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BMO SPACES ASSOCIATED TO GENERALIZED PARABOLIC SECTIONS

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Abstract. Parabolic sections were introduced by Huang[1] to study the parabolic Monge-Ampère equation. In this note, we introduce the generalized parabolic sections $\mathcal{P}$ and define $\mathrm{BMO}^q_{\mathcal{P}}$ spaces related to these sections. We then establish the John-Nirenberg type inequality and verify that all $\mathrm{BMO}^q_{\mathcal{P}}$ are equivalent for $q \geq 1$.

Key words: $\mathrm{BMO}^q_{\mathcal{P}}$, *generalized parabolic section, John-Nirenberg's inequality*

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1 Introduction

In 1996, Caffarelli and Gutiérrez[1] studied the real variable theory related to the Monge-Ampère equation. They define sections to be a family of convex sets $\mathcal{F} = \{S(x,t) : x \in \mathbf{R}^n \text{ and } t > 0\}$ in $\mathbf{R}^n$ satisfying certain axioms of affine invariance. In term of these sections, they set up a variant of the Calderón-Zygmund decomposition by using the covering lemma and the doubling condition of a Borel measure μ; this decomposition is very important in studying the linearized Monge-Ampère equation[2]. As an application, they defined $\mathrm{BMO}_{\mathcal{F}}(\mathbf{R}^n)$ and showed the John-Nirenberg type inequality. Hardy space $H^1_{\mathcal{F}}(\mathbf{R}^n)$ associated to sections was established by Ding

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and Lin[4]. They also showed that the dual space of $H^1_{\mathcal{F}}(\mathbf{R}^n)$ is just the space $\mathrm{BMO}_{\mathcal{F}}(\mathbf{R}^n)$ defined in [1] and the Monge-Ampère singular integral operator is bounded from $H^1_{\mathcal{F}}(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$.

On the other hand, to study the parabolic Monge-Ampère equation, Huang[5] defined the parabolic sections and proved the Besicovitch type covering lemma and Caldrón-Zygmund decomposition associated with these sections.

So a natural question arises: is there a theory of Hardy and BMO type spaces associated to the parabolic sections? In the present note, we want to deal with this problem. More precisely, we introduce the generalized parabolic sections $\mathcal{P}$ and define $\mathrm{BMO}^q_{\mathcal{P}}$ spaces associated to these sections. We then establish the John-Nirenberg type inequality and verify that all $\mathrm{BMO}^q_{\mathcal{P}}$ are equivalent for $q \geq 1$. We remark that Hardy spaces for the generalized parabolic sections have been developed in [6].

Now we give the definition and basic properties of the generalized parabolic sections. Suppose $\varphi(t) : [0,\infty) \to [0,\infty)$ is a monotonic increasing function satisfying

$$\varphi(0) = 0, \qquad \lim_{t\to\infty} \varphi(t) = \infty, \qquad \varphi(2t) \leq C\varphi(t),$$

where C is a constant depending only on φ. Define the generalized parabolic sections by $Q_\varphi(z,r) = S(x,r) \times (t-\varphi(r)/2, t+\varphi(r)/2)$, where S is the (elliptic) sections. Note that if $\varphi(t) = t$, then this definition coincides with that used in [5]. Since we can choose $\varphi(t)$ to be any polynomial in t with nonnegative coefficients and without constant term, thus our definition of parabolic sections are more general. Throughout this paper, we will work for a fixed function φ described as above. Thus we use $Q(z,t)$ to denote the generalized parabolic section without specifying φ. The generalized parabolic sections have the following properties.

(A) There exist positive constants K_1, K_2, K_3, ε_1 and ε_2 with the following property: Given two sections $Q(z_0,r_0)$, $Q(z,r)$ with $r \leq r_0$ and T_p an affine transformation that normalizes $Q(z_0,r_0)$, if

$$Q(z_0,r_0) \cap Q(z,r) \neq \emptyset,$$

then there exists $z' = (x',t') \in B(0,K_3)$ such that

$$\begin{aligned} B\left(x', K_2(\tfrac{r}{r_0})^{\varepsilon_2}\right) \times \left(t' - \tfrac{\varphi(r)}{2r_0}, t' + \tfrac{\varphi(r)}{2r_0}\right) &\subset T_p(Q(z,r)) \\ &\subset B\left(x', K_1(\tfrac{r}{r_0})^{\varepsilon_1}\right) \times \left(t' - \tfrac{\varphi(r)}{2r_0}, t' + \tfrac{\varphi(r)}{2r_0}\right), \end{aligned} \tag{1.1}$$

and

$$T_p z = (Tx, t') \in B\left(x', \frac{1}{2}K_2(\frac{r}{r_0})^{\varepsilon_2}\right) \times \{t'\}.$$

(B) There exists $\delta > 0$ such that for a section $Q(z_0,r)$ and $z \notin Q(z_0,r)$, if T_p is an affine transformation that normalizes $Q(z_0,r)$, then

$$K(T_p(z),\varepsilon^{\delta}) \cap T_p(Q(z_0,(1-\varepsilon)r)) = \emptyset, \qquad \text{for} \qquad 0<\varepsilon<1.$$

(C) $\cap_{r>0}Q(z,r) = \{z\}$ and $\cup_{r>0}Q(z,r) = \mathbf{R}^{n+1}$.

The measure $\mathcal{M}$ is defined by $d\mathcal{M} = d\mu \otimes dt$, where μ is the Borel measure on $\mathbf{R}^n$ used in [5] and assumed to have the doubling property

$$\mu(S(x,2t)) \le C\mu(S(x,t)) \quad \text{for all section} \quad S(x,t)$$

and satisfy $\mu(\mathbf{R}^n) = \infty$. From this, we can easily see

$$\mathcal{M}(Q(x,2r)) \le C'\mathcal{M}(Q(x,r)) \quad \text{and} \quad \mathcal{M}(\mathbf{R}^{n+1}) = \infty. \tag{1.2}$$

Now we are ready to give the definition of the space $BMO^q_{\mathcal{P}}$. Below the generalized parabolic sections are often called parabolic sections for simplicity. For $1 \le q < \infty$, we say that $f \in \mathrm{BMO}^q_{\mathcal{P}}$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{\mathcal{M}(Q)}\int_Q |f(z)-f_Q|^q \mathrm{d}\mathcal{M}(z)\right)^{1/q} \le C \tag{1.2}$$

for every parabolic section Q, where

$$f_Q = \mathcal{M}(Q)^{-1}\int_Q f(z)\mathrm{d}\mathcal{M}(z).$$

The "Norm" of f in $\mathrm{BMO}^q_{\mathcal{P}}$ is the smallest constant appearing in (1.3) , and it is denoted by $\|f\|_{q,*}$. When $q=1$, we use $\mathrm{BMO}_{\mathcal{P}}$ and $\|\cdot\|_*$ to denote $\mathrm{BMO}^1_{\mathcal{P}}$ and $\|\cdot\|_{1,*}$ respectively.

Our first result is the following John-Nirenberg type inequality.

Theorem 1.1. *There exist positive constants C_1, C_2 which depend only on the measure $\mathcal{M}$ such that for every $f \in \mathrm{BMO}_{\mathcal{P}}$ and continuous we have*

$$\frac{1}{\mathcal{M}(Q)}\int_Q \exp\left(C_1\frac{|f(z)-f_Q|}{\| f \|_*}\right)\mathrm{d}\mathcal{M}(z) \le C_2$$

for every section Q.

Our second result is the equivalence of all $\mathrm{BMO}^q_{\mathcal{P}}$ spaces which is used in proving the dual theorem in [6].

Theorem 1.2. *For any* $1<q<\infty$, $\mathrm{BMO}^q_{\mathcal{P}}(\mathbf{R}^{n+1}) = \mathrm{BMO}_{\mathcal{P}}(\mathbf{R}^{n+1})$.

Finally, we would like to point out that the main idea used in this paper is taken from [3] and [1].

2 Proof of Theorem 1.1

We may assume that $\|f\|_* = 1$. We shall show that there exist positive numbers $\varepsilon < 1$ and M depending only on the measure $\mathcal{M}$ such that

$$\mathcal{M}\{z \in Q : |f(z) - f_Q| > t\} \leq \varepsilon_0 \mathcal{M}\{z \in Q : |f(z) - f_Q| > t - M\} \tag{2.1}$$

for every parabolic section Q and every $t > M$. Let us fix a parabolic section Q and set

$$A = \{z \in Q : |f(z) - f_Q| > t\}, \quad B = \{z \in Q : |f(z) - f_Q| > t - M\}.$$

The following Calderón-Zygmund decomposition is showed in [1] for the sections, but the argument also works for the generalized parabolic sections. Similar decomposition is also obtained in [5] to prove the parabolic Harnack inequality.

Theorem 2.1. *Given a bounded open set O and $0 < \lambda < 1$, there exists a family of parabolic sections $\mathcal{P}$=$\{Q(z_k, r_k)\}_{k=1}^{\infty}$ with the following properties:*

(a) $\dfrac{\delta}{C_1} \leq \dfrac{\mu(Q_k \cap O)}{\mu(Q_k)} \leq \delta$, *$C_1 > 0$ depending only on C in (A).*

(b) $O \subset \cup_k Q_k$.

(c) $\mathcal{M}(O) \leq \delta_0 \mathcal{M}(\cup_k Q_k)$ *where $\delta_0 = \delta_0(\delta, C_2) < 1$ and C_2 is a constant depending only on the parameters in (A), (B), (C) and dimension.*

(d) *If $\tau > 0$ is sufficiently small and $Q_k^{\tau} = Q(z_k, (1-\tau)t_k)$, then*

$$\sum_k \chi_{Q(z_k,(1-\varepsilon)r_k)}(z) \leq K \log \frac{1}{\tau},$$

and

$$\frac{\delta}{c_2} < \frac{\mathcal{M}(Q_k^{\tau} \cap O)}{\mathcal{M}(Q_k^{\tau})} \leq \delta,$$

where K is a constant depending only on the constants in (A) and (B) and c_2 depends only on the doubling constant in (1.2).

Let $0 < \delta < 1$ and $\{Q(z_k, r_k)\}_{k=1}^{\infty}$ be the decomposition of the set A given by Theorem 2.1. We then have

(1) There exists $C_1 > 0$, depending only on the doubling constant of M such that

$$\frac{\delta}{c_1} < \frac{\mathcal{M}(Q_k \cap A)}{\mathcal{M}(Q_k)} \leq \delta;$$

(2) $A \subset \cup_{k=1}^{\infty} Q_k$;

(3) For $0 < \tau$ sufficiently small the family Q_k^{τ}= $Q_k(z_k, (1-\tau)t_k)$, $k = 1, 2, \cdots$, has bounded overlaps; i.e.,

$$\sum_k \chi_{Q_k^{\tau}}(z) \leq K \log \frac{1}{\tau},$$

and

$$\frac{\delta}{c_2} < \frac{\mathcal{M}(Q_k^\tau \cap A)}{\mathcal{M}(Q_k^\tau)} \le \delta.$$

Pick $\varepsilon > 0$ sufficiently small such that

$$\varepsilon \le \min\left\{\frac{\delta}{c_1}, \frac{\delta}{c_2}\right\} < \delta \le 1 - \varepsilon.$$

Then

$$\varepsilon < \frac{\mathcal{M}(Q_k \cap A)}{Q_k}, \frac{\mathcal{M}(Q_k^\tau \cap A)}{Q_k^\tau} < 1 - \varepsilon.$$

We claim that

$$\frac{\mathcal{M}(Q_k^\tau \cap B)}{Q_k^\tau} > 1 - \frac{\varepsilon}{2}, \qquad \forall k \ge 1.$$

Denote $g(z) = f(z) - f_Q$ and note that (2.2) is equivalent to

$$\frac{\mathcal{M}(Q_k^\tau \cap \{z \in Q : |g(z)| \le t - M\})}{Q_k^\tau} \le \frac{\varepsilon}{2}.$$

Suppose on the contrary that the claim is false. Then there exists m such that

$$\frac{\mathcal{M}(Q_m^\tau \cap \{z \in Q : |g(z)| \le t - M\})}{Q_m^\tau} > \frac{\varepsilon}{2}. \tag{2.3}$$

Note that for any parabolic section Q', we have

$$g(z) - g_{Q'} = f(z) - f_{Q'}$$

and consequently, $\|g\|_* \le \|f\|_* \le 1$.

Let

$$\bar{g}_m = \frac{1}{\mathcal{M}(Q_m^\tau)} \int_{Q_m^\tau} g(z) \mathrm{d}\mathcal{M}.$$

We have the following possible cases:

(1) $t - \frac{M}{2} \le |\bar{g}_m| < t$.

(2) $t - M < |\bar{g}_m| < t - \frac{M}{2}$.

(3) $|\bar{g}_m| > t$.

(4) $|\bar{g}_m| < t - M$.

In the first case we have

$$\begin{aligned} 1 &\ge \frac{1}{\mathcal{M}(Q_m^\tau)} \int_{Q_m^\tau} |g(z) - \bar{g}_m| \mathrm{d}\mathcal{M} \\ &\ge \frac{1}{\mathcal{M}(Q_m^\tau)} \int_{Q_m^\tau \cap \{z \in Q : |g(z)| \le t - M\}} \big||g(z)| - |\bar{g}_m|\big| \mathrm{d}\mathcal{M} \\ &\ge \frac{M}{2} \frac{\mathcal{M}(Q_m^\tau \cap \{z \in Q : |g(z)| \le t - M\})}{\mathcal{M}(Q_m^\tau)} \ge \frac{M}{2}\frac{\varepsilon}{2} \end{aligned}$$

by (2.3). In the second case we have

$$1 \geq \frac{1}{\mathcal{M}(Q_m^\tau)} \int_{Q_m^\tau} |g(z) - \bar{g}_m| \mathrm{d}\mathcal{M} \geq \frac{1}{\mathcal{M}(Q_m^\tau)} \int_{Q_m^\tau \cap A} \big| |g(z)| - |\bar{g}_m| \big| \mathrm{d}\mathcal{M}$$
$$\geq \frac{M}{2} \frac{\mathcal{M}(Q_m^\tau \cap A)}{\mathcal{M}(Q_m^\tau)} \geq \frac{M}{2}\varepsilon.$$

by the property (3) of the decomposition. In case (3), we have

$$\begin{aligned} 1 &\geq \frac{1}{\mathcal{M}(Q_m^\tau)} \int_{Q_m^\tau} |g(z) - \bar{g}_m| \mathrm{d}\mathcal{M} \\ &\geq \frac{1}{\mathcal{M}(Q_m^\tau)} \int_{Q_m^\tau \cap \{z \in Q : |g(z)| \leq t - M\}} \big| |g(z)| - |\bar{g}_m| \big| \mathrm{d}\mathcal{M} \\ &\geq M \frac{\mathcal{M}(Q_m^\tau \cap \{z \in Q : |g(z)| \leq t - M\})}{\mathcal{M}(Q_m^\tau)} \geq M \frac{\varepsilon}{2} \end{aligned}$$

by (2.3). Finally, in case (4) we have

$$1 \geq \frac{1}{\mathcal{M}(Q_m^\tau)} \int_{Q_m^\tau} |g(z) - \bar{g}_m| \mathrm{d}\mathcal{M}(x) \geq \frac{1}{\mathcal{M}(Q_m^\tau)} \int_{Q_m^\tau \cap A} \big| |g(z)| - |\bar{g}_m| \big| \mathrm{d}\mathcal{M}(x)$$
$$\geq M \frac{\mathcal{M}(Q_m^\tau \cap A)}{\mathcal{M}(Q_m^\tau)} \geq M\varepsilon$$

obtained by the property (3) of the decomposition. Therefore, if M is large enough (depending only on the size of ε) we get a contradiction which proves the claim (2.2).

We now write

$$\mathcal{M}(Q_k^\tau \cap B) = \mathcal{M}(Q_k^\tau \cap \{z \in Q : t - M < |g(z)| \leq t\}) + \mathcal{M}(Q_m^\tau \cap A).$$

Hence, by (2.2) and the choice of ε we obtain

$$(1 - \frac{\varepsilon}{2})\mathcal{M}(Q_k^\tau) < \mathcal{M}(Q_k^\tau \cap \{z \in Q : t - M < |g(z)| \leq t\}) + (1 - \varepsilon)\mathcal{M}(Q_m^\tau),$$

i.e.,

$$\mathcal{M}(Q_k^\tau \cap \{z \in Q : t - M < |g(z)| \leq t\}) > (1 - \frac{\varepsilon}{2})\mathcal{M}(Q_k^\tau).$$

Summing the last inequality over k, using Theorem 2.1 and the bounded overlaps we obtain

$$\begin{aligned} &K \log \frac{1}{\tau} (\mathcal{M}(\{z \in Q : |g(z)| > t - M\}) - \mathcal{M}(\{z \in Q : |g(z)| > t\})) \\ &= K \log \frac{1}{\tau} \mathcal{M}(\{z \in Q : t - M < |g(z)| < t\}) \\ &\geq \sum_k \mathcal{M}(Q_k^\tau \cap \{z \in Q : t - M < |g(z)| \leq t\}) \\ &> \frac{\varepsilon}{2} \sum_k \mathcal{M}(Q_k^\tau) \geq \frac{\varepsilon}{2} \mathcal{M}(\bigcup_{k=1}^\infty Q_k^\tau) \geq \frac{\varepsilon}{2} \delta_0^{-1} \mathcal{M}(A). \end{aligned}$$

Hence,

$$\mathcal{M}(A) \leq \varepsilon_0 \mathcal{M}(B)$$

with $\varepsilon_0 = \left(1+\frac{\varepsilon}{2}(\delta_0 K \log\frac{1}{\tau})^{-1}\right)^{-1}$.

The inequality (2.1) implies that

$$\mathcal{M}(\{z\in Q : |g(z)| > M+kM\}) \leq \varepsilon_0^k \mathcal{M}(\{z\in Q : |g(z)| > M\}) \leq \varepsilon_0^k \mathcal{M}(Q) \tag{2.4}$$

for $k = 0, 1, \cdots$.

We write

$$\begin{aligned} &\int_Q \exp(\alpha|g(z)|)\mathrm{d}\mathcal{M} = \alpha\int_0^\infty e^{\alpha t}\mathcal{M}(\{z\in Q : |g(z)| > t\})\mathrm{d}t \\ = \ & \alpha\int_0^M e^{\alpha t}\mathcal{M}(\{z\in Q : |g(z)| > t\})\mathrm{d}t + \alpha\int_M^\infty e^{\alpha t}\mathcal{M}(\{z\in Q : |g(z)| > t\})\mathrm{d}t \\ = \ & I + II. \end{aligned}$$

Clearly $I \leq C\mathcal{M}(Q)$. On the other hand,

$$\begin{aligned} II &= e^{\alpha M}\int_0^\infty e^{\alpha t}\mathcal{M}(\{z\in Q : |g(z)| > M+t\})\mathrm{d}t \\ &= e^{\alpha M}\sum_{k=0}^\infty \int_{kM}^{(k+1)M} e^{\alpha t}\mathcal{M}(\{z\in Q : |g(z)| > M+t\})\mathrm{d}t \\ &\leq Me^{2\alpha M}\sum_{k=0}^\infty e^{\alpha kM}\mathcal{M}(\{z\in Q : |g(z)| > M+t\}) \\ &\leq Me^{2\alpha M}\sum_{k=0}^\infty e^{\alpha kM}\varepsilon_0^k\mathcal{M}(Q). \end{aligned}$$

Since $\varepsilon_0 < 1,\ \varepsilon_0 = e^{-\lambda_0}$ with $\lambda_0 > 0$, by taking $\alpha \leq \lambda_0$ the series converges and we obtain the desired result.

3 Proof of Theorem 1.2

By Hölder's inequality, it is easy to see $\mathrm{BMO}^q_{\mathcal{P}}(\mathbf{R}^{n+1}) \subset BMO_{\mathcal{P}}(\mathbf{R}^{n+1})$. On the other hand, we assume that $f \in \mathrm{BMO}_{\mathcal{P}}(\mathbf{R}^{n+1})$ with $\|f\|_* = 1$. Then we can take positive numbers $\varepsilon_0 < 1$ and Γ depending only on the constant in (1.2) and the constants in the properties (A) and (B) of the parabolic sections, such that, for any parabolic section $Q \in \mathcal{P}$ and each $k = 0, 1, 2, \cdots$, by (2.4)

$$\mathcal{M}(\{z\in Q : |f(z) - m_Q(f)| > \Gamma + k\Gamma\}) \leq \varepsilon_0^k \mathcal{M}(\{z\in Q : |f(z) - m_Q(f)| > \Gamma\}). \tag{3.1}$$

Thus

$$\begin{aligned}
&\frac{1}{\mathcal{M}(Q)}\int_Q |f(z)-m_Q(f)|\mathrm{d}\mathcal{M}(z)\\
&\quad=\frac{q}{\mathcal{M}(Q)}\int_0^\infty \alpha^{q-1}\mathcal{M}(\{z\in Q:|f(z)-m_Q(f)|>\alpha\})\mathrm{d}\alpha\\
&\quad=\frac{q}{\mathcal{M}(Q)}\int_0^\Gamma \alpha^{q-1}\mathcal{M}(\{z\in Q:|f(z)-m_Q(f)|>\alpha\})\mathrm{d}\alpha\\
&\qquad+\frac{q}{\mathcal{M}(Q)}\int_\Gamma^\infty \alpha^{q-1}\mathcal{M}(\{z\in Q:|f(z)-m_Q(f)|>\alpha\})\mathrm{d}\alpha\\
&\quad:=I_1+I_2.
\end{aligned}$$

We have

$$I_1\le \frac{q}{\mathcal{M}(Q)}\int_0^\Gamma \alpha^{q-1}\mathrm{d}\alpha\cdot\mathcal{M}(Q)=\Gamma^q<\infty.$$

On the other hand, by (3.1) and $\varepsilon_0<1$,

$$\begin{aligned}
I_2&=\frac{q}{\mathcal{M}(Q)}\int_0^\infty(\alpha+\Gamma)^{q-1}\mathcal{M}(\{z\in Q:|f(z)-m_Q(f)|>\alpha+\Gamma\})\mathrm{d}\alpha\\
&=\frac{q}{\mathcal{M}(Q)}\sum_{k=0}^\infty\int_{k\Gamma}^{(k+1)\Gamma}(\alpha+\Gamma)^{q-1}\mathcal{M}(\{z\in Q:|f(z)-m_Q(f)|>\alpha+\Gamma\})\mathrm{d}\alpha\\
&\le\frac{q}{\mathcal{M}(Q)}\sum_{k=0}^\infty((k+1)\Gamma+\Gamma)^{q-1}\mathcal{M}(\{z\in Q:|f(z)-m_Q(f)|>k\Gamma+\Gamma\})\cdot\Gamma\\
&\le\frac{q}{\mathcal{M}(Q)}\sum_{k=0}^\infty(k+2)^{q-1}\Gamma^q\varepsilon_0^k\mathcal{M}(\{z\in Q:|f(z)-m_Q(f)|>\Gamma\})\\
&\le q\Gamma^q\sum_{k=0}^\infty(k+2)^{q-1}\varepsilon_0^k\le C_q\Gamma^q.
\end{aligned}$$

Then by (3.2), we conclude that $\mathrm{BMO}^q_{\mathcal{P}}(\mathbf{R}^{n+1})\supset\mathrm{BMO}_{\mathcal{P}}(\mathbf{R}^{n+1})$ and Theorem 1.2 follows.

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