then A-statistical convergence coincides with statistical convergence.

2. Taking A as the identity matrix, A-statistical convergence coincides with ordinary convergence, i.e.

$$st_A - \lim_n x_n = \lim x_n = L.$$

2 Construction of Operator

Here we introduce a general family of $q-$Bernstein polynomials and compute the rate of convergence with help of modulus of continuity and Lipschitz class. Before introducing the operators, we mention certain definitions based on q-integers, for the DETAILS, see [10] and [11]. For each nonnegative integer k, the $q-$integer $[k]$ and the $q-$factorial $[k]!$ are respectively defined by

$$[k] := \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1 \end{cases}.$$

and

$$[k]! := \begin{cases} [k]\,[k-1] \cdots [1], & k \geq 1, \\ 1, & k = 0. \end{cases}$$

For the integers $n,\ k$ satisfying $n \geq k \geq 0$, the $q-$binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}\ .$$

We use the following notations:

$$(a+b)_q^n := \prod_{s=0}^{n-1}(a+q^s b), \quad n \in \mathbf{N}, \quad a,b \in \mathbf{R}, \tag{2.1}$$

$$(1+a)_q^\infty := \prod_{s=0}^{\infty}(1+q^s a), \qquad a \in \mathbf{R}, \tag{2.2}$$

$$(1+a)_q^t := \frac{(1+a)_q^\infty}{(1+q^t a)_q^\infty}, \qquad a,t \in \mathbf{R}. \tag{2.3}$$

Note that the infinite product (2.2) is convergent if $q \in (0,1)$ and

$$(t;q)_0 := 1, (t;q)_n := \prod_{j=0}^{n-1}(1-q^j t), (t;q)_\infty := \prod_{j=0}^{\infty}(1-q^j t).$$

Also it can be seen that

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Let $a_n(t)$ be a sequence of functions defined on the interval $[0,1]$ s.t. $a_n(t) \in (0,1]$ for all $n \in \mathbf{N}$ and $t \in [0,1]$.

For $f \in C[0,1]$ and $q \in (0,1]$, we define the $q-$Bernstein polynomial with help of $a_n(t)$ as:

$$\Psi_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{a_n(q)[k]}{[n]}\right) p_{n,k}(q;x), \tag{2.4}$$

here

$$p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)_q^{n-k-1}.$$

Obviously for $a_n(q) = 1$ in (2.4), we get the classic $q-$Bernstein polynomial introduced by Phillips[7]. M.A. Ozarslan, O. Duman[6] also introduced similar type of generalization for Meyer-Konig Zeller type operators.

Lemma 1. *For all $x \in [0,1]$, $n \in \mathbf{N}$ and $q \in (0,1)$, we have*

$$\Psi_{n,q}(e_0;x) = 1, \tag{2.5}$$

$$\Psi_{n,q}(e_1;x) = x a_n(q), \tag{2.6}$$

$$\Psi_{n,q}(e_2;x) = a_n^2(q)\left(x^2 - \frac{x^2}{[n]} + \frac{x}{[n]}\right). \tag{2.7}$$

Proof. Clearly (2.5) exists. A direct calculation yields that

$$\begin{aligned}\Psi_{n,q}(e_1;x) &= a_n(q)\sum_{k=1}^{n}\frac{[n-1]!}{[k-1]![n-k]!}x^k(1-x)_q^{n-k}\\ &= a_n(q)x\sum_{k=0}^{n-1}\begin{bmatrix} n-1\\ k\end{bmatrix}x^k(1-x)_q^{(n-1)-k}\\ &= a_n(q)x.\end{aligned}$$

Also

$$\begin{aligned}\Psi_{n,q}(e_2;x) &= a_n^2(q)\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}\frac{[k]^2}{[n]^2}x^k(1-x)_q^{n-k}\\ &= a_n^2(q)\sum_{k=0}^{n-1}\begin{bmatrix} n-1\\ k\end{bmatrix}\frac{(q[k]+1)}{[n]}x^{k+1}(1-x)_q^{n-k-1}\\ &= a_n^2(q)\left(q\sum_{k=0}^{n-2}\frac{[n-1]}{[n]}\begin{bmatrix} n-2\\ k\end{bmatrix}x^{k+2}(1-x)_q^{n-k-2}\right.\\ &\quad\left.+\sum_{k=0}^{n-1}\frac{1}{[n]}\begin{bmatrix} n-1\\ k\end{bmatrix}x^{k+1}(1-x)_q^{n-k-1}\right)\\ &= a_n^2(q)\left(\frac{[n-1]q}{[n]}x^2 + \frac{x}{[n]}\right)\\ &= a_n^2(q)\left(x^2 - \frac{x^2}{[n]} + \frac{x}{[n]}\right).\end{aligned}$$

Hence the result folows.

Remark 1. One can observe that the central moments of $\Psi_{n,q}(f;.)$ are given by

$$\begin{aligned}\Psi_{n,q}(c_1;x) &= x(a_n(q)-1),\\ \Psi_{n,q}(c_2;x) &= x^2(a_n(q)-1)^2 + \frac{a_n^2(q)}{[n]}(x-x^2),\end{aligned}$$

where $c_1 = t-x$ and $c_2 = (t-x)^2$.

Bohman-Korovkin type theorem [3] may be read as follows:

Theorem A. Let $A = (a_{jn})_{j,n}$ be a non-negative regular summability matrix and let $(L_n)_n$ be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$, then for all $f \in C[a,b]$, we have

$$st_A - \lim_n \|L_n f - f\| = 0$$

if and only if

$$st_A - \lim_n \|L_n f_v - f_v\| = 0, \quad \text{for all} \quad v = 0, 1, 2,$$

where

$$f_v(t) = t^v \quad \text{for all} \quad v = 0, 1, 2.$$

Now, in the above definition of the operator (2.4), we replace the fixed q with a sequence $(q_n)_{n\in\mathbf{N}}$, such that $q_n \in (0,1]$ and satisfying the conditions

$$st_A - \lim_n a_n(q_n) = 1 \quad \textit{and} \quad st_A - \lim_n q_n = 1. \tag{2.8}$$

Theorem 1. *Let $(q_n)_{n\in\mathbf{N}}$ be a sequence satisfying* (2.8). *Then for all $f \in C[0,a]$, $0 < a < 1$, we have*

$$st_A - \lim_n \left\|\Psi_{n,q}(f;\cdot) - f\right\| = 0.$$

Proof. It is clear that

$$st_A - \lim_n \left\|\Psi_{n,q}(e_0;x) - e_0\right\| = 0. \tag{2.9}$$

Based on the equation (2.6), we get

$$\|\Psi_{n,q_n}(e_1,x) - e_1(x)\| = x(a_n(q_n) - 1) \leq a_n(q_n) - 1. \tag{2.10}$$

For every $\varepsilon > 0$, we define two sets as follows:

$$T_0 := \{n : \|\Psi_{n,q_n}(e_1,x) - e_1(x)\| \geq \varepsilon\} \quad \text{and} \quad T_1 = \{n : a_n(q_n) - 1 \geq \varepsilon\}.$$

Then by (2.10), one can observe that $T_0 \subseteq T_1$, hence for all $j \in \mathbf{N}$, we get

$$0 \leq \sum_{n\in T_0} a_{jn} \leq \sum_{n\in T_1} a_{jn};$$

since $st_A - \lim_n a_n(q_n) = 1$, we get

$$\sum_{n\in T_0} a_{jn} = 0. \tag{2.11}$$

Taking the limit $j \to \infty$ gives

$$st_A - \lim_n \left\|\Psi_{n,q}(e_0;x) - e_0\right\| = 0. \tag{2.12}$$

By the equation (2.7), we have

$$\|\Psi_{n,q_n}(e_2,x) - e_2(x)\| \leq (a_n^2(q_n) - 1) + \frac{1}{[n]}. \tag{2.13}$$

For every $\varepsilon > 0$, we define the sets as follows:

$$\begin{aligned} S_0 &= \{n : \|\Psi_{n,q_n}(e_2,x) - e_2(x)\| \geq \varepsilon\}, \\ S_1 &= \{n : a_n^2(q_n) - 1 \geq \varepsilon\}, \\ S_2 &= \{n : \frac{1}{[n]} \geq \varepsilon\}. \end{aligned}$$

Then by (2.13), one can observe that $S_0 \subseteq S_1 \subseteq S_2$, hence for all $j \in \mathbf{N}$, we get

$$0 \leq \sum_{n \in S_0} a_{jn} \leq \sum_{n \in S_1} a_{jn} + \sum_{n \in S_2} a_{jn}.$$

Since $st_A - \lim_n a_n^2(q_n) = 1$, $st_A - \lim_n \frac{1}{[n]} = 0$, consequently

$$\sum_{n \in S_0} a_{jn} = 0. \tag{2.14}$$

Taking the limit $j \to \infty$ gives

$$st_A - \lim_n \left\| \Psi_{n,q}(e_2;x) - e_2 \right\| = 0. \tag{2.15}$$

Finally, using (2.9), (2.12) and (2.15) the proof follows from theorem A.

Remark 2. By replacing A with Cesàro matrix of order one (C_1), we get the statistical convergence of the operator and replacing A with the identity matrix we get the simple convergence.

Recall the concept of modulus of continuity of $f(x) \in [0,a]$, denoted by $\omega(f,\delta)$, is defined by

$$\omega(f,\delta) = \sup_{|x-y| \leq \delta, x,y \in [0,a]} |f(x) - f(y)|. \tag{2.16}$$

The modulus of continuity possesses the following property (see [5])

$$\omega(f,\lambda\delta) \leq (1+\lambda)\omega(f,\delta). \tag{2.17}$$

Corollary 2. *Let $(q_n)_n$ be a sequence satisfying* (2.8) *. Then*

$$|\Psi_{n,q}(f;x) - f| \leq 2\omega(f,\sqrt{\delta_n}) \tag{2.18}$$

for all $f \in C[0,1]$, where

$$\delta_n = \Psi_{n,q}\left((t-x)^2;x\right). \tag{2.19}$$

Proof. By the linearity and monotonicity of $\Psi_{n,q}$, we get

$$|\Psi_{n,q}(f;x) - f| \leq \Psi_{n,q}(|f(t) - f(x)|;x)$$

also

$$|f(t)-f(x)| \leq \omega(f,\delta)\left(1+\frac{1}{\delta}(t-x)^2\right).$$

Therefore, we obtain

$$|\Psi_{n,q}(f;x)-f| \leq \omega(f,\delta)\left(1+\frac{1}{\delta}\Psi_{n,q}((t-x)^2;x)\right).$$

By Remark 1, we get

$$\Psi_{n,q}\left((t-x)^2;x\right) \leq (a_n(q_n)-1)^2 x^2 + \frac{(a_n^2(q_n))}{[n]}.$$

Since $a_n(q_n)$ satisfies (2.8), we get

$$\lim_{n\to\infty} \Psi_{n,q}\left((t-x)^2;x\right) = 0. \tag{2.20}$$

So, letting $\delta_n = \Psi_{n,q}\left((t-x)^2;x\right)$ and taking $\delta = \sqrt{\delta_n}$, we finally get

$$|\Psi_{n,q}(f;x)-f| \leq 2\omega(f,\sqrt{\delta_n}).$$

As usual, a function $f \in \mathrm{Lip}_M(\alpha)$, ($M>0$ and $0<\alpha\leq 1$), if the inequality

$$|f(t)-f(x)| \leq M|t-x|^{\alpha} \tag{2.21}$$

holds for all $t,x \in [0,1]$.

In the following theorem, we will compute the rate of convergence by mean of Lipschitz class.

Corollary 3. *For all $f \in Lip_M(\alpha)$ and $x \in [0,1]$, we have*

$$|\Psi_{n,q}(f;x)-f| \leq M\delta_n^{\alpha/2} \tag{2.22}$$

where $\delta_n = \Psi_{n,q}(|t-x|^2;x)$.

Proof. Using inequality (2.13) and Hölder inequality with $p=\frac{2}{\alpha}, q=\frac{2}{2-\alpha}$, we get

$$\begin{aligned} |\Psi_{n,q}(f;x)-f| &\leq \Psi_{n,q}(|f(t)-f(x)|;x) \\ &\leq M\Psi_{n,q}(|t-x|^{\alpha};x) \\ &\leq M\Psi_{n,q}(|t-x|^2;x)^{\alpha/2}. \end{aligned}$$

Taking $\delta_n = \Psi_{n,q}(|t-x|^2;x)$, we get

$$|M_{n,q}(f;x)-f| \leq M\delta_n^{\alpha/2}.$$

Remark 3. By Corollary 2 or Corollary 3, we find that $\Psi_{n,q}(f;.)$ converges to f uniformly on [0,1].

Let us recall concept of Peetre's type $K-$functional (see [2]). Define

$$C^2[0,a] := \{f \in C[0,a] : f^{'}, f^{''} \in C[0,a]\},$$

then $C^2[0,a]$ is a normed linear space with the norm defined as

$$\|f\|_{C^2[0,a]} := \|f\| + \|f^{'}\| + \|f^{''}\|.$$

Peetre's type $K-$functional is defined as (see[9])

$$K(f,\delta) := \inf_{g\in C^2[0,a]}\{\|f-g\| + \delta\|g\|_{C^2[0,a]}\}.$$

In the following theorem we estimate the rate of A-statistical convergence by means of Peetre's type K-functional.

Theorem 4. *Let $(q_n)_{n\in\mathbf{N}}$ be a sequence satisfying* (2.8). *Then for all $f \in C[0,a]$, $0 < a < 1$, we have*

$$st_A - \lim_n \left\|\Psi_{n,q}(f;\cdot) - f\right\| \le 2K(f;\delta_n),$$

where

$$\delta_n = \frac{1}{2}\{(a_n(q_n)-1) + \frac{1}{4}\{(a_n(q_n)-1)^2 + \frac{a_n(q_n)^2}{[n]}\}\}.$$

Proof. Let $g \in C^2[0,a]$, then

$$g(t) - g(x) = g^{'}(x)(t-x) + \int_x^t g^{''}(s)(t-s)\mathrm{d}s.$$

Therefore

$$|\Psi_{n,q}(g;x) - g(x)| \le \|g^{'}\|\Psi_{n,q}(c_1;x) + \frac{\|g^{''}\|}{2}\Psi_{n,q}(c_2;x),$$

where $\Psi_{n,q}(c_1;x)$ and $\Psi_{n,q}(c_2;x)$ are first and second central moments, we get

$$\begin{aligned} |\Psi_{n,q}(g;x) - g(x)| &\le x(a_n(q_n)-1)\|g^{'}\| + \frac{1}{2}\{x^2(a_n(q_n)-1)^2 + \frac{a_n(q_n)^2}{[n]}(x-x^2)\}\|g^{''}\| \\ &\le \{x(a_n(q_n)-1) + \frac{1}{2}\{x^2(a_n(q_n)-1)^2 + \frac{a_n(q_n)^2}{[n]}(x-x^2)\}\}\|g\|_{C^2[0,a]}. \end{aligned}$$

As $|\Psi_{n,q}(f;x)| \le \|f(x)\|$, we can write

$$\begin{aligned} &|\Psi_{n,q}(f;x) - f(x)| \le |\Psi_{n,q}(f-g;x) - f(x)| + |\Psi_{n,q}(g;x) - g(x)| + |f(x) - g(x)| \\ &\le 2\|g-f\|_{C^2[0,a]} + |\Psi_{n,q}(g;x) - g(x)| \\ &\le 2\left[\|g-f\|_{C^2[0,a]} + \{\frac{x}{2}(a_n(q_n)-1) + \frac{1}{4}\{x^2(a_n(q_n)-1)^2 + \frac{a_n(q_n)^2}{[n]}(x-x^2)\}\}\|g\|_{C^2[0,a]}\right] \\ &\le 2\left[\|g-f\|_{C^2[0,a]} + \{\frac{1}{2}(a_n(q_n)-1) + \frac{1}{4}\{(a_n(q_n)-1)^2 + \frac{a_n(q_n)^2}{[n]}\}\}\|g\|_{C^2[0,a]}\right]. \end{aligned}$$

By letting δ_n as that given in the statement of Theorem and on taking infimum over $g \in C^2[0,a]$ on the right hand side of the above inequality we get

$$|\Psi_{n,q}(f;x) - f(x)| \leq 2K(f,\delta_n).$$

Remark 4. Since $st_A - \lim_n a_n^2(q_n) = 1$, $st_A - \lim_n \frac{1}{[n]} = 0$, one can observe that $st_A - \lim_n \delta_n = 0$, the above theorem gives the rate of A-satatistical convergence of $\Psi_{n,q}(f;x)$ to f.

3 A rth Order Generalization of Operator

In this section, we introduce a generalization of the positive linear operator $\Psi_{n,q}$, by using the method introduced by Popova and Kirov[3]. Let us consider the space $C(r,f)[0,1]$ of all continuous functions for which the rth order derivative exists and continuous on $[0,1]$. The rth order generalization of $\Psi_{n,q}$is as follows:

$$\Psi_{n,r,q}(f;x) = \sum_{k=0}^{n}\sum_{i=0}^{r} p_{n,k}(q;x) f^{(i)}(\varphi_{n,k}(q)) \frac{(x-\varphi_{n,k}(q))^i}{i!}, \tag{3.1}$$

where $f \in C(r,f)[0,1]$, $x \in [0,1)$ and $\varphi_{n,k}(q) = \dfrac{a_n(q)[k]}{[n]}$. For $x = 1$, we define $\Psi_{n,r,q}(f;x) = f(1)$. Clearly for $r = 0$, $\Psi_{n,r,q}(f;x) = \Psi_{n,q}(f;x)$.

We prove some approximation theorems for $\Psi_{n,r,q}(f;x)$ as follows.

Theorem 5. *For $f \in C(r,f)[0,1]$ s.t. $f^{(r)} \in \mathrm{Lip}_M(\alpha)$ and for any $n \in \mathbf{N}$, $x \in [0,1]$ and $r \in \mathbf{N}$, we have*

$$\left|\Psi_{n,r,q}(f;x) - f(x)\right| \leq \frac{M\alpha B(\alpha,r)}{(r-1)!(\alpha+r)} |\Psi_{n,q}(g;x)|, \tag{3.2}$$

where $g(y) = |y-x|^{\alpha+r}$for each $x \in [0,1]$ and $B(\alpha,r)$ denotes the beta function.

Proof. Take $x \in [0,1)$, as for $x = 1$ the result is trivial. Consider

$$f(x) - \Psi_{n,r,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;x) f(x) - \Psi_{n,r,q}(f;x).$$

From the definition of $\Psi_{n,r,q}(f;x)$ (see (3.1)), we get

$$f(x) - \Psi_{n,r,q}(f;x) = \sum_{k=0}^{n} p_{n,k}(q;x) \left(f(x) - \sum_{i=0}^{r} f^{(i)}(\varphi_{n,k}(q)) \frac{(x-\varphi_{n,k}(q))^i}{i!} \right). \tag{3.3}$$

By Taylor's formula, we can write

$$\begin{aligned} &f(x) - \sum_{i=0}^{r} f^{(i)}(\varphi_{n,k}(q)) \frac{(x-\varphi_{n,k}(q))^i}{i!} \\ &= \frac{(x-\varphi_{n,k}(q))^r}{(r-1)!} \int_0^1 (1-t)^{r-1} \left(f^{(r)}(\varphi_{n,k}(q) + t(x-\varphi_{n,k}(q))) - f^{(r)}(\varphi_{n,k}(q)) \right) \mathrm{d}t. \end{aligned} \tag{3.4}$$

As $f^{(r)} \in Lip_M(\alpha)$, we obtain

$$\left|f^{(r)}(\varphi_{n,k}(q)+t(x-\varphi_{n,k}(q)))-f^{(r)}(\varphi_{n,k}(q))\right| \le Mt^{\alpha}|x-\varphi_{n,k}(q)|^{\alpha}. \tag{3.5}$$

Using the equations (3.4) and (3.5), we get

$$\left|f(x)-\sum_{i=0}^{r} f^{(i)}(\varphi_{n,k}(q))\frac{(x-\varphi_{n,k}(q))^i}{i!}\right| \le \frac{|x-\varphi_{n,k}(q)|^{\alpha+r}}{(r-1)!}\int_0^1 (1-t)^{r-1}t^{\alpha}\mathrm{d}t.$$

Also

$$\int_0^1 (1-t)^{r-1}t^{\alpha}\mathrm{d}t = \frac{\alpha B(\alpha,r)}{\alpha+r}.$$

Using the above facts we get

$$\left|f(x)-\sum_{i=0}^{r} f^{(i)}(\varphi_{n,k}(q))\frac{(x-\varphi_{n,k}(q))^i}{i!}\right| \le \frac{M\alpha B(\alpha,r)}{(r-1)!(\alpha+r)}|x-\varphi_{n,k}(q)|^{\alpha+r}. \tag{3.6}$$

Finally by the equations (3.3) and (3.6), we get the desired result.

Remark 5. In the above theorem we observe the following:

1. $g \in C[0,1]$ and $g(x)=0$.

2. $g \in \mathrm{Lip}_1(\alpha)$ as $|g(y)-g(x)| \le |y-x|^{\alpha}$ for $x,y \in [0,1]$.

Corollary 6. *Let $x \in [0,1]$ and $r \in \mathbf{N}$, then for $f \in C(r,f)[0,1]$ s.t. $f^{(r)} \in \mathrm{Lip}_M(\alpha)$ and for any $n \in \mathbf{N}$, we have*

$$\left|\Psi_{n,r,q}(f;x)-f(x)\right| \le \frac{2M\alpha B(\alpha,r)}{(r-1)!(\alpha+r)}\omega(g;\sqrt{\delta_n}). \tag{3.7}$$

Using Remark 5, Theorem 3 and Corollary 2 we get the result immediately.

Corollary 7. *Let $x \in [0,1]$ and $r \in \mathbf{N}$, then for $f \in C(r,f)[0,1]$ s.t. $f^{(r)} \in \mathrm{Lip}_M(\alpha)$ and for any $n \in \mathbf{N}$, we have*

$$\left|\Psi_{n,r,q}(f;x)-f(x)\right| \le \frac{M\alpha B(\alpha,r)}{(r-1)!(\alpha+r)}\delta_n^{\alpha/2}. \tag{3.8}$$

Again by using Remark 5, Theorem 5 and Corollary 3 we get the results immediately.

Corollary 8. *Let $x \in [0,1]$ and $r \in \mathbf{N}$, then for $f \in C(r,f)[0,1]$ s.t. $f^{(r)} \in \mathrm{Lip}_M(\alpha)$ and for any $n \in \mathbf{N}$, we have*

$$\left|\Psi_{n,r,q}(f;x)-f(x)\right| \le \frac{2M\alpha B(\alpha,r)}{(r-1)!(\alpha+r)}K(g;\delta_n). \tag{3.9}$$

Theorem 9. *Let q in (3.1) be replaced by the sequence $(q_n)_{n\in\mathbf{N}}$ satisfying (2.8), then $\Psi_{n,r,q_n}(f;\cdot)$ converges to f uniformly on $[0,1]$.*

Proof. The result is directly obtained by using Corollary 6 or 7.

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