

# REPRESENTATION OF GENERALIZED TRANSLATIONS OF THE PRODUCT OF TWO FUNCTIONS ON SIGNED HYPERGROUPS ON THE REAL LINE

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**Abstract.** In this paper, we consider the generalized translations associated with the Dunkl and the Jacobi-Dunkl differential-difference operators on the real line which provide the structure of signed hypergroups on  $\mathbf{R}$ . Especially, we study the representation of the generalized translations of the product of two functions for these signed hypergroups.

**Key words:** *Bessel function, Dunkl operator, generalized translation, signed hypergroup, Jacobi-Dunkl operator, Jacobi function*

**AMS (2010) subject classification:** 33C45, 33C10, 43A62, 42A38

## 1 Introduction

The generalized translation operators are introduced by Delsarte and Levitan (see [4], [9]) and an interesting harmonic analysis for them is developed, since they permit to define the convolution and this concept allows to introduce the so called hypergroups (see [2]) and signed hypergroups (see [13], [12]).

A natural question arises : how the translation of the product of two functions may be represented in the framework of signed hypergroups?

We begin by the following remark. If we consider even functions on  $\mathbf{R}$ , the appropriate translation in this situation is given by

$$T_y f(x) = \frac{1}{2}[f(x+y) + f(x-y)].$$

In this case, we have

$$T_y(fg)(x) = T_y f(x)T_y g(x) + \frac{1}{4}[f(x+y) - f(x-y)][g(x+y) - g(x-y)]. \quad (1)$$

We observe that in contrast with the group situation, we do not have

$$T_y(fg)(x) = T_y f(x)T_y g(x),$$

and we remark the appearance of other terms. In [10] the authors have given analogous of the representation (1) in the context of Bessel-Kingman and Jacobi hypergroups on the half real line.

The aim of this paper is to extend analogous of the representation given in [10] in the context of signed hypergroups. More precisely, we are interested in establishing representations of  $T_y(fg)$ , where  $T_y$  is the generalized translation operator defined in the Bessel-Dunkl and Jacobi-Dunkl signed hypergroups.

The structure of these signed hypergroups is derived from differential-difference operators on  $\mathbf{R}$  of the form

$$\Lambda f(x) = f'(x) + \frac{A'(x)}{A(x)} \left( \frac{f(x) - f(-x)}{2} \right).$$

The Dunkl operator on  $\mathbf{R}$  corresponds to the function  $A(x) = |x|^{2\alpha+1}$ ,  $\alpha > -\frac{1}{2}$ . The Jacobi-Dunkl operator corresponds to the function  $A(x) = 2^{2\rho} (\sinh |x|)^{2\alpha+1} (\cosh x)^{2\beta+1}$ ,  $\rho = \alpha + \beta + 1$ .

M. Rösler<sup>[11]</sup>, ( resp. N. Ben Salem and A. Ould Ahmed Salem<sup>[11]</sup>, have established a product formula for the eigenfunctions of the Dunkl operator, ( resp. the Jacobi-Dunkl operator), which leads to generalized translations and uniformly bounded convolutions of point measures and generate structures of signed hypergroup on  $\mathbf{R}$ .

The paper is organized as follows.

The first part is devoted to the study of the representation of the translation of product of two functions for the translation associated to the Dunkl operator.

The second part deals with a similar study for the translation associated to the Jacobi-Dunkl operator.

## **2 Representation of Translation of the Product of Two Functions for Bessel-Dunkl Signed Hypergroup**

### **2.1 Preliminary**

In this subsection, we recall the basic notions and we give a summarized background for Bessel-Kingman and Bessel-Dunkl signed hypergroups which we need in this study.

First of all, we introduce the following notations.

Let  $\alpha \in \mathbf{R}$ ,  $\alpha \geq -\frac{1}{2}$ , the Bessel operator noted  $\Delta_\alpha$  is defined by

$$\Delta_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx}.$$

For  $\lambda \in \mathbf{C}$ , the normalized Bessel function  $j_\alpha$  given by

$$j_\alpha(x) = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{n! \Gamma(n + \alpha + 1)}, \quad x \in \mathbf{R} \tag{2}$$

is the eigenfunction of  $\Delta_\alpha$  satisfying

$$\begin{cases} \Delta_\alpha u = -\lambda^2 u, \\ u(0) = 1, \quad u'(0) = 0. \end{cases}$$

For  $\alpha > -\frac{1}{2}$ , the functions  $j_\alpha$  satisfy the following product formula

$$j_\alpha(x)j_\alpha(y) = c_\alpha \int_0^\pi j_\alpha(\sqrt{x^2 + y^2 - 2xy \cos \theta}) \sin^{2\alpha} \theta \, d\theta, \quad x, y \geq 0,$$

where

$$c_\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\frac{1}{2})\Gamma(\alpha + \frac{1}{2})}.$$

This product formula permits to define the translation for the Bessel-Kingman hypergroup denoted  $\sigma_y^\alpha f$  which is defined by

$$\sigma_y^\alpha f(x) = c_\alpha \int_0^\pi f(\sqrt{x^2 + y^2 - 2xy \cos \theta}) \sin^{2\alpha} \theta \, d\theta, \quad x, y \geq 0. \tag{3}$$

**Notations** We denote by

$$\begin{aligned} dm_\alpha(x) &= \frac{x^{2\alpha+1}}{2^\alpha \Gamma(\alpha + 1)} \, dx, \quad x \geq 0, \\ \sigma_y^{(m,\alpha)} f(x) &= (-1)^m c_{m,\alpha} \int_0^\pi f(\sqrt{x^2 + y^2 - 2xy \cos \theta}) C_m^\alpha(\cos \theta) \sin^{2\alpha} \theta \, d\theta, \end{aligned}$$

where

$$c_{m,\alpha} = \frac{\Gamma(\alpha)\Gamma(\alpha + 1)\Gamma(m + 1)}{2^{1-2\alpha} \pi \Gamma(m + 2\alpha)}.$$

$C_m^\gamma$  the Gegenbauer polynomial of degree  $m$  and of index  $\gamma$ .

It is well known that these polynomials verify the following proprieties (see [14], (7.33.1), (7.33.2), (7.33.3)):

i) For  $\gamma > 0$ , we have

$$\max_{-1 \leq x \leq 1} |C_m^\gamma(x)| = \frac{\Gamma(m+2\gamma)}{\Gamma(m+1)\Gamma(2\gamma)}. \tag{4}$$

ii) For  $-\frac{1}{2} < \gamma < 0$ , we have

• If  $m$  is even,

$$\max_{-1 \leq x \leq 1} |C_m^\gamma(x)| = |P_m^{(\gamma,\gamma)}(0)| = \left| \frac{\Gamma(\frac{m}{2} + \gamma)}{\Gamma(\gamma)\Gamma(\frac{m}{2} + 1)} \right|. \tag{5}$$

• If  $m$  is odd,

$$\max_{-1 \leq x \leq 1} |C_m^\gamma(x)| < |2\gamma|(m+2\gamma)^{-\frac{1}{2}} \left| \frac{\Gamma(\gamma + \frac{(m-1)}{2} + 1)}{\Gamma(\gamma+1)\Gamma(\frac{(m-1)}{2} + 1)} \right|. \tag{6}$$

iii) For all  $m, n \in \mathbf{N}$  (see [5], p. 177), we have

$$\int_0^\pi C_m^\gamma(\cos \theta) C_n^\gamma(\cos \theta) (\sin \theta)^{2\gamma} d\theta = 0, \quad \text{if } m \neq n,$$

$$\int_0^\pi (C_m^\gamma(\cos \theta))^2 (\sin \theta)^{2\gamma} d\theta = \frac{2^{1-2\gamma} \pi \Gamma(m+2\gamma)}{m! (\gamma+m) (\Gamma(\gamma))^2}, \quad \text{if } m = n. \tag{7}$$

For  $f \in L^1((0, \infty), dm_\alpha(x))$ , the Bessel transform noted  $\mathcal{H}_\alpha(f)$  is defined by

$$\mathcal{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(x) j_\alpha(\lambda x) dm_\alpha(x).$$

For  $f \in L^1((0, \infty), dm_\alpha(x))$  and  $\mathcal{H}_\alpha(f) \in L^1((0, \infty), dm_\alpha(x))$ , the inversion formula for the Bessel transform is given by

$$f(x) = \int_0^{+\infty} \mathcal{H}_\alpha(f)(\lambda) j_\alpha(\lambda x) dm_\alpha(x), \quad m_\alpha \text{ a. e.}$$

It is shown in [10] that for all  $\alpha > -\frac{1}{2}, \alpha \neq 0$ , under some conditions imposing on functions  $f$  and  $g$  that

$$\sigma_y^\alpha(fg)(x) = \sigma_y^\alpha f(x) \sigma_y^\alpha g(x) + \sum_{m=1}^{+\infty} \frac{(m+\alpha)\Gamma(m+2\alpha)}{\alpha\Gamma(2\alpha)\Gamma(m+1)} \sigma_y^{(m,\alpha)} f(x) \sigma_y^{(m,\alpha)} g(x). \tag{8}$$

We denote by

$$d\mu_\alpha(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} |x|^{2\alpha+1} dx, \quad x \in \mathbf{R}.$$

The Dunkl operator on  $\mathbf{R}$  of index  $\alpha$ ,  $\alpha > -\frac{1}{2}$  is defined by

$$\mathcal{D}_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x}, \quad f \in C^1(\mathbf{R}). \tag{9}$$

For  $\lambda \in \mathbf{C}$ , the function

$$e_\alpha(x) = j_\alpha(ix) + \frac{x}{2(\alpha + 1)} j_{\alpha+1}(ix), \quad x \in \mathbf{R}$$

is the eigenfunction of  $\mathcal{D}_\alpha$  satisfying

$$\begin{cases} \mathcal{D}_\alpha u = \lambda u, \\ u(0) = 1. \end{cases}$$

It was shown in [11] that the functions  $e_\alpha(\lambda \cdot)$ ,  $\lambda \in \mathbf{C}$ , satisfy on  $\mathbf{R}$ , the product formula

$$e_\alpha(\lambda x) e_\alpha(\lambda y) = \int_{\mathbf{R}} e_\alpha(\lambda z) d\mu_{x,y}^\alpha(z). \tag{10}$$

where

$$d\mu_{x,y}^\alpha(z) = \begin{cases} w_\alpha(x,y,z) d\mu_\alpha(z), & \text{if } xy \neq 0, \\ \delta_x, & \text{if } y = 0, \\ \delta_y, & \text{if } x = 0 \end{cases}$$

and  $w_\alpha(x,y,z)$  is a continuous function on  $]-|x|-|y|, -||x|-|y|| \cup [||x|-|y||, |x|+|y|]$ , uniformly bounded, with support  $[-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]$ .

The formula (10) can be written in the following form

$$e_\alpha(\lambda x) e_\alpha(\lambda y) = c_\alpha \left[ \int_0^\pi e_\alpha^\text{e}(\lambda Z_{x,y}(\theta)) h^\text{e}(x,y,\theta) \sin^{2\alpha} \theta d\theta + \int_0^\pi e_\alpha^\text{o}(\lambda Z_{x,y}(\theta)) h^\text{o}(x,y,\theta) \sin^{2\alpha} \theta d\theta \right],$$

where  $e_\alpha^\text{e}$  is the even part of  $e_\alpha$  namely, we have

$$e_\alpha^\text{e}(z) = j_\alpha(iz),$$

and  $e_\alpha^\text{o}$  is the odd part of  $e_\alpha$ , it is given by

$$\begin{aligned} e_\alpha^\text{o}(z) &= \frac{z}{2(\alpha + 1)} j_{\alpha+1}(iz), \\ Z_{x,y}(\theta) &= \sqrt{x^2 + y^2 - 2|xy| \cos \theta}, \\ h^\text{e}(x,y,\theta) &= 1 - \text{sgn}(xy) \cos \theta \end{aligned}$$

and

$$h^o(x, y, \theta) = \begin{cases} \frac{(x+y)(1-\operatorname{sgn}(xy)\cos\theta)}{Z_{x,y}(\theta)}, & \text{if } (x, y) \neq 0, \\ 0, & \text{if } (x, y) = 0. \end{cases}$$

The product formula (10) permits to define the generalized translation operator  $T_y^\alpha$  associated with  $\mathcal{D}_\alpha, y \in \mathbf{R}$ , by

$$\begin{aligned} T_y^\alpha f(x) &= \int_{\mathbf{R}} f(z) d\mu_{x,y}^\alpha(z) \\ &= c_\alpha \left[ \int_0^\pi f_e(Z_{x,y}(\theta)) h^e(x, y, \theta) \sin^{2\alpha} \theta d\theta + \int_0^\pi f_o(Z_{x,y}(\theta)) \times h^o(x, y, \theta) \sin^{2\alpha} \theta d\theta \right], \end{aligned}$$

where  $f_e$  and  $f_o$  are respectively the even and the odd parts of  $f$ .

The characteristic propriety of the translation  $T_y^\alpha$  is that

$$T_y^\alpha e_\alpha(x) = e_\alpha(x) e_\alpha(y).$$

This generalized translation provides the real line with a structure of signed hypergroup and the functions  $e_\alpha$  are the characters of this signed hypergroup.

For  $f \in L^1(\mathbf{R}, d\mu_\alpha(x))$ , the Dunkl transform on  $\mathbf{R}$  is defined by

$$\begin{aligned} \mathcal{F}_\alpha f(\lambda) &= \int_{\mathbf{R}} f(x) e_\alpha(-i\lambda x) d\mu_\alpha(x), \quad \lambda \in \mathbf{C} \\ &= \mathcal{H}_\alpha(f_e)(\lambda) + i\lambda \mathcal{H}_\alpha(J(f_o))(\lambda), \end{aligned} \tag{11}$$

where  $J$  is the integral operator defined on  $L^1(\mathbf{R}, d\mu_\alpha(x))$  by

$$Jf(x) = \int_{-\infty}^x f(t) dt, \quad x \in \mathbf{R}.$$

For  $f \in L^1(\mathbf{R}, d\mu_\alpha(x))$ , we have the following propriety:

$$\mathcal{F}_\alpha(T_y^\alpha f)(\lambda) = \mathcal{F}_\alpha f(\lambda) e_\alpha(i\lambda y).$$

Let  $f \in L^1(\mathbf{R}, d\mu_\alpha(x))$  such that  $\mathcal{F}_\alpha f \in L^1(\mathbf{R}, d\mu_\alpha(\lambda))$ , then we have the following inversion formula :

$$f(x) = \int_{\mathbf{R}} \mathcal{F}_\alpha f(\lambda) e_\alpha(i\lambda x) d\mu_\alpha(\lambda), \mu_\alpha \quad \text{a. e.} \tag{12}$$

## 2.2 Representation of Translation of the Product of Two Functions for Dunkl Transform

It is clear that if  $x = 0$  and  $f$  and  $g$  are two functions having translation,  $T_y^\alpha(fg)(0) = f(y)g(y)$ , then we will consider in the following that  $x \neq 0$ .

In the following we suppose that  $\alpha \neq 0$ , the case  $\alpha = 0$  will be studied earlier.

To derive an explicit formula of  $T_y^\alpha(fg)$  under some conditions imposing on  $f$  and  $g$ , we define the following transforms:

$$T_{y,e}^{(m,\alpha)} f(x) = (-1)^m c_{m,\alpha} \int_0^\pi [f_e(Z_{x,y}(\theta))h^e(x,y,\theta) + f_o(Z_{x,y}(\theta))h^o(x,y,\theta)] C_m^\alpha(\cos \theta) \sin^{2\alpha} \theta \, d\theta, \tag{13}$$

$$T_{y,o}^{(m,\alpha)} f(x) = (-1)^m c_{m,\alpha+1} \int_0^\pi \frac{Z_{x,y}(\theta)}{x} [f_e(Z_{x,y}(\theta))h^o(x,y,\theta) + f_o(Z_{x,y}(\theta))h^e(x,y,\theta)] \times C_m^{\alpha+1}(\cos \theta) \sin^{2\alpha} \theta \, d\theta. \tag{14}$$

The orthogonality propriety of the Gegenbauer polynomials leads to get the following proposition.

**Proposition 2.1.** *For an arbitrary polynomial  $f$  of degree  $\leq m - 1$ , we have  $T_{y,e}^{(m,\alpha)} f = 0$  and  $T_{y,o}^{(m,\alpha)} f = 0$ .*

**Proposition 2.2.** *For  $\alpha > -\frac{1}{2}$ ,  $\alpha \neq 0$  and  $f$  a convenient function, we have*

$$|T_{y,e}^{(m,\alpha)} f(x)| \leq c m^{\max(0,-\alpha)} (\sigma_{|y|}^\alpha(|f_e|)(|x|) + \sigma_{|y|}^\alpha(|f_o|)(|x|))$$

and

$$|T_{y,o}^{(m,\alpha)} f(x)| \leq c \frac{|x| + |y|}{|x|} (\sigma_{|y|}^\alpha(|f_e|)(|x|) + \sigma_{|y|}^\alpha(|f_o|)(|x|)),$$

where  $c$  is a positive constant.

*Proof.* It is clear that

$$|h^e(x,y,\theta)| \leq 2, \quad x, y \in \mathbf{R}, \quad \theta \in [0, \pi]$$

and it is shown in [11] that

$$|h^o(x,y,\theta)| \leq 2, \quad x, y \in \mathbf{R}, \quad \theta \in [0, \pi].$$

If  $\alpha > 0$ , from (4) we deduce that

$$c_{m,\alpha} |C_m^\alpha(\cos \theta)| \leq c_\alpha$$

and if  $-\frac{1}{2} < \alpha < 0$ , from (5) and (6), we have

$$c_{m,\alpha} |C_m^\alpha(\cos \theta)| \leq c m^{-\alpha} c_\alpha,$$

where  $c$  is a positive constant which may change from line to line throughout this paper.

Then, for  $\alpha > -\frac{1}{2}$ , we obtain

$$\begin{aligned} |T_{y,e}^{(m,\alpha)} f(x)| &\leq cm^{\max(0,-\alpha)} c_\alpha \left( \int_0^\pi f_e(Z_{x,y}(\theta)) \sin^{2\alpha} \theta \, d\theta + \int_0^\pi f_o(Z_{x,y}(\theta)) \sin^{2\alpha} \theta \, d\theta \right) \\ &= cm^{\max(0,-\alpha)} (\sigma_{|y|}^\alpha(|f_e|)(|x|) + \sigma_{|y|}^\alpha(|f_o|)(|x|)). \end{aligned}$$

On the other hand, if  $\alpha > -\frac{1}{2}$  then

$$c_{m,\alpha+1} |C_m^{\alpha+1}(\cos \theta)| \leq cc_\alpha,$$

and

$$\left| \frac{Z_{x,y}(\theta)}{x} \right| \leq \frac{|x| + |y|}{|x|},$$

then we obtain

$$\begin{aligned} |T_{y,o}^{(m,\alpha)} f(x)| &\leq c \frac{|x| + |y|}{|x|} \left( \int_0^\pi f_o(Z_{x,y}(\theta)) \sin^{2\alpha} \theta \, d\theta + \int_0^\pi f_e(Z_{x,y}(\theta)) \sin^{2\alpha} \theta \, d\theta \right) \\ &= c \frac{|x| + |y|}{|x|} (\sigma_{|y|}^\alpha(|f_o|)(|x|) + \sigma_{|y|}^\alpha(|f_e|)(|x|)). \end{aligned}$$

The main purpose of this section is to show, under some conditions imposing on the functions  $f$  and  $g$ , the following representation of  $T_y^\alpha(fg)$  :

$$\begin{aligned} T_y^\alpha(fg)(x) &= \sum_{m=0}^{+\infty} \frac{(m + \alpha)\Gamma(m + 2\alpha)}{\alpha\Gamma(2\alpha)\Gamma(m + 1)} T_{y,e}^{(m,\alpha)} g(x) \sigma_y^{(m,\alpha)}(f_e)(x) \\ &+ \sum_{m=0}^{+\infty} \frac{(\alpha + \frac{1}{2})\Gamma(m + 2\alpha + 2)(m + \alpha + 1)}{(\alpha + 1)^2\Gamma(2\alpha + 2)\Gamma(m + 1)} T_{y,o}^{(m,\alpha)} g(x) \sigma_y^{(m,\alpha+1)}(f_o)(x). \end{aligned} \tag{15}$$

**Lemma 2.3.** For arbitrary  $f$  having translations  $T_y^\alpha f$  and  $\alpha > -\frac{1}{2}$ ,  $\alpha \neq 0$ , we have

$$\begin{aligned} T_y^\alpha(fe_\alpha(\lambda.))(x) &= \sum_{m=0}^{+\infty} \frac{\Gamma(m + 2\alpha)(m + \alpha)}{\alpha\Gamma(2\alpha)\Gamma(m + 1)} \sigma_y^{(m,\alpha)} j_\alpha(\lambda.)(x) T_{y,e}^{(m,\alpha)} f(x) \\ &+ \sum_{m=0}^{+\infty} \frac{i\lambda x}{2(\alpha + 1)} \frac{(\alpha + \frac{1}{2})\Gamma(m + 2\alpha + 2)(m + \alpha + 1)}{(\alpha + 1)^2\Gamma(2\alpha + 2)\Gamma(m + 1)} \sigma_y^{(m,\alpha+1)} j_{\alpha+1}(\lambda.)(x) T_{y,o}^{(m,\alpha)} f(x). \end{aligned} \tag{16}$$

*Proof.* From the definition of  $T_y^\alpha$  we have

$$\begin{aligned} T_y^\alpha(fe_\alpha(i\lambda.))(x) &= c_\alpha \int_0^\pi [f_e(Z_{x,y}(\theta)) e_\alpha^e(i\lambda Z_{x,y}(\theta)) + f_o(Z_{x,y}(\theta)) e_\alpha^o(i\lambda Z_{x,y}(\theta))] \\ &\times h^e(x,y,\theta) \sin^{2\alpha} \theta \, d\theta + c_\alpha \int_0^\pi [f_o(Z_{x,y}(\theta)) e_\alpha^e(i\lambda Z_{x,y}(\theta)) + f_e(Z_{x,y}(\theta)) e_\alpha^o(i\lambda Z_{x,y}(\theta))] \end{aligned}$$



$$\times h^o(x, y, \theta) \sin^{2\alpha} \theta \, d\theta. \tag{17}$$

Now

$$e_\alpha^e(i\lambda Z_{x,y}(\theta)) = j_\alpha(\lambda Z_{x,y}(\theta))$$

and

$$e_\alpha^o(i\lambda Z_{x,y}(\theta)) = \frac{i\lambda Z_{x,y}(\theta)}{2(\alpha + 1)} j_{\alpha+1}(\lambda Z_{x,y}(\theta)).$$

From ([10], p 118-119) we have

$$j_\alpha(\lambda Z_{x,y}(\theta)) = \sum_{m=0}^{+\infty} (-1)^m \frac{(m + \alpha)}{\alpha} \sigma_y^{(m,\alpha)} j_\alpha(\lambda \cdot)(x) C_m^\alpha(\cos \theta), \tag{18}$$

then

$$e_\alpha^e(i\lambda Z_{x,y}(\theta)) = \sum_{m=0}^{+\infty} (-1)^m \frac{(m + \alpha)}{\alpha} \sigma_y^{(m,\alpha)} j_\alpha(\lambda \cdot)(x) C_m^\alpha(\cos \theta) \tag{19}$$

and

$$e_\alpha^o(i\lambda Z_{x,y}(\theta)) = \frac{i\lambda Z_{x,y}(\theta)}{2(\alpha + 1)} \sum_{m=0}^{+\infty} (-1)^m \frac{(m + \alpha + 1)}{\alpha + 1} \sigma_y^{(m,\alpha+1)} j_{\alpha+1}(\lambda \cdot)(x) C_m^{\alpha+1}(\cos \theta). \tag{20}$$

On the other hand, we have

$$\sum_{m=0}^{+\infty} \left| (-1)^m \frac{(m + \alpha)}{\alpha} \sigma_y^{(m,\alpha)} j_\alpha(\lambda \cdot)(x) C_m^\alpha(\cos \theta) \right| \leq c \exp(\lambda \sqrt{|xy|}) \tag{21}$$

and

$$\sum_{m=0}^{+\infty} \left| (-1)^m \frac{(m + \alpha + 1)}{\alpha + 1} \sigma_y^{(m,\alpha+1)} j_{\alpha+1}(\lambda \cdot)(x) C_m^{\alpha+1}(\cos \theta) \right| \leq c \exp(\lambda \sqrt{|xy|}).$$

Now from (17), (19) and (20), the dominated convergence theorem allows integration term by term and then gives the expression (16).

**Theorem 2.4.** *Let  $\alpha > -\frac{1}{2}$ ,  $\alpha \neq 0$ .*

i) *If  $f \in L^1(\mathbf{R}, d\mu_\alpha(x))$ ,  $\mathcal{F}_\alpha(f) \in L^1(\mathbf{R}, d\mu(\lambda))$  and  $g$  is a polynomial, then formula (15) holds.*

ii) *If  $f \in L^1(\mathbf{R}, d\mu_\alpha(x))$ ,  $\mathcal{F}_\alpha(f)$  has compact support and  $g$  has translation  $T_y^\alpha g$ , then formula (15) holds.*

*Proof.* It is noted that, under the conditions in (i) or in (ii) on  $f$ , we have  $\mathcal{F}_\alpha(f) \in L^1(\mathbf{R}, d\mu_\alpha(\lambda))$  and then the inversion formula

$$f(x) = \int_{\mathbf{R}} \mathcal{F}_\alpha f(\lambda) e_\alpha(i\lambda x) \, d\mu_\alpha(\lambda)$$

holds, so that we can write

$$\begin{aligned}
 T_y^\alpha(fg)(x) &= \int_{\mathbf{R}} f(z)g(z)d\mu_{x,y}^\alpha(z) \\
 &= \int_{\mathbf{R}} \left[ \int_{\mathbf{R}} \mathcal{F}_\alpha f(\lambda) e_\alpha(i\lambda z) d\mu_\alpha(\lambda) \right] g(z) d\mu_{x,y}^\alpha(z). \quad (22)
 \end{aligned}$$

From Fubini's theorem and Lemma 2.3 we obtain

$$\begin{aligned}
 T_y^\alpha(fg)(x) &= \int_{\mathbf{R}} \mathcal{F}_\alpha f(\lambda) T_y^\alpha(g \cdot e_\alpha(i\lambda \cdot))(x) d\mu_\alpha(\lambda) \\
 &= \int_{\mathbf{R}} \mathcal{F}_\alpha f(\lambda) \left[ \sum_{m=0}^{+\infty} \frac{\Gamma(m+2\alpha)(m+\alpha)}{\alpha\Gamma(2\alpha)\Gamma(m+1)} \sigma_y^{(m,\alpha)} j_\alpha(\lambda \cdot)(x) T_{y,e}^{(m,\alpha)} g(x) \right. \\
 &\quad \left. + \sum_{m=0}^{+\infty} \frac{i\lambda x}{2(\alpha+1)} \frac{(\alpha+\frac{1}{2})\Gamma(m+2\alpha+2)(m+\alpha+1)}{(\alpha+1)^2\Gamma(2\alpha+2)\Gamma(m+1)} \sigma_y^{(m,\alpha+1)} j_{\alpha+1}(\lambda \cdot)(x) T_{y,o}^{(m,\alpha)} g(x) \right] d\mu_\alpha(\lambda). \quad (23)
 \end{aligned}$$

If  $g$  is a polynomial, then by Proposition 2.1,  $T_{y,e}^{(m,\alpha)} g = 0$  and  $T_{y,o}^{(m,\alpha)} g = 0$  for sufficient large  $m$ , and hence, using Lemma 2.3,  $T_y^\alpha(g \cdot e_\alpha(i\lambda \cdot))(x)$  has the expression (16) with finite terms only. Integration term-by-term for (23) shows that

$$\begin{aligned}
 T_y^\alpha(fg)(x) &= \sum_{m=0}^{+\infty} \frac{\Gamma(m+2\alpha)(m+\alpha)}{\alpha\Gamma(2\alpha)\Gamma(m+1)} T_{y,e}^{(m,\alpha)} g(x) \int_{\mathbf{R}} \mathcal{F}_\alpha(f)(\lambda) \sigma_y^{(m,\alpha)} j_\alpha(\lambda \cdot)(x) d\mu_\alpha(\lambda) \\
 &\quad + \sum_{m=0}^{+\infty} \frac{(\alpha+\frac{1}{2})\Gamma(m+2\alpha+2)(m+\alpha+1)}{(\alpha+1)^2\Gamma(2\alpha+2)\Gamma(m+1)} T_{y,o}^{(m,\alpha)} g(x) \\
 &\quad \cdot \int_{\mathbf{R}} \frac{i\lambda x}{2(\alpha+1)} \mathcal{F}_\alpha(f)(\lambda) \sigma_y^{(m,\alpha+1)} j_{\alpha+1}(\lambda \cdot)(x) d\mu_\alpha(\lambda). \quad (24)
 \end{aligned}$$

Using Fubini's theorem in the last integrals above, we find

$$\begin{aligned}
 T_y^\alpha(fg)(x) &= \sum_{m=0}^{+\infty} \frac{\Gamma(m+2\alpha)(m+\alpha)}{\alpha\Gamma(2\alpha)\Gamma(m+1)} T_{y,e}^{(m,\alpha)} g(x) \sigma_y^{(m,\alpha)} \left( \int_{\mathbf{R}} \mathcal{F}_\alpha(f)(\lambda) j_\alpha(\lambda \cdot) d\mu_\alpha(\lambda) \right)(x) \\
 &\quad + \sum_{m=0}^{+\infty} \frac{(\alpha+\frac{1}{2})\Gamma(m+2\alpha+2)(m+\alpha+1)}{(\alpha+1)^2\Gamma(2\alpha+2)\Gamma(m+1)} T_{y,o}^{(m,\alpha)} g(x) \sigma_y^{(m,\alpha+1)} \\
 &\quad \cdot \left( \int_{\mathbf{R}} \frac{i\lambda x}{2(\alpha+1)} \mathcal{F}_\alpha(f)(\lambda) j_{\alpha+1}(\lambda \cdot) d\mu_\alpha(\lambda) \right)(x).
 \end{aligned}$$

From (11) we have

$$\begin{aligned} \int_{\mathbf{R}} \mathcal{F}_\alpha(f)(\lambda) j_\alpha(\lambda x) \, d\mu_\alpha(\lambda) &= \int_{\mathbf{R}} \mathcal{H}_\alpha(f_e)(\lambda) j_\alpha(\lambda x) \, d\mu_\alpha(\lambda) \\ &\quad + \int_{\mathbf{R}} i\lambda \mathcal{H}_\alpha(J(f_o))(\lambda) j_\alpha(\lambda x) \, d\mu_\alpha(\lambda) \\ &= \int_{\mathbf{R}} \mathcal{H}_\alpha(f_e)(\lambda) j_\alpha(\lambda x) \, d\mu_\alpha(\lambda) \\ &= \int_0^{+\infty} \mathcal{H}_\alpha(f_e)(\lambda) j_\alpha(\lambda x) \, dm_\alpha(\lambda) \\ &= f_e(x) \end{aligned}$$

due to the inversion formula for the Bessel transform.

On the other hand

$$\begin{aligned} \int_{\mathbf{R}} \frac{i\lambda x}{2(\alpha+1)} \mathcal{F}_\alpha(f)(\lambda) j_{\alpha+1}(\lambda x) \, d\mu_\alpha(\lambda) &= \int_{\mathbf{R}} \frac{i\lambda x}{2(\alpha+1)} \mathcal{H}_\alpha(f_e)(\lambda) j_{\alpha+1}(\lambda x) \, d\mu_\alpha(\lambda) \\ &\quad + \int_{\mathbf{R}} i\lambda \mathcal{H}_\alpha(J(f_o))(\lambda) \frac{i\lambda x}{2(\alpha+1)} j_{\alpha+1}(\lambda x) \, d\mu_\alpha(\lambda) \\ &= \int_0^{+\infty} \mathcal{H}_\alpha(J(f_o))(\lambda) \frac{d}{dx} j_\alpha(\lambda x) \, dm_\alpha(\lambda) \\ &= \frac{d}{dx} \left( \int_0^{+\infty} \mathcal{H}_\alpha(J(f_o))(\lambda) j_\alpha(\lambda x) \, dm_\alpha(\lambda) \right). \end{aligned}$$

As in above

$$\int_{\mathbf{R}} \frac{i\lambda x}{2(\alpha+1)} \mathcal{F}_\alpha(f)(\lambda) j_{\alpha+1}(\lambda \cdot)(x) \, d\mu_\alpha(\lambda) = \frac{d}{dx} J(f_o)(x) = f_o(x).$$

Thus (15) follows readily from (24) under the conditions in (i).

Now we show (15) under the conditions of part (ii). We first note that  $T_y^\alpha(g \cdot e_\alpha(\lambda \cdot))(x)$  has the same expression as (16), and by Proposition 2.2 and the expression (9) in [10] we have

$$\begin{aligned} \sum_{m=0}^{+\infty} \left| \frac{(-1)^m \Gamma(m+2\alpha)(m+\alpha)}{\alpha \Gamma(2\alpha) \Gamma(m+1)} T_{y,e}^{(m,\alpha)} g(x) \sigma_y^{(m,\alpha)} j_\alpha(\lambda \cdot)(x) \right| \\ \leq c(\sigma_{|y|}^\alpha(|g_e|)(|x|) + \sigma_{|y|}^\alpha(|g_o|)(|x|)) \sum_{m=0}^{+\infty} \frac{m^{2\alpha} m^{\max(0,-\alpha)}}{\Gamma(m+\alpha+1)^2} \left( \frac{\lambda^2 |xy|}{4} \right)^m. \end{aligned}$$

In the same way as (21), we have

$$\begin{aligned} \sum_{m=0}^{+\infty} \left| \frac{(-1)^m \Gamma(m+2\alpha)(m+\alpha)}{\alpha \Gamma(2\alpha) \Gamma(m+1)} T_{y,e}^{(m,\alpha)} g(x) \sigma_y^{(m,\alpha)} j_\alpha(\lambda \cdot)(x) \right| \\ \leq c(\sigma_{|y|}^\alpha(|g_e|)(|x|) + \sigma_{|y|}^\alpha(|g_o|)(|x|)) \exp(\lambda \sqrt{|xy|}). \end{aligned}$$

Also we have

$$\sum_{m=0}^{+\infty} \left| \frac{(-1)^m (\alpha + \frac{1}{2}) \Gamma(m + 2\alpha + 2) (m + \alpha + 1)}{(\alpha + 1)^2 \Gamma(2\alpha + 2) \Gamma(m + 1)} T_{y,o}^{(m,\alpha)} g(x) \sigma_y^{(m,\alpha+1)} j_{\alpha+1}(\lambda \cdot)(x) \right|$$

$$\leq c \frac{|x| + |y|}{|x|} (\sigma_{|y|}^\alpha(|f_e|)(|x|) + \sigma_{|y|}^\alpha(|f_o|)(|x|)) \exp(\lambda \sqrt{|xy|}).$$

Since  $\mathcal{F}_\alpha f$  has compact support, the integral in (23) is taken over a finite interval of  $\lambda$ , on which the function  $\exp(\lambda \sqrt{|xy|})$  is a bounded function of  $\lambda$ . By the dominated convergence theorem, integration term-by-term in (23), as we operate for (24), gives the formula (15).

*Remark.* If  $\alpha = 0$ , we denote

$$T_{y,e}^{(m,0)} f(x) = \frac{(-1)^m}{\pi} \int_0^\pi [f_e(Z_{x,y}(\theta)) h^e(x, y, \theta) + f_o(Z_{x,y}(\theta)) h^o(x, y, \theta)] \cos m\theta \, d\theta,$$

$$T_{y,o}^{(m,0)} f(x) = \frac{(-1)^m}{\pi} \int_0^\pi \frac{Z_{x,y}(\theta)}{x} [f_e(Z_{x,y}(\theta)) h^o(x, y, \theta) + f_o(Z_{x,y}(\theta)) h^e(x, y, \theta)] C_m^1(\cos \theta) \, d\theta$$

and we have

$$\sigma_y^{(m,0)} f(x) = \frac{(-1)^m}{\pi} \int_0^\pi f(Z_{x,y}(\theta)) \cos m\theta \, d\theta,$$

$$e_0^e(i\lambda Z_{x,y}(\theta)) = 2 \sum_{m=0}^{+\infty} (-1)^m \sigma_y^{(m,0)} j_0(\lambda \cdot)(x) \cos m\theta,$$

$$e_0^o(i\lambda Z_{x,y}(\theta)) = 2 \sum_{m=0}^{+\infty} (-1)^m \frac{i\lambda Z_{x,y}(\theta)}{2} \sigma_y^{(m,1)} j_1(\lambda \cdot)(x) C_m^1(\cos \theta).$$

From this formula we get

$$T_y^0(f e_0(i\lambda \cdot))(x) = 2 \sum_{m=0}^{+\infty} \sigma_y^{(m,0)} j_0(\lambda \cdot)(x) T_{y,e}^{(m,0)} f(x) + 2 \sum_{m=0}^{+\infty} \frac{i\lambda x}{2} \sigma_y^{(m,1)} j_1(\lambda \cdot)(x) T_{y,o}^{(m,0)} f(x).$$

Then from this formula and the inversion formula, we obtain the following representation :

$$T_y^0(fg)(x) = 2 \sum_{m=0}^{+\infty} \sigma_y^{(m,0)} f_e(x) T_{y,e}^{(m,0)} g(x) + 2 \sum_{m=0}^{+\infty} \sigma_y^{(m,1)} f_o(x) T_{y,o}^{(m,0)} g(x).$$

### 3 Representation of Translation of the Product of Two Functions for Jacobi-Dunkl Transforms

#### 3.1 Preliminary

First, we recall some results in Harmonic Analysis associated with the Jacobi operator. Let  $\alpha \geq \beta \geq -\frac{1}{2}$ , the Jacobi operator noted  $\Delta_{\alpha,\beta}$  is defined on  $(0, \infty)$  by

$$\begin{aligned} \Delta_{\alpha,\beta} &= \frac{d^2}{dx^2} [(2\alpha + 1) \coth x + (2\beta + 1) \tanh x] \frac{d}{dx} \\ &= \frac{1}{A_{\alpha,\beta}(x)} \frac{d}{dx} (A_{\alpha,\beta}(x) \frac{d}{dx}), \end{aligned}$$

where

$$A_{\alpha,\beta}(x) = 2^{2\rho} (\sinh x)^{2\alpha+1} (\cosh x)^{2\beta+1}$$

and

$$\rho = \alpha + \beta + 1.$$

For  $\lambda \in \mathbb{C}$ , the Jacobi function  $\varphi_\lambda^{(\alpha,\beta)}$  given by

$$\varphi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1 \left( \frac{\rho - i\lambda}{2}, \frac{\rho + i\lambda}{2}; \alpha + 1; -\sinh^2 x \right)$$

( ${}_2F_1$  denotes the Gauss hypergeometric function), is the eigenfunction of  $\Delta_{\alpha,\beta}$  satisfying

$$\begin{cases} \Delta_{\alpha,\beta} u = -(\lambda^2 + \rho^2)u, \\ u(0) = 1, \quad u'(0) = 0. \end{cases}$$

In the following, we suppose that  $\alpha > \beta > -\frac{1}{2}$ . The functions  $\varphi_\lambda^{(\alpha,\beta)}$  satisfy the following product formula

$$\varphi_\lambda^{(\alpha,\beta)}(x) \varphi_\lambda^{(\alpha,\beta)}(y) = \int_0^1 \int_0^\pi \varphi_\lambda^{(\alpha,\beta)}(Z_{x,y}(r, \psi)) \, dm_{\alpha,\beta}(r, \psi), \quad x, y \geq 0,$$

where

$$dm_{\alpha,\beta}(r, \psi) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha - \beta)\Gamma(\beta + \frac{1}{2})} (1 - r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin \psi)^{2\beta} \, dr d\psi.$$

This product formula permits to define the translation for the Jacobi-Dunkl hypergroup denoted  $\tau_y^{\alpha,\beta} f$  which is defined by

$$\tau_y^{\alpha,\beta} f(x) = \int_0^1 \int_0^\pi f(Z_{x,y}(r, \psi)) \, dm_{\alpha,\beta}(r, \psi), \quad x, y \geq 0, \tag{25}$$

where

$$Z_{x,y}(r, \psi) = \arg \cosh(|\cosh x \cosh y + r \exp(i\psi) \sinh x \sinh y|).$$

**Notations.** We denote by

$$\begin{aligned} d\mu_{\alpha,\beta}(x) &= A_{\alpha,\beta}(|x|) dx, \quad x \in \mathbf{R}, \\ c(\lambda) &= \frac{2^{\rho-i\lambda} \Gamma(\alpha+1) \Gamma(i\lambda)}{\Gamma(\frac{\rho+i\lambda}{2}) \Gamma(\frac{\alpha-\beta+1+i\lambda}{2})}, \quad \lambda \in \mathbf{C} \setminus \{i\mathbf{N}\}, \\ dv_{\alpha,\beta}(\lambda) &= \frac{1}{2\pi} |c(\lambda)|^{-2} d\lambda. \\ \tau_y^{(k,l,\alpha,\beta)} f(x) &= (-1)^{k+l} \int_0^1 \int_0^\pi f(Z_{x,y}(r, \psi)) R_{k,l}^{(\alpha,\beta)}(r, \psi) dm_{\alpha,\beta}(r, \psi), \end{aligned}$$

where  $\{R_{k,l}^{(\gamma,\eta)}\}$ ,  $k, l \in \mathbf{N} \setminus \{0\}$ ,  $k \geq l$  is the orthogonal system on  $L^2([0, 1] \times [0, \pi], dm_{\gamma,\eta}(r, \psi))$  defined by

$$R_{k,l}^{(\gamma,\eta)}(r, \psi) = R_l^{(\gamma-\eta-1, \beta+k-l)}(2r^2-1) r^{k-l} R_{k-l}^{(\eta-\frac{1}{2}, \eta-\frac{1}{2})}(\cos \psi) \tag{26}$$

and  $R_n^{(\gamma,\eta)}(x)$  are the Jacobi polynomials with  $R_n^{(\gamma,\eta)}(1) = 1$ .

For  $k \geq l$ ,  $R_{k,l}^{(\gamma,\eta)}$  verify the following propriety (see [7]),

$$|R_{k,l}^{(\gamma,\eta)}(r, \psi)| \leq ck^d, \tag{27}$$

where  $c$  and  $d$  are nonnegative constants depending only on  $\gamma$  and  $\eta$ .

For  $f \in L^1((0, \infty), d\mu_{\alpha,\beta}(x))$ , the Jacobi transform noted  $\mathcal{F}(f)$  is defined by

$$\mathcal{F}(f)(\lambda) = \int_0^{+\infty} f(x) \varphi_\mu^{(\alpha,\beta)}(x) d\mu_{\alpha,\beta}(x).$$

For  $f \in L^1((0, \infty), d\mu_{\alpha,\beta}(x))$  and  $\mathcal{F}f \in L^1((0, \infty), dv_{\alpha,\beta}(\lambda))$ , the inversion formula for the Jacobi transform is given by

$$f(x) = \int_0^{+\infty} \mathcal{F}f(\lambda) \varphi_\lambda^{(\alpha,\beta)}(x) dv_{\alpha,\beta}(\lambda), \quad v_{\alpha,\beta} \text{ a.e.}$$

It is shown in [10] that for all  $\alpha > \beta > -\frac{1}{2}$ , under some conditions imposing on the functions  $f$  and  $g$  we have

$$\tau_y^{\alpha,\beta}(fg)(x) = \tau_y^\alpha f(x) \tau_y^\beta g(x) + \sum_{k=1}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha,\beta)} \tau_y^{(k,l,\alpha,\beta)} f(x) \tau_y^{(k,l,\alpha,\beta)} g(x), \tag{28}$$

where

$$\Pi_{k,l}^{(\alpha,\beta)} = \frac{(2k-2l+2\beta)(k+l+\alpha)(\alpha-\beta)_l(2\beta+1)_{k-l}(\alpha+1)_k}{(k-l+2\beta)(k+\alpha)l!(k-l)!(\beta+1)_k}.$$

For  $\alpha > \beta > -\frac{1}{2}$ , the Jacobi-Dunkl operator on  $\mathbf{R}$  is defined by

$$\Lambda_{\alpha,\beta} f(x) = f'(x) + [(2\alpha+1) \coth x + (2\beta+1) \tanh x] \left( \frac{f(x) - f(-x)}{2} \right), \quad f \in C^1(\mathbf{R}). \tag{29}$$

For  $\lambda \in \mathbf{C}$ , the Jacobi-Dunkl kernel  $\Psi_\lambda^{\alpha,\beta}$  is the unique  $C^\infty$ -solution on  $\mathbf{R}$  of the differential-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta} u = i\lambda u, & \lambda \in \mathbf{C}, \\ u(0) = 1. \end{cases} \tag{30}$$

It is given by

$$\Psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{(\alpha,\beta)}(x) - \frac{i}{\lambda} \frac{d}{dx} \varphi_\mu^{\alpha,\beta}(x), & \text{if } \lambda \in \mathbf{C} \setminus \{0\}, \\ 1, & \text{if } \lambda = 0, \end{cases} \quad \forall x \in \mathbf{R}, \tag{31}$$

with  $\lambda^2 = \rho^2 + \mu^2$ .

The function  $\Psi_\lambda^{\alpha,\beta}$  can be written as follows

$$\Psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{(\alpha,\beta)}(x) + \frac{i\lambda}{2(\alpha+1)} \cosh x \sinh x \varphi_\mu^{\alpha+1,\beta+1}(x).$$

We put

$$h^e(x, y, r, \psi) = 1 + r \cos \psi.$$

$$\delta(x, y, r, \psi) = \sinh(x+y)(\cosh x \cosh y + r \cos \psi \cosh(x+y) + r^2 \sinh x \sinh y).$$

$$h^o(x, y, r, \psi) = \begin{cases} \frac{\delta(x,y,r,\psi)}{\cosh Z_{x,y}(r,\psi) \sqrt{(\cosh Z_{x,y}(r,\psi))^2 - 1}}, & \text{if } x \neq -y, \\ 0, & \text{if } x = -y. \end{cases}$$

It is shown in [1] that the functions  $\Psi_\lambda^{\alpha,\beta}$ ,  $\lambda \in \mathbf{C}$ , satisfy on  $\mathbf{R}$ , the product formula

$$\Psi_\lambda^{\alpha,\beta}(x) \Psi_\lambda^{\alpha,\beta}(y) = \int_{\mathbf{R}} \Psi_\lambda^{\alpha,\beta}(u) d\mu_{x,y}^{\alpha,\beta}(u), \tag{32}$$

where

$$d\mu_{x,y}^{\alpha,\beta}(u) = \begin{cases} \mathcal{K}_{\alpha,\beta}(x,y,u) d\mu_{\alpha,\beta}(u), & \text{if } xy \neq 0, \\ \delta_x, & \text{if } y = 0, \\ \delta_y, & \text{if } x = 0 \end{cases} \tag{33}$$

and  $\mathcal{K}_{\alpha,\beta}$  is a continuous function on  $]-|x|-|y|, -||x|-|y|| \cup [||x|-|y||, |x|+|y|[$  with a support in  $I_{x,y} = [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|]$ , which is not necessarily positive.

The formula (33) can be written in the following form

$$\Psi_\lambda^{\alpha,\beta}(x) \Psi_\lambda^{\alpha,\beta}(y) = \int_0^1 \int_0^\pi \Psi_{\lambda,e}^{\alpha,\beta}(Z_{x,y}(r,\psi)) h^e(x,y,r,\psi) dm_{\alpha,\beta}(r,\psi)$$

$$+ \int_0^1 \int_0^\pi \Psi_{\lambda,o}^{\alpha,\beta}((Z_{x,y}(r, \psi))h^o(x, y, r, \psi) \, dm_{\alpha,\beta}(r, \psi), \tag{34}$$

where  $\Psi_{\lambda,e}^{\alpha,\beta}$  is the even part of  $\Psi_\lambda^{\alpha,\beta}$  defined by

$$\Psi_{\lambda,e}^{\alpha,\beta}(x) = \phi_\mu^{\alpha,\beta}(x)$$

and  $\Psi_{\lambda,o}^{\alpha,\beta}$  is the odd part of  $\Psi_\lambda^{\alpha,\beta}$ , it is given by

$$\Psi_{\lambda,o}^{\alpha,\beta}(x) = \frac{-i}{\lambda} \frac{d}{dx} \phi_\mu^{\alpha,\beta}(x).$$

For  $f$  a convenient function, the product formula (35) allows to define the generalized translation operator  $T_y^{\alpha,\beta}$  associated with  $\Lambda_{\alpha,\beta}$ ,  $y \in \mathbf{R}$ , by

$$\begin{aligned} T_y^{\alpha,\beta} f(x) &= \int_{\mathbf{R}} f(z) \, d\mu_{x,y}^{\alpha,\beta}(z) \\ &= \int_0^1 \int_0^\pi f_e(Z_{x,y}(r, \psi))h^e(x, y, r, \psi) \, dm_{\alpha,\beta}(r, \psi) \\ &\quad + \int_0^1 \int_0^\pi f_o(Z_{x,y}(r, \psi))h^o(x, y, r, \psi) \, dm_{\alpha,\beta}(r, \psi). \end{aligned} \tag{35}$$

The characteristic propriety of the translation  $T_y^{\alpha,\beta}$  is that

$$T_y^{\alpha,\beta} \Psi_\lambda^{\alpha,\beta}(x) = \Psi_\lambda^{\alpha,\beta}(x) \Psi_\lambda^{\alpha,\beta}(y).$$

This generalized translation provides the real line with the structure of signed hypergroup, and the functions  $\Psi_\lambda^{\alpha,\beta}$  are the characters of this signed hypergroup.

We denote by

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2 - \rho^2}|c(\sqrt{\lambda^2 - \rho^2})|^2} \, ll_{\mathbf{R} \setminus ]-\rho, \rho[}(\lambda) \, d\lambda.$$

For  $f \in L^1(\mathbf{R}, d\mu_{\alpha,\beta}(x))$ , the Jacobi-Dunkl transform on  $\mathbf{R}$  is defined by

$$\mathcal{F}_{\alpha,\beta} f(\lambda) = \int_{\mathbf{R}} f(x) \Psi_{-\lambda}^{\alpha,\beta}(x) \, d\mu_{\alpha,\beta}(x), \quad \lambda \in \mathbf{C}.$$

For  $f \in L^1(\mathbf{R}, d\mu_{\alpha,\beta}(x))$ , we have the following propriety:

$$\mathcal{F}_{\alpha,\beta}(T_y^{\alpha,\beta} f)(\lambda) = \mathcal{F}_\alpha f(\lambda) \Psi_\lambda^{\alpha,\beta}(y).$$

Let  $f \in L^1(\mathbf{R}, d\mu_{\alpha,\beta}(x))$  such that  $\mathcal{F}_{\alpha,\beta} f \in L^1(\mathbf{R}, d\sigma(\lambda))$ , then we have the following inversion formula:

$$f(x) = \int_{\mathbf{R}} \mathcal{F}_{\alpha,\beta} f(\lambda) \Psi_\lambda^{\alpha,\beta}(x) \, d\sigma(\lambda), \quad \sigma \text{ a. e.} \tag{36}$$



**3.2 Representation of Translation of the Product of Two Functions for Jacobi-Dunkl Transforms**

It is clear that if  $x = 0$  and  $f, g$  are two functions having the translation,  $T_y^{\alpha,\beta}(fg)(0) = f(y)g(y)$ , then we will consider in the following that  $x \neq 0$ .

Analogously to those in Section 2, to derive an explicit formula of  $T_y^{\alpha,\beta}(fg)$  for given  $f$  and  $g$ , we define the following transforms:

$$T_{y,e}^{(k,l,\alpha,\beta)} f(x) = (-1)^{(k+l)} \int_0^1 \int_0^\pi [f_e(Z_{x,y}(r, \psi))h^e(x, y, r, \psi) + f_o(Z_{x,y}(r, \psi))h^o(x, y, r, \psi)] \times R_{k,l}^{(\alpha,\beta)}(r, \psi) dm_{\alpha,\beta}(r, \psi), \tag{37}$$

$$T_{y,o}^{(k,l,\alpha,\beta)} f(x) = (-1)^{(k+l)} \int_0^1 \int_0^\pi \frac{\sinh(Z_{x,y}(r, \psi)) \cosh(Z_{x,y}(r, \psi))}{\sinh x \cosh x} [f_e(Z_{x,y}(r, \psi))h^o(x, y, r, \psi) + f_o(Z_{x,y}(r, \psi))h^e(x, y, r, \psi)] R_{k,l}^{(\alpha+1,\beta+1)}(r, \psi) dm_{\alpha,\beta}(r, \psi). \tag{38}$$

**Proposition 3.1.** *Let  $\alpha > \beta > -\frac{1}{2}$ , then we have the following inequalities*

$$|T_{y,e}^{(k,l,\alpha,\beta)} f(x)| \leq ck^d (\tau_y^{\alpha,\beta}(|f_e|)(x) + \tau_y^{\alpha,\beta}(|f_o|)(x))$$

and

$$|T_{y,o}^{(k,l,\alpha,\beta)} f(x)| \leq ck^d |\coth x| \cosh^2 y (\tau_y^{\alpha,\beta}(|f_e|)(x) + \tau_y^{\alpha,\beta}(|f_o|)(x)),$$

where  $c$  and  $d$  are nonnegative constants depending only on  $\alpha$  and  $\beta$ .

*Proof.* It is clear that

$$|h^e(x, y, r, \psi)| \leq 2, \quad x, y \in \mathbf{R}, \quad r \in [0, 1], \quad \psi \in [0, \pi].$$

It is shown in [1] that

$$|h^o(x, y, r, \psi)| \leq 2, \quad x, y \in \mathbf{R}, \quad r \in [0, 1], \quad \psi \in [0, \pi].$$

From (27), for all  $\alpha > \beta > -\frac{1}{2}$ , there exists  $c$  and  $d$  two nonnegative constants depending only on  $\alpha$  and  $\beta$  such that

$$|R_{k,l}^{(\alpha,\beta)}(r, \psi)| \leq ck^d, \\ |R_{k,l}^{(\alpha+1,\beta+1)}(r, \psi)| \leq ck^d,$$

then we obtain

$$|T_{y,e}^{(k,l,\alpha,\beta)} f(x)| \leq ck^d \int_0^1 \int_0^\pi |f_e|(Z_{x,y}(r, \Psi)) dm_{\alpha,\beta}(r, \Psi) + \int_0^1 \int_0^\pi |f_o|(Z_{x,y}(r, \Psi)) dm_{\alpha,\beta}(r, \Psi)$$

which gives the first inequality.

On the other hand, we have

$$\left| \frac{\sinh(Z_{x,y}(r, \Psi)) \cosh(Z_{x,y}(r, \Psi))}{\sinh x \cosh x} \right| \leq 4 |\coth x| \cosh^2 y,$$

then we obtain the second inequality.

Now by the orthogonality of  $R_{k,l}^{(\alpha,\beta)}$  on  $L^2([0, 1] \times [0, \pi], dm_{\alpha,\beta}(r, \Psi))$ , it is easy to prove the following proposition.

**Proposition 3.2.** *If  $f(x) = g(\exp(x))$ , where  $g$  is a polynomial of degree  $\leq (k - 1)$ , we have  $T_{y,e}^{(k,l,\alpha,\beta)} = 0$  and  $T_{y,o}^{(k,l,\alpha,\beta)} = 0$ .*

We shall show that, under some conditions,  $T_y^{\alpha,\beta}(fg)$  has the following representation:

$$T_y^{\alpha,\beta}(fg)(x) = \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha,\beta)} T_y^{(k,l,\alpha,\beta)} f(x) T_{y,e}^{(k,l,\alpha,\beta)} g(x) + \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha+1,\beta+1)} T_y^{(k,l,\alpha+1,\beta+1)} f(x) T_{y,o}^{(k,l,\alpha,\beta)} g(x). \tag{39}$$

**Lemma 3.3.** *For arbitrary  $f$  having translations  $T_y^{\alpha,\beta} f$ , we have*

$$T_y^{\alpha,\beta}(f\Psi_\lambda^{\alpha,\beta})(x) = \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha,\beta)} T_y^{(k,l,\alpha,\beta)} \varphi_\mu^{(\alpha,\beta)}(x) T_{y,e}^{(k,l,\alpha,\beta)} f(x) + \sum_{k=0}^{+\infty} \sum_{l=0}^k \frac{i\lambda}{2(\alpha+1)} \cosh x \sinh x \Pi_{k,l}^{(\alpha+1,\beta+1)} T_y^{(k,l,\alpha+1,\beta+1)} \varphi_\mu^{(\alpha+1,\beta+1)}(x) T_{y,o}^{(k,l,\alpha,\beta)} f(x). \tag{40}$$

*Proof.* From the definition of  $T_y^{\alpha,\beta}$ , we can write

$$T_y^{\alpha,\beta}(f\Psi_\lambda^{\alpha,\beta})(x) = \int_0^1 \int_0^\pi [f_e(Z_{x,y}(r, \Psi)) \Psi_{\lambda,e}^{\alpha,\beta}(Z_{x,y}(r, \Psi)) + f_o(Z_{x,y}(r, \Psi)) \Psi_{\lambda,o}^{\alpha,\beta}(Z_{x,y}(r, \Psi))] \times h^\epsilon(x, y, r, \Psi) dm_{\alpha,\beta}(r, \Psi) + \int_0^1 \int_0^\pi [f_e(Z_{x,y}(r, \Psi)) \Psi_{\lambda,o}^{\alpha,\beta}(Z_{x,y}(r, \Psi)) + f_o(Z_{x,y}(r, \Psi)) \Psi_{\lambda,e}^{\alpha,\beta}(Z_{x,y}(r, \Psi))] h^\circ(x, y, r, \Psi) dm_{\alpha,\beta}(r, \Psi). \tag{41}$$

Now

$$\Psi_{\lambda,e}^{\alpha,\beta}(Z_{x,y}(r,\psi)) = \varphi_{\mu}^{(\alpha,\beta)}(Z_{x,y}(r,\psi))$$

and

$$\Psi_{\lambda,o}^{\alpha,\beta}(Z_{x,y}(r,\psi)) = \frac{i\lambda}{2(\alpha+1)} \sinh(Z_{x,y}(r,\psi)) \cosh(Z_{x,y}(r,\psi)) \varphi_{\mu}^{(\alpha+1,\beta+1)}(Z_{x,y}(r,\psi)).$$

From ([10], p 123), we have the following addition formula

$$\varphi_{\mu}^{(\alpha,\beta)}(Z_{x,y}(r,\psi)) = \sum_{k=0}^{+\infty} \sum_{l=0}^k (-1)^{k+l} T_y^{(k,l,\alpha,\beta)} \varphi_{\mu}^{(\alpha,\beta)}(x) \Pi_{k,l}^{(\alpha,\beta)} R_{k,l}^{(\alpha,\beta)}(r,\psi), \tag{42}$$

then

$$\Psi_{\lambda,e}^{\alpha,\beta}(Z_{x,y}(r,\psi)) = \sum_{k=0}^{+\infty} \sum_{l=0}^k (-1)^{k+l} T_y^{(k,l,\alpha,\beta)} \varphi_{\mu}^{(\alpha,\beta)}(x) \Pi_{k,l}^{(\alpha,\beta)} R_{k,l}^{(\alpha,\beta)}(r,\psi) \tag{43}$$

and

$$\begin{aligned} \Psi_{\lambda,o}^{\alpha,\beta}(Z_{x,y}(r,\psi)) &= \frac{i\lambda}{2(\alpha+1)} \sinh(Z_{x,y}(r,\psi)) \cosh(Z_{x,y}(r,\psi)) \times \\ &\sum_{k=0}^{+\infty} \sum_{l=0}^k (-1)^{k+l} T_y^{(k,l,\alpha+1,\beta+1)} \varphi_{\mu}^{(\alpha+1,\beta+1)}(x) \Pi_{k,l}^{(\alpha+1,\beta+1)} R_{k,l}^{(\alpha+1,\beta+1)}(r,\psi). \end{aligned} \tag{44}$$

From the absolute and uniform convergence of the series (43) and (44) with respect to  $(r, \psi) \in [0, 1] \times [0, \pi]$  (see [7], Theorem 2.1), the dominated convergence theorem allows integration term by term and then shows the lemma.

**Theorem 3.4.** Let  $\alpha > \beta > -\frac{1}{2}$ .

i) If  $f \in L^1(\mathbf{R}, d\mu_{\alpha,\beta}(x))$ ,  $\mathcal{F}_{\alpha,\beta}(f) \in L^1(\mathbf{R}, d\sigma(\lambda))$ ,  $g = h(\exp x)$ , and  $h$  is a polynomial, then the formula (39) holds.

ii) If  $f \in L^1(\mathbf{R}, d\mu_{\alpha,\beta}(x))$ ,  $\mathcal{F}_{\alpha,\beta}(f)$  has compact support and  $g$  has translations  $T_y^{\alpha,\beta} g$ , then the formula (39) holds.

*Proof.* From ([6], Cor. 9),  $|c(\mu)|^{-2} \leq c(1+\mu)^{2\alpha+1}$  for  $0 < \mu < \infty$ , where  $c$  is a nonnegative constant. Hence, if  $\mathcal{F}_{\alpha,\beta}(f)$  has compact support, then  $\mathcal{F}_{\alpha,\beta}(f) \in L^1(\mathbf{R}, d\sigma(\lambda))$ . The inversion formula

$$f(x) = \int_{\mathbf{R}} \mathcal{F}_{\alpha,\beta} f(\lambda) \psi_{\lambda}^{\alpha,\beta}(x) d\sigma(\lambda)$$

holds. So that we can write

$$\begin{aligned} T_y^{\alpha,\beta}(fg)(x) &= \int_{\mathbf{R}} f(z)g(z) d\mu_{x,y}^{\alpha,\beta}(z) \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}} \mathcal{F}_{\alpha,\beta} f(\lambda) \psi_{\lambda}^{\alpha,\beta}(z) d\sigma(\lambda) g(z) d\mu_{x,y}^{\alpha,\beta}(z). \end{aligned}$$

From Fubini's theorem and Lemma 3.3, we obtain

$$\begin{aligned}
 T_y^{\alpha,\beta}(fg)(x) &= \int_{\mathbf{R}} \mathcal{F}_{\alpha,\beta}f(\lambda) T_y^{\alpha,\beta}(g\Psi_{\lambda}^{\alpha,\beta})(x) \, d\sigma(\lambda) \\
 &= \int_{\mathbf{R}} \mathcal{F}_{\alpha,\beta}f(\lambda) \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha,\beta)} T_y^{(k,l,\alpha,\beta)} \varphi_{\mu}^{(\alpha,\beta)}(x) T_{y,e}^{(k,l,\alpha,\beta)} f(x) \\
 &\quad + \sum_{k=0}^{+\infty} \sum_{l=0}^k \frac{i\lambda}{2(\alpha+1)} \cosh x \sinh x \Pi_{k,l}^{(\alpha+1,\beta+1)} T_y^{(k,l,\alpha+1,\beta+1)} \\
 &\quad \varphi_{\mu}^{(\alpha+1,\beta+1)}(x) T_{y,o}^{(k,l,\alpha,\beta)} f(x) \, d\sigma(\lambda). \tag{45}
 \end{aligned}$$

If  $g = h(\exp x)$ , where  $h$  is a polynomial, then by Proposition 3.2,  $T_{y,e}^{(k,l,\alpha,\beta)} = 0$  and  $T_{y,o}^{(k,l,\alpha,\beta)} = 0$  for sufficient large  $k$ , and hence using Lemma 3.3,  $T_y^{\alpha}(g\Psi_{\lambda}^{\alpha,\beta})(x)$  has the expression (40) with finite terms only. Integration term-by-term for (45) shows that

$$\begin{aligned}
 T_y^{\alpha,\beta}(fg)(x) &= \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha,\beta)} T_{y,e}^{(k,l,\alpha,\beta)} g(x) \int_{\mathbf{R}} \mathcal{F}_{\alpha,\beta}f(\lambda) T_y^{(k,l,\alpha,\beta)} \varphi_{\mu}^{(\alpha,\beta)}(x) \, d\sigma(\lambda) \\
 &\quad + \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha+1,\beta+1)} T_{y,o}^{(k,l,\alpha,\beta)} g(x) \int_{\mathbf{R}} \frac{i\lambda}{2(\alpha+1)} \sinh x \cosh x \\
 &\quad \cdot \mathcal{F}_{\alpha,\beta}f(\lambda) T_y^{(k,l,\alpha+1,\beta+1)} \varphi_{\mu}^{(\alpha+1,\beta+1)}(x) \, d\sigma(\lambda). \tag{46}
 \end{aligned}$$

Using Fubini's theorem to the last integrals above, we obtain

$$\begin{aligned}
 T_y^{\alpha,\beta}(fg)(x) &= \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha,\beta)} T_{y,e}^{(k,l,\alpha,\beta)} g(x) T_y^{(k,l,\alpha,\beta)} \left( \int_{\mathbf{R}} \mathcal{F}_{\alpha,\beta}f(\lambda) \varphi_{\mu}^{(\alpha,\beta)} \, d\sigma(\lambda) \right)(x) \\
 &\quad + \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha+1,\beta+1)} T_{y,o}^{(k,l,\alpha,\beta)} g(x) T_y^{(k,l,\alpha+1,\beta+1)} \\
 &\quad \left( \int_{\mathbf{R}} \frac{i\lambda}{2(\alpha+1)} \sinh x \cosh x \mathcal{F}_{\alpha,\beta}f(\lambda) \varphi_{\mu}^{(\alpha+1,\beta+1)} \, d\sigma(\lambda) \right)(x).
 \end{aligned}$$

From [3] we have

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = 2\mathcal{F}f_e(\lambda) + 2i\lambda\mathcal{F}(Jf_o)(\lambda),$$

then

$$\begin{aligned}
 \int_{\mathbf{R}} \mathcal{F}_{\alpha,\beta}(f)(\lambda) \varphi_{\mu}^{(\alpha,\beta)}(x) \, d\sigma(\lambda) &= 2 \int_{\mathbf{R}} \mathcal{F}(f_e)(\lambda) \varphi_{\mu}^{(\alpha,\beta)}(x) \, d\sigma(\lambda) \\
 &= \int_0^{+\infty} \mathcal{F}(f_e)(\mu) \varphi_{\mu}^{(\alpha,\beta)}(x) \, d\nu_{\alpha,\beta}(\mu) = f_e(x)
 \end{aligned}$$

due to the inversion formula for Jacobi transform.

On the other hand

$$\begin{aligned} & \int_{\mathbf{R}} \frac{i\lambda}{2(\alpha+1)} \sinh x \cosh x \mathcal{F}_{\alpha,\beta} f(\lambda) \varphi_{\mu}^{(\alpha+1,\beta+1)}(x) d\sigma(\lambda) \\ &= 2 \int_{\mathbf{R}} \left(\frac{-i}{\lambda}\right) \mathcal{F}(f_e)(\mu) \frac{d}{dx} \varphi_{\mu}^{(\alpha,\beta)}(x) d\sigma(\lambda) \\ &\quad + 2i\lambda \int_{\mathbf{R}} \left(\frac{-i}{\lambda}\right) \mathcal{F}(J(f_o))(\mu) \frac{d}{dx} \varphi_{\mu}^{(\alpha,\beta)}(x) d\sigma(\lambda) \\ &= 2 \int_{\mathbf{R}} \mathcal{F}(Jf_o)(\mu) \frac{d}{dx} \varphi_{\mu}^{(\alpha,\beta)}(x) d\sigma(\lambda) \\ &= \int_0^{+\infty} \mathcal{F}(J(f_o))(\mu) \frac{d}{dx} \varphi_{\mu}^{(\alpha,\beta)}(x) d\nu_{\alpha,\beta}(\mu) \\ &= \frac{d}{dx} \left( \int_0^{+\infty} \mathcal{F}(J(f_o))(\mu) \varphi_{\mu}^{(\alpha,\beta)}(x) d\nu_{\alpha,\beta}(\mu) \right). \end{aligned}$$

As in above

$$\int_{\mathbf{R}} \frac{i\lambda}{2(\alpha+1)} \sinh x \cosh x \mathcal{F}_{\alpha,\beta} f(\lambda) \varphi_{\mu}^{(\alpha+1,\beta+1)}(x) d\sigma(\lambda) = \frac{d}{dx} J(f_o)(x) = f_o(x).$$

Thus (39) follows readily from (46) under the conditions in (i).

Now we shall show (39) under the conditions of part (ii). From ([10], p 124) and Proposition 3.1, we have

$$\begin{aligned} & \left| \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha,\beta)} T_y^{(k,l,\alpha,\beta)} \varphi_{\mu}^{(\alpha,\beta)}(x) T_{y,e}^{(k,l,\alpha,\beta)} g(x) \right| \\ & \leq c(\tau_y^{\alpha,\beta}(|f_e|)(x) + \tau_y^{\alpha,\beta}(|f_o|)(x)) \sum_{k=0}^{+\infty} \sum_{l=0}^k k^{d+2\alpha+1} |T_y^{(k,l,\alpha,\beta)} \varphi_{\mu}^{(\alpha,\beta)}(x)| \end{aligned}$$

and

$$\begin{aligned} & \left| \sum_{k=0}^{+\infty} \sum_{l=0}^k \Pi_{k,l}^{(\alpha+1,\beta+1)} T_y^{(k,l,\alpha+1,\beta+1)} \varphi_{\mu}^{(\alpha,\beta)}(x) T_{y,o}^{(k,l,\alpha,\beta)} g(x) \right| \\ & \leq c |\coth x| \cosh^2 y (\tau_y^{\alpha,\beta}(|f_e|)(x) + \tau_y^{\alpha,\beta}(|f_o|)(x)) \\ & \quad \times \sum_{k=0}^{+\infty} \sum_{l=0}^k k^{d+2\alpha+1} |T_y^{(k,l,\alpha+1,\beta+1)} \varphi_{\mu}^{(\alpha+1,\beta+1)}(x)| \end{aligned}$$

which converges uniformly for  $\mu$  in compact subsets of  $\mathbf{R}$ . Then by the dominated convergence theorem, integration term-by-term for (45), as we operate for (46), gives the formula (39).

*Remark.* The two cases  $\alpha = \beta > -\frac{1}{2}$  or  $\alpha > \beta = -\frac{1}{2}$  are related by the quadratic transformation

$$\Psi_{\lambda}^{(\alpha,-\frac{1}{2})}(2x) = \Psi_{2\lambda}^{(\alpha,\alpha)}(x),$$

then it suffices to study the representation of translation of two functions for one case.

If  $\alpha = \beta$ , from [8] we have

$$dm_{\alpha,\alpha}(r, \psi) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \delta(1 - r) \sin^{2\alpha} \psi \, dr \, d\psi,$$

then

$$\begin{aligned} T_y^{\alpha,\alpha} f(x) &= \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \left[ \int_0^\pi f_e(Z_{x,y}(1, \psi)) h^e(x, y, 1, \psi) \sin^{2\alpha} \psi \, d\psi \right. \\ &\quad \left. + \int_0^\pi f_o(Z_{x,y}(1, \psi)) h^o(x, y, 1, \psi) \sin^{2\alpha} \psi \, d\psi \right] \end{aligned}$$

and the addition formula (42) degenerates to a single series ( $l = 0$ ), therefore

$$\begin{aligned} T_{y,e}^{(k,0,\alpha,\alpha)} f(x) &= (-1)^k \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_0^\pi [f_e(Z_{x,y}(1, \psi)) h^e(x, y, 1, \psi) \\ &\quad + f_o(Z_{x,y}(1, \psi)) h^o(x, y, 1, \psi)] \times R_{k,0}^{(\alpha,\beta)}(1, \psi) \sin^{2\alpha} \psi \, d\psi \end{aligned}$$

and

$$\begin{aligned} T_{y,o}^{(k,0,\alpha,\alpha)} f(x) &= (-1)^k \int_0^\pi \frac{\sinh(Z_{x,y}(1, \psi)) \cosh(Z_{x,y}(1, \psi))}{\sinh x \cosh x} [f_e(Z_{x,y}(1, \psi)) h^o(x, y, 1, \psi) \\ &\quad + f_o(Z_{x,y}(1, \psi)) h^e(x, y, 1, \psi)] R_{k,0}^{(\alpha+1,\alpha+1)}(1, \psi) \sin^{2\alpha} \psi \, d\psi, \end{aligned}$$

finally we obtain the following representation

$$\begin{aligned} T_y^{\alpha,\alpha}(fg)(x) &= \sum_{k=0}^{+\infty} \Pi_{k,0}^{(\alpha,\alpha)} T_y^{(k,0,\alpha,\alpha)} f(x) T_{y,e}^{(k,0,\alpha,\alpha)} g(x) \\ &\quad + \sum_{k=0}^{+\infty} \Pi_{k,0}^{(\alpha+1,\alpha+1)} T_y^{(k,l,\alpha+1,\alpha+1)} f(x) T_{y,o}^{(k,0,\alpha,\alpha)} g(x). \end{aligned}$$

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