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A UNIFIED THREE POINT APPROXIMATING SUBDIVISION SCHEME

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Abstract. In this paper, we propose a three point approximating subdivision scheme, with three shape parameters, that unifies three different existing three point approximating schemes. Some sufficient conditions for subdivision curve C^0 to C^3 continuity and convergence of the scheme for generating tensor product surfaces for certain ranges of parameters by using Laurent polynomial method are discussed. The systems of curve and surface design based on our scheme have been developed successfully in garment CAD especially for clothes modelling.

Key words: *approximating subdivision scheme, shape parameters, Laurent polynomial*

AMS (2010) subject classification: 65D17, 65D07, 65D05

1 Introduction

In recent years, the subdivision scheme became one of the most popular methods of creating geometric object in computer aided geometric design and in animation industry. Their popularity is due to the facts that subdivision algorithms are easy to implement and suitable for computer applications. If the limit curve / surface approximates the initial control polygon and that after subdivision, the newly generated control points are not in the limit curve / surface, the scheme is said to be approximating. It is called interpolating if after subdivision, the control points of the original control polygon and the new generated control points are interpolated on the limit curve / surface. The important schemes for applications should allow to control the shape of the limit curve and be capable of reproducing families of curves widely used in computer graphics. A wide variety of schemes that has been proposed in the literature which posses shape parameters [2,4,5,8,9,15] are interpolating[3,11,16], presented approximating subdivision schemes with tension

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parameters. Zhijie Cai[17] developed the systems of curve and surface design for garment CAD by taking nonuniform control points. Hassan et al[6] proposed approximating three point binary scheme which has C^3 continuity. Kai Hormann et al[10] offered the dual three-point scheme which has quadratic precision. Recently, Siddiqi et al [2] presented a new three point approximating C^2 subdivision scheme. The smoothness of subdivision schemes using Laurents polynomial method has been discussed by [1,5,13] and [6]. The aim of this work is to offer an approximating three point scheme that captures Hassan et al[6], Kai Hormann et al[10] and Siddiqi et al[12] schemes and offer more flexibility in curve and surface drawing. Here the proposed scheme is C^3 as well as accompanied with three shape parameters, which helps in designing more than a single parameter. It also provides more flexibility in designing, especially in cloth modeling.

2 Preliminaries

A general form of univariate subdivision scheme S which maps a polygon $f^k = \{f_i^k\}_{i\in\mathbf{Z}}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbf{Z}}$ is defined by

$$\begin{cases} f_{2i}^{k+1} = \sum\limits_{j\in\mathbf{Z}} a_{2j} f_{i-j}^k, \\ f_{2i+1}^{k+1} = \sum\limits_{j\in\mathbf{Z}} a_{2j+1} f_{i-j}^k, \end{cases} \tag{2.1}$$

where the set $a = \{a_i | i \in \mathbf{Z}\}$ of coefficients is called mask of the subdivision scheme. A necessary condition for the uniform convergence of the subdivision scheme (2.1) is that

$$\sum_{j\in\mathbf{Z}} a_{2j} = \sum_{j\in\mathbf{Z}} a_{2j+1} = 1. \tag{2.2}$$

For analysis of the subdivision scheme with mask a, it is very practical to consider the z-transform of the mask,

$$a(z) = \sum_{i\in\mathbf{Z}} a_i z^i, \tag{2.3}$$

which is usually called the symbol/Laurent polynomial of the scheme. From (2.2) and (2.3) the Laurent polynomial of a convergent subdivision scheme satisfies

$$a(-1) = 0 \quad \text{and} \quad a(1) = 2. \tag{2.4}$$

This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of associated Laurent polynomial $a^{(1)}(z)$ which can be defined as follows:

$$a^{(1)}(z) = \frac{2z}{1+z} a(z).$$

The subdivision scheme S_1 with symbol $a^{(1)}(z)$ is related to scheme S with symbol $a(z)$ by [5], Chapter 3, Theorem 2-3.

Since there are two rules for computing the values at next refinement level, one with even coefficients of the mask and the other with odd coefficients of the mask, we define the norm

$$\|S\|_\infty = \max\left\{\sum_{j\in\mathbf{Z}}|a_{2j}|, \sum_{j\in\mathbf{Z}}|a_{2j+1}|\right\},$$

$$\left\|\left(\frac{1}{2}S_n\right)^L\right\|_\infty = \max\left\{\sum_{j\in\mathbf{Z}}\left|b^{[n,L]}_{i+2^L j}\right| : i=0,1,\cdots,2^L-1\right\}, \tag{2.5}$$

where

$$b^{[n,L]}(z) = \frac{1}{2^L}\prod_{j=0}^{L-1}a^{(n)}(z^{2^j}), \quad a^{(n)}(z) = \frac{2z}{1+z}a^{(n-1)}(z).$$

In case of bivariate scheme, if $a(z_1,z_2) = \sum_{i,j\in\mathbf{Z}} a_{ij}z_1^i z_2^j$ is Laurent polynomial of scheme, then the necessary condition for convergence of the scheme is

$$a(1,1) = 4, \quad a(-1,1) = a(1,-1) = a(-1,-1) = 0. \tag{2.6}$$

3 A Three Point Approximating Curve Scheme

Given a polygon $f^k = \{f_i^k\}_{i\in\mathbf{Z}}$, the refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i\in\mathbf{Z}}$ at the level $k+1$ is given by the following recursive relation

$$\begin{aligned} f_{2i}^{k+1} &= \psi f_{i-1}^k + \phi f_i^k + \eta f_{i+1}^k, \\ f_{2i+1}^{k+1} &= \eta f_{i-1}^k + \phi f_i^k + \psi f_{i+1}^k, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \eta &= \frac{1}{16}\mu + \frac{\lambda}{2}(1-\omega)^2 - \frac{3}{32}\nu, \\ \phi &= \frac{10}{16}\mu + \frac{\lambda}{2}(1+2\omega-2\omega^2) + \frac{30}{32}\nu, \\ \psi &= \frac{5}{16}\mu + \frac{\lambda}{2}\omega^2 + \frac{5}{32}\nu, \end{aligned}$$

and $\lambda = 1-\mu-\nu$.

Remark 3.1. If we set $\nu = 0, \mu = 1$ in (3.1), we get Hassan et al[6] scheme. If we take $\mu = 0, \nu = 1$, we have the dual three point scheme [10]. If we take $\mu = \nu = 0$, then this scheme coincides with Siddiqi et al[12] subdivision scheme.

3.1 Smootheness Analysis of Three Point Curve Scheme

In this section, we analyze the smoothness of the proposed scheme S by taking $\mu = 4\nu + 1$, for all ν. For simplicity of computation we take $\nu = 1/2$. The Laurent polynomial of proposed scheme S is

$$\begin{aligned} a(z) &= \left(\frac{1}{16}\mu + \frac{\lambda}{2}(1-\omega)^2 - \frac{3}{32}\nu\right) z^{-3} + \left(\frac{5}{16}\mu + \frac{\lambda}{2}\omega^2 + \frac{5}{32}\nu\right) z^{-2} \\ &\quad + \left(\frac{10}{16}\mu + \frac{\lambda}{2}(1+2\omega-2\omega^2) + \frac{30}{32}\nu\right) z^{-1} + \left(\frac{10}{16}\mu + \frac{\lambda}{2}(1+2\omega-2\omega^2) + \frac{30}{32}\nu\right) \\ &\quad + \left(\frac{5}{16}\mu + \frac{\lambda}{2}\omega^2 + \frac{5}{32}\nu\right) z + \left(\frac{1}{16}\mu + \frac{\lambda}{2}(1-\omega)^2 - \frac{3}{32}\nu\right) z^2. \end{aligned}$$

This implies

$$a(z) = z^{-1}(1+z)\xi_1(z), \tag{3.2}$$

where

$$\begin{aligned} \xi_1(z) &= \left(\frac{1}{16}\mu + \frac{\lambda}{2}(1-\omega)^2 - \frac{3}{32}\nu\right) z^2 + \left(\frac{1}{4}\mu - \frac{\lambda}{2}(1-2\omega) + \frac{1}{4}\nu\right) z \\ &\quad + \left(\frac{3}{8}\mu + \lambda(1-\omega^2) + \frac{11}{16}\nu\right) + \left(\frac{1}{4}\mu - \frac{\lambda}{2}(1-2\omega) + \frac{1}{4}\nu\right) z^{-1} \\ &\quad + \left(\frac{1}{16}\mu + \frac{\lambda}{2}(1-\omega)^2 - \frac{3}{32}\nu\right) z^{-2}. \end{aligned}$$

From (2.5) for $L = n = 1$ and (3.2) we have

$$\frac{1}{2}a^{(1)}(z) = \frac{z}{1+z}a(z) = \xi_1(z). \tag{3.3}$$

For C^0 continuity of S we require that the Laurent polynomial $a(z)$ satisfies (2.4), which it does, and $\left\|\frac{1}{2}S_1\right\|_\infty < 1$. The norm of the scheme $\frac{1}{2}S_1$ is

$$\begin{aligned} \left\|\frac{1}{2}S_1\right\|_\infty &= \max\left\{2\left|\frac{1}{16}\mu + \frac{\lambda}{2}(1-\omega)^2 - \frac{3}{32}\nu\right| + \left|\frac{3}{8}\mu + \lambda(1-\omega^2) + \frac{11}{16}\nu\right|,\right. \\ &\qquad \left. 2\left|\frac{1}{4}\mu - \frac{\lambda}{2}(1-2\omega) + \frac{1}{4}\nu\right|\right\} < 1, \end{aligned}$$

for $0.65 < \omega < 0.85$. Therefore, $\frac{1}{2}S_1$ is contractive. Hence by [5], Chapter 3, Theorem 3, S is C^0.

By (3.3) the Laurent polynomial of the scheme S_1 can be written as

$$a^{(1)}(z) = 2z^{-1}(1+z)\xi_2(z), \tag{3.4}$$

where

$$\xi_2(z)=\left(\frac{1}{16}\mu+\frac{\lambda}{2}(1-\omega)^2-\frac{3}{32}\nu\right)z^2+\left(\frac{3}{16}\mu-\frac{\lambda}{2}(2-4\omega+\omega^2)+\frac{11}{32}\nu\right)z$$
$$+\left(\frac{3}{16}\mu+\frac{\lambda}{2}(4-4\omega-\omega^2)+\frac{11}{32}\nu\right)+\left(\frac{1}{16}\mu+\frac{\lambda}{2}(-5+6\omega+\omega^2)-\frac{3}{32}\nu\right)z^{-1}.$$

Utilizing (2.5) for $n=2$ & $L=1$ and (3.4) we get

$$\frac{1}{2}a^{(2)}(z)=\frac{z}{1+z}a^{(1)}(z)=2\xi_2(z). \tag{3.5}$$

Now for C^1 continuity we first need $a^{(1)}(z)$ to satisfy (2.4), which it does for $\lambda(3-4\omega)=0$, and this implies $\omega=3/4$ and $\left\|\left(\frac{1}{2}S_2\right)^L\right\|_\infty<1$. We empirically calculate the norm of $\frac{1}{2}S_2$ for $L=1$ and see that

$$\left\|\frac{1}{2}S_2\right\|_\infty=2\max\left\{\left|\frac{1}{16}\mu+\frac{\lambda}{2}(-5+6\omega+\omega^2)-\frac{3}{32}\nu\right|+\left|\frac{3}{16}\mu-\frac{\lambda}{2}(2-4\omega+\omega^2)\right.\right.$$
$$\left.\left.+\frac{11}{32}\nu\right|,\left|\frac{3}{16}\mu+\frac{\lambda}{2}(4-4\omega-\omega^2)+\frac{11}{32}\nu\right|+\left|\frac{1}{16}\mu+\frac{\lambda}{2}(1-\omega)^2-\frac{3}{32}\nu\right|\right\}<1.$$

Therefore, $\frac{1}{2}S_2$ is contractive. Hence by [[5], Chapter 3, Theorem 4], S is C^1. Now from (3.5) the Laurent polynomial of the scheme S_2 is

$$a^{(2)}(z)=4z^{-1}(1+z)\xi_3(z), \tag{3.6}$$

where

$$\begin{aligned}\xi_3(z) &= \left(\frac{1}{16}\mu+\frac{\lambda}{2}(1-\omega)^2-\frac{3}{32}\nu\right)z^2+\left(\frac{2}{16}\mu+\frac{\lambda}{2}(-3+6\omega\right.\\ &\left.-2\omega^2)+\frac{14}{32}\nu\right)z^1+\left(\frac{1}{16}\mu+\frac{\lambda}{2}(7-10\omega+\omega^2)-\frac{3}{32}\nu\right).\end{aligned}$$

With the choice of $n=3$ & $L=1$, we have the following from (2.5) and (3.6)

$$\frac{1}{2}a^{(3)}(z)=\frac{z}{1+z}a^{(2)}(z)=4\xi_3(z). \tag{3.7}$$

For C^2 continuity we require that $a^{(2)}(z)$ satisfy (2.4), which is true for $\lambda(3-4\omega)=0$ and $\left\|\left(\frac{1}{2}S_3\right)^L\right\|_\infty<1$. The norm of $\frac{1}{2}S_3$ for $L=1$ is

$$\begin{aligned}\left\|\frac{1}{2}S_3\right\|_\infty &= 4\max\left\{\left|\frac{1}{16}\mu+\frac{\lambda}{2}(7-10\omega+\omega^2)-\frac{3}{32}\nu\right|+\left|\frac{1}{16}\mu+\frac{\lambda}{2}(1-\omega)^2-\frac{3}{32}\nu\right|,\right.\\ &\left.\left|\frac{2}{16}\mu+\frac{\lambda}{2}(-3+6\omega-2\omega^2)+\frac{14}{32}\nu\right|\right\}<1.\end{aligned}$$

Therefore, $\frac{1}{2}S_3$ is contractive. Hence by [[5], Chapter 3, Theorem 4], S is C^2. Now from (3.7) the Laurent polynomial of the scheme S_3 can be written as

$$a^{(3)}(z) = 8z^{-1}(1+z)\xi_4(z),$$

where

$$\xi_4(z) = \left(\frac{1}{16}\mu + \frac{\lambda}{2}(1-\omega)^2 - \frac{3}{32}\nu\right)z^2 + \left(\frac{1}{16}\mu + \frac{\lambda}{2}(-4+8\omega-3\omega^2) + \frac{17}{32}\nu\right)z.$$

With the choice of $n=4$ & $L=1$, we have

$$\frac{1}{2}a^{(4)}(z) = \frac{z}{1+z}a^{(2)}(z) = 8\xi_4(z). \tag{3.8}$$

For C^3 continuity we require that $a^{(3)}(z)$ satisfy (2.4), it does for $\frac{\lambda}{2}(4\omega^2-18\omega+11) - \frac{20}{32}\nu = 0$ & $\omega = 3/4$, and

$$\begin{aligned}\left\|\frac{1}{2}S_4\right\|_\infty = 8\max\Bigg\{&\left|\frac{1}{16}\mu + \frac{\lambda}{2}(1-\omega)^2 - \frac{3}{32}\nu\right|,\\ &\left|\frac{1}{16}\mu + \frac{\lambda}{2}(-4+8\omega-3\omega^2) + \frac{17}{32}\nu\right|\Bigg\} < 1.\end{aligned}$$

Therefore $\frac{1}{2}S_4$ is contractive. Hence by [5], Chapter 3, Theorem 4, S is C^3.

Theorem 3.1. *The univariate subdivision scheme presented by* (3.1) *is* C^3 *for* $\omega = 3/4$, $\nu = 1/2$ *and* $\mu = 4\nu+1$.

Remark 3.2. Since for $\nu = 0, \mu = 1$, our scheme coincides with Hassan et al[6]. So the subdivided points lie on the the quartic (See [7]) then by [5], Chapter 2, Theorem 3 the maximum approximation order of proposed scheme is 5.

4 A Three Point Approximating Surface Scheme

In the previous Section, we have discussed the curve scheme. In this Section, we extend this scheme to tensor product surfaces. Starting from polygon $f^k = \{f^k_{i,j}\}_{i,j\in\mathbf{Z}}$, the refined polygon $f^{k+1} = \{f^{k+1}_{i,j}\}_{i,j\in\mathbf{Z}}$ at level $k+1$ is defined by tensor product of (3.1)

$$\begin{aligned} f^{k+1}_{2i,2j} &= \psi\{\psi f^k_{i-1,j-1} + \phi f^k_{i,j-1} + \eta f^k_{i+1,j-1}\} + \phi\{\psi f^k_{i-1,j} + \phi f^k_{i,j} + \eta f^k_{i+1,j}\}\\ &\quad +\eta\{\psi f^k_{i-1,j+1} + \phi f^k_{i,j+1} + \eta f^k_{i+1,j+1}\}, \end{aligned}$$

$$\begin{aligned} f^{k+1}_{2i+1,2j} &= \psi\{\eta f^k_{i-1,j-1} + \phi f^k_{i,j-1} + \psi f^k_{i+1,j-1}\} + \phi\{\eta f^k_{i-1,j} + \phi f^k_{i,j} + \psi f^k_{i+1,j}\}\\ &\quad +\eta\{\eta f^k_{i-1,j+1} + \phi f^k_{i,j+1} + \psi f^k_{i+1,j+1}\}, \end{aligned}$$

$$\begin{aligned} f^{k+1}_{2i,2j+1} &= \eta\{\psi f^k_{i-1,j-1}+\phi f^k_{i,j-1}+\eta f^k_{i+1,j-1}\}+\phi\{\psi f^k_{i-1,j}+\phi f^k_{i,j}+\eta f^k_{i+1,j}\} \\ &\quad +\psi\{\psi f^k_{i-1,j+1}+\phi f^k_{i,j+1}+\eta f^k_{i+1,j+1}\}, \end{aligned}$$

$$\begin{aligned} f^{k+1}_{2i+1,2j+1} &= \eta\{\eta f^k_{i-1,j-1}+\phi f^k_{i,j-1}+\psi f^k_{i+1,j-1}\}+\phi\{\eta f^k_{i-1,j}+\phi f^k_{i,j}+\psi f^k_{i+1,j}\} \\ &\quad +\psi\{\eta f^k_{i-1,j+1}+\phi f^k_{i,j+1}+\psi f^k_{i+1,j+1}\}. \end{aligned} \tag{4.1}$$

4.1 Convergence of Three Point Surface Scheme

The Laurent polynomial of scheme (4.1) is

$$a(z_1,z_2)=a(z_1)a(z_2),$$

where $a(z)$ is defined in (3.2). This implies

$$a(z_1,z_2)=(1+z_1)(1+z_2)q(z_1,z_2), \tag{4.2}$$

where

$$q(z_1,z_2)=\frac{1}{1024}z_1^{-3}z_2^{-3}\{L(z_1)L(z_2)\},$$

$$\begin{aligned} L(z) &= (16\lambda\omega^2-32\lambda\omega+16\lambda+2\mu-3\nu)z^4+(32\lambda\omega-16\lambda+8\mu+8\nu)z^3 \\ &\quad +(-32\lambda\omega^2-64\lambda\omega+32\lambda+22\nu+12\mu)z^2+(32\lambda\omega-16\lambda+8\nu+8\mu)z \\ &\quad +(16\lambda\omega^2-32\lambda\omega+16\lambda+2\mu-3\nu). \end{aligned}$$

Since $q(z_1,z_2)=q(z_2,z_1)$ then contractivity of only one scheme S_{a_1} or S_{a_2} is needed in order to prove the convergence of $a(z_1,z_2)$ (See [5]). From (4.2) the Laurent polynomial of the scheme S_{a_1} is

$$a_1(z_1,z_2) \quad = \quad (1+z_2)q(z_1,z_2).$$

This implies

$$a_1(z_1,z_2)=\left(\frac{1}{32}z_1^{-3}a(z_1)\right)\left(\frac{1}{32}z_2^{-3}(1+z_2)a(z_2)\right).$$

Since

$$\begin{aligned} \left\|\frac{1}{2}S_{a_1}\right\|_\infty &= \frac{1}{32}\max\{2|16\lambda\omega^2-32\lambda\omega+16\lambda+2\mu-3\nu|+|-32\lambda\omega^2 \\ &\quad -64\lambda\omega+32\lambda+12\mu+22\nu|,2|32\lambda\omega-16\lambda+8\mu+8\nu|\}<1, \end{aligned} \tag{4.3}$$

for $-0.25<\omega<0.875$, $\mu=4\nu+1$ and $\nu=0.05$, so S_{a_1} is contractive. By [5], Chapter 3, Theorem 6, S_a is convergent.

Theorem 4.1. *A proposed bivariate subdivision scheme defined by* (4.1) *is convergent for* $-0.25 < \omega < 0.875$*,* $\mu = 4\nu + 1$ *and* $\nu = 0.05$.

4.2 Applications of Proposed Scheme

We give several examples for illustration of curve and surface schemes. Figures 1 & 2 show curves and surfaces with different values of shape parameters. Finally, we give examples of our results in garment CAD. In Figures 3 & 4, cloth models generated by three point scheme is shown, in which control points and three times iteration have been used to generate the models. The versatility of scheme has been presented by showing different figures according to different values of parameters. It is noted that these parametric values are independent of parametric values under continuity conditions.

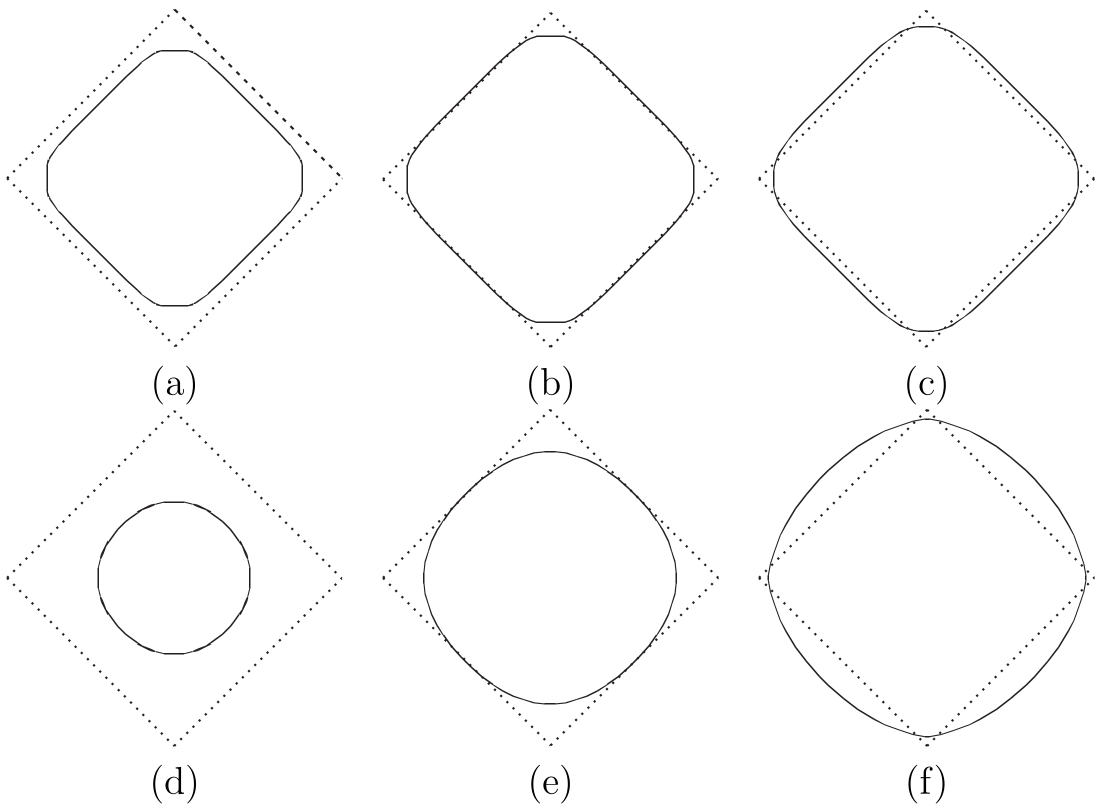

Fig. 1. Dashed lines show initial polygon and other curves are constructed by our scheme with parameters (a) $\mu = 0.0$, $\nu = 0.25$, $w = 0.5613$ (b) $\mu = 0.0$, $\nu = 0.55$, $w = 0.5398$ (c) $\mu = 0.0$, $\nu = 0.70$, $w = 0.50$ (d) $\mu = 0.95$, $\nu = 0.0$, $w = 2.70$ (e) $\mu = 0.0$, $\nu = 0.25$, $w = 0.75$ (f) $\mu = 0.0$, $\nu = 0.95$, $w = 1.50$

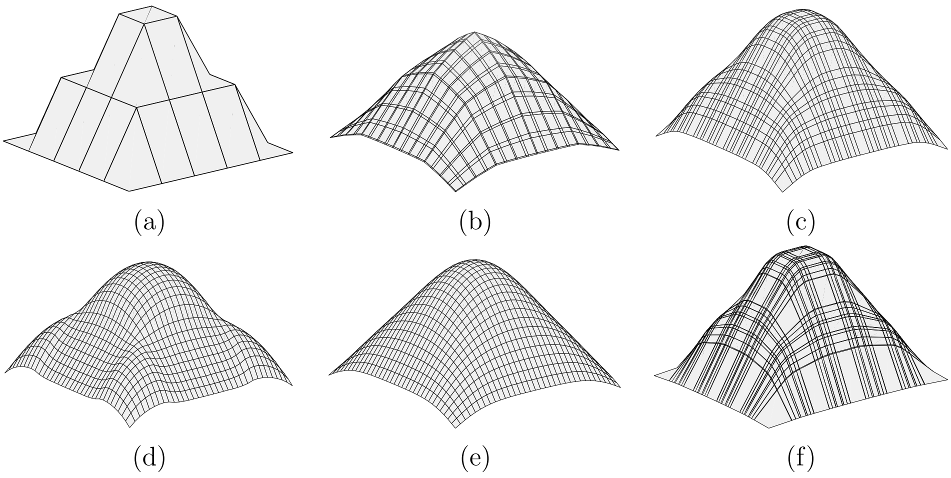

Fig. 2. (a) Initial polygon (b) Constructed by our scheme with parameters $\mu = 0.45$, $\nu = 0.30$, $w = 1.40$ (c) $\mu = 0.05$, $\nu = 0.20$, $w = 0.65$ (d) $\mu = 0.05$, $\nu = 0.90$, $w = 0.65$ (e) $\mu = 0.90$, $\nu = 0.10$, $w = 0.65$ (f) $\mu = 0.00$, $\nu = 0.05$, $w = 0.4740$

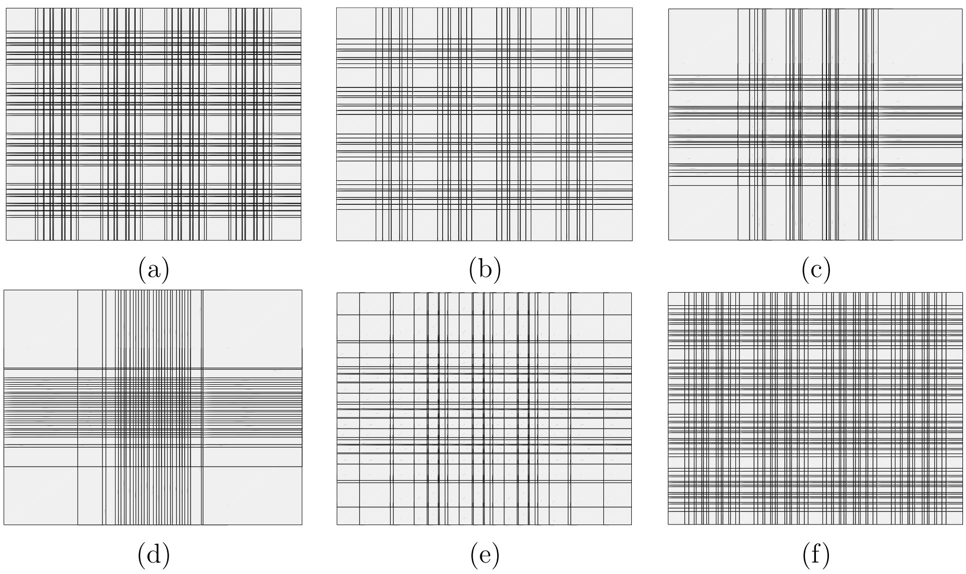

Fig. 3. lothes model surface generated by our scheme (a) $\mu = 0.05$, $\nu = 0.20$, $w = 0.50$ (b) $\mu = 0.00$, $\nu = 0.00$, $w = 0.70$ (c) $\mu = 0.00$, $\nu = 0.00$, $w = 0.251$ (d) $\mu = 0.00$, $\nu = 0.05$, $w = 0.00$ (e) $\mu = 0.45$, $\nu = 0.30$, $w = -0.99$ (f) $\mu = 0.05$, $\nu = 0.20$, $w = 0.60$

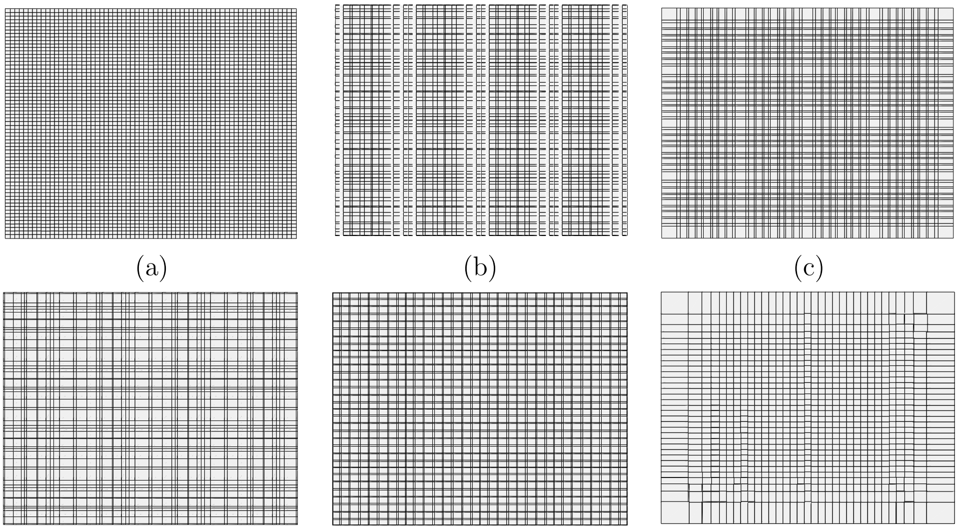

Fig. 4. Clothes model surface (a) $\mu = 0.05$, $\nu = 0.90$,$w = 0.65$ (b) $\mu = 0.05$, $\nu = 0.90$, $w = 2.75$ (c) $\mu = 0.05$, $\nu = 0.90$, $w = -1.78$ (d) $\mu = 0.45$, $\nu = 0.30$, $w = 1.75$ (e) $\mu = 0.45$, $\nu = 0.30$, $w = 1.43$ (f) $\mu = 0.10$, $\nu = 0.75$, $w = -1.10$

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